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On an inverse problem for a nonlinear loaded parabolic-hyperbolic equation of fractional order

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Abstract. The work is devoted to the study of the unique solvability of an inverse problem for a nonlinear parabolic-hyperbolic equation with the Caputo fractional derivative and nonlinear loads. The inverse problem is well posed in the mixed domain with characteristic triangle on the hyperbolic part. With the aid of main functional relations and nonlinear gluing condition, the investigation of the problem is reduced to a nonlinear integral equation. The unique solvability of the integral equation is proven by the method of successive approximations.

Keywords: Inverse problem, nonlinear parabolic-hyperbolic equation, fractional derivative, nonlinear loaded terms, nonlinear gluing condition, unique solvability.

MSC (2020):34K37, 35M10

1. Introduction

In this paper we will study an inverse problem on identifying the right hand sides of the following parabolic-hyperbolic equation of fractional order with nonlinear loaded term:

$$f(x)=\begin{cases} \beta u_{xx}-{}_CD_{0y}^{\alpha}u+\gamma u^2+p_1\left(x,u(x,0)\right), & y>0,\\ u_{xx}-u_{yy}+p_2\left(x,u(x,0)\right), & y<0,\end{cases} \tag{1.1}$$

where $\beta,\gamma=const$,

$${}_CD_{0y}^{\alpha}f(y)=\frac{1}{\Gamma(1-\alpha)}\int_0^y (y-t)^{-\alpha}f'(t)dt,\ \ 0<\alpha<1,$$

$p_i(x,z)\,(i=1,2)$ are given functions, $0<\alpha<1$.

Many different types of inverse problems for fractional-order PDEs are well studied. In this area some recent progress has been made in the series of articles; see, for example, [1]-[8]. The inverse problems on identifying the right-hand sides of the parabolic and parabolic-hyperbolic equations of fractional order from an additional over-determination condition have important applications in various areas of applied sciences and engineering. However, inverse problems for parabolic-hyperbolic equations of fractional order with nonlinear loads have been studied relatively less, see [9],[10].

In our paper, the aim is to study the inverse problem for equation (1.1) in a mixed domain with characteristic triangle.

2. Problem statement

Let Ω be a domain, bounded with segments: $A_1A_2=\{(x,y):\,x=l,\ 0<y<h\}$, $B_1B_2=\{(x,y):\ x=0,\ 0<y<h\}$, $B_2A_2=\{(x,y):\ y=h,\ 0<x<l\}$ at $y>0$, and with characteristics: $A_1C:\,x-y=l$, $B_1C:\,x+y=0$ of equation (1.1) for $y<0$, where $A_1(l;0)$, $A_2\left(l;h\right),\ B_1(0;0),\ B_2(0;h)$ and $C\left(\frac{l}{2};\frac{-l}{2}\right)$.

We consider the domains: $\Omega^+=\Omega\cap(y>0)$, $\Omega^-=\Omega\cap(y<0)$, $I=\{x:\,0<x<l\}$, $I_1=\{x:\,0<x<\frac{l}{2}\}$, $I_2=\{x:\,\frac{l}{2}<x<l\}$.

Problem. To find functions $f(x)\in C(0,l)\cap L_1(0,l)$ and $u(x,y)$ from the class W such that

$$W=\Big\{u\in C(\bar\Omega)\cap C^2(\Omega^-),u_{xx}\in C\left(\Omega^+\right), {}_CD^{\alpha}_{oy}u\in C\left(\Omega^+\cup I\right),$$
$$u(x,y)\in AC[0,h]\ \ \forall x\in[0,l],\ u\in C^1\big(\bar\Omega^-\setminus A_1B_1\big),u_y\in C\left(\Omega^-\cup I\right)\Big\},\quad (2.1)$$

satisfying the boundary conditions:

$$u(0,y)=0,\ \ u(l,y)=0,\ \ 0\le y\le h; \tag{2.2}$$

$$u(x,-x)=\varphi(x),\ \ \ 0\le x\le \frac{l}{2}; \tag{2.3}$$

$$u_n(x,y)\,|_{B_1C}=\psi_1(x),\ \ 0\le x\le\frac{l}{2},\ \ u_n(x,y)\,|_{A_1C}=\psi_2(x),\ \ \frac{l}{2}\le x\le l; \tag{2.4}$$

and nonlinear gluing condition:

$$\lim_{y\to+0}y^{1-\alpha}u_y(x,y)=\lambda(x)u_y(x,-0)+r(x,u(x,0))\ \ 0<x<l, \tag{2.5}$$

where $\varphi(x)$, $\psi_i(x)\,(i=1,2)$, $\lambda(x)$ are given functions.

We will study equation (1.1) in the class (2.1) with the conditions (2.2), (2.3), (2.4) and (2.5) when the following conditions are fulfilled:

$$\varphi(x)\in C^1\left[0,\frac{l}{2}\right]\cap C^2\left(0,\frac{l}{2}\right); \tag{2.6}$$

$$\lambda(x)\in C^1[0,l],\ \psi_1(x)\in C\left[0,\frac{l}{2}\right]\cap C^1\left(0,\frac{l}{2}\right),\ \psi_2(x)\in C\left[\frac{l}{2},l\right]\cap C^1\left[\frac{l}{2},l\right), \tag{2.7}$$

$$|p_i(x,z)|\le p_{0i}\,|z|,\quad |\Gamma(\alpha)r(x,z)|\le r_0\,|z|, \tag{2.8}$$

$$|p_i(x,z_1)-p_i(x,z_2)|\le L_i\,|z_1-z_2|,\ \ |r(x,z_1)-r(x,z_2)|\le L_0\,|z_1-z_2|, \tag{2.9}$$

$$\psi_1\left(\frac{l}{2}\right)=\psi_2\left(\frac{l}{2}\right),\quad \psi_1'\left(\frac{l}{2}\right)=-\psi_2'\left(\frac{l}{2}\right), \tag{2.10}$$

where $L_0,\ L_i,\ p_{0i},\ \ r_0=\text{const}>0,\ \ i=1,2.$

3. Main functional relations

Taking into account the condition (2.5), the general solution to equation (1.1) in the domain Ω^- we write as follows:

$$u(x,y)=F_1(x+y)+F_2(x-y)+\omega(x), \tag{3.1}$$

where $F_1(x,y)$ and $F_2(x,y)$ are any twice continuously differentiable functions, $\omega(x)$ is arbitrary solution to a differential equation

$$\omega''(x)=f(x)-p_2\left(x,u(x,0)\right). \tag{3.2}$$

We assume, that

$$\omega(x)=\begin{cases}\omega_1(x), & 0\le x\le\frac{l}{2},\\ \omega_2(x), & \frac{l}{2}\le x\le l.\end{cases}$$

Taking into account the condition (3.1), from the solution (3.1) we get

$$\omega_1''(x) = \sqrt{2}\psi_1'(x), \quad 0 \le x \le \frac{l}{2}; \quad \omega_1''(x) = -\sqrt{2}{\psi_2}'(x), \quad \frac{l}{2} \le x \le l. \tag{3.3}$$

We set

$$f(x) = \begin{cases} f_1(x), & 0 < x \le \frac{l}{2}, \\ f_2(x), & \frac{l}{2} \le x < l. \end{cases} \tag{3.4}$$

If $f_1\left(\frac{l}{2}\right) = f_2\left(\frac{l}{2}\right)$, then, by virtue of (3.2) and (3.3), we find

$$f_1(x) = p_2\left(x, u(x,0)\right) + \sqrt{2}{\psi_1}'(x), \quad 0 \le x \le \frac{l}{2}, \tag{3.5}$$

$$f_2(x) = p_2\left(x, u(x,0)\right) - \sqrt{2}{\psi_2}'(x), \quad \frac{l}{2} \le x \le l. \tag{3.6}$$

Moreover, integrating equations (3.3), we have

$$\omega(x) = \begin{cases} \sqrt{2}\int\limits_0^x \psi_1(t)dt + c_1 x, & 0 \le x \le \frac{l}{2}, \\ \sqrt{2}\int\limits_x^l \psi_2(t)dt + c_2(l-x), & \frac{l}{2} \le x \le l, \end{cases} \tag{3.7}$$

where c_1 and c_2 are any constants.

The function $\omega(x)$ must be a twice continuously differentiable function for $0 < x < l$. Hence, we find c_1 and c_2:

$$c_1 = \frac{\sqrt{2}}{l}\left(-\int\limits_0^{l/2}\psi_1(t)dt + \int\limits_{l/2}^{l}\psi_2(t)dt\right) - \frac{\sqrt{2}}{2}\left(\psi_1\left(\frac{l}{2}\right) + \psi_2\left(\frac{l}{2}\right)\right),$$

$$c_2 = \frac{\sqrt{2}}{l}\left(\int\limits_0^{l/2}\psi_1(t)dt - \int\limits_{l/2}^{l}\psi_2(t)dt\right) - \frac{\sqrt{2}}{2}\left(\psi_1\left(\frac{l}{2}\right) + \psi_2\left(\frac{l}{2}\right)\right).$$

Substituting the values of c_1 and c_2 equations (3.7), we find the function $\omega(x)$ in explicit form:

$$\omega(x) = \begin{cases} \sqrt{2}\int\limits_0^x \psi_1(t)dt - \frac{\sqrt{2}x}{l}\left(\int\limits_0^{l/2}\psi_1(t)dt - \int\limits_{l/2}^{l}\psi_2(t)dt\right) - \frac{\sqrt{2}x}{2}\left(\psi_1\left(\frac{l}{2}\right) + \right. \\ \left. +\psi_2\left(\frac{l}{2}\right)\right), \ 0 \le x \le \frac{l}{2}; \\ \sqrt{2}\int\limits_x^l \psi_2(t)dt + \frac{\sqrt{2}(l-x)}{l}\left(\int\limits_0^{l/2}\psi_1(t)dt - \int\limits_{l/2}^{l}\psi_2(t)dt\right) - \\ -\frac{\sqrt{2}(l-x)}{2}\left(\psi_1\left(\frac{l}{2}\right) + \psi_2\left(\frac{l}{2}\right)\right), \ \frac{l}{2} \le x \le l. \end{cases} \tag{3.8}$$

Note that the solution to equation (1.1) with initial data

$$u(x,0) = \tau(x), \ 0 \le x \le l \quad \text{and} \quad u_y(x,0) = \nu^-(x), \quad 0 < x < l \tag{3.9}$$

in the domain Ω^- has a form:

$$u(x,y) = \frac{1}{2}\big(\tau(x-y) + \tau(x+y)\big) - \frac{1}{2}\big(\omega(x-y) + \omega(x+y)\big) - \frac{1}{2}\int\limits_{x+y}^{x-y}\nu(t)dt + \omega(x). \tag{3.10}$$

By virtue of the condition (2.3), from (3.10) we find

$$\nu^{-}(x)=\tau'(x)-\omega'(x)+\omega'\left(\frac{x}{2}\right)-\varphi'\left(\frac{x}{2}\right). \tag{3.11}$$

Further, we consider the functions in (3.9), $\lim\limits_{y\to+0} y^{1-\alpha}u_y(x,y)=\nu^{+}(x)$, gluing condition (2.5), ${}_cD_{0y}^{\alpha}f(y)=D_{0y}^{\alpha-1}f'(y)$ and well known equality $\lim\limits_{y\to 0} D_{0y}^{\alpha-1}f'(y)=\Gamma(\alpha)\lim\limits_{y\to 0} y^{1-\alpha}f'(y)$. Then from equation (1.1) at $y\to+0$ we obtain [11]:

$$\beta\,\tau''(x)-\Gamma(\alpha)\,\lambda(x)\nu^{-}(x)-\Gamma(\alpha)\,r\left(x,\tau(x)\right)+\gamma\,\tau^2(x)+p_1\left(x,\tau(x)\right)=f(x). \tag{3.12}$$

Hence, substituting the functional relation (3.11) into (3.12), we have

$$\tau''(x)+\overline{\lambda}(x)\,\tau'(x)=F(x,\tau(x))+f_0(x), \tag{3.13}$$

where $\overline{\lambda}(x)=-\frac{\Gamma(\alpha)}{\beta}\lambda(x)$,

$$F(x,\tau(x))=-\frac{\gamma}{\beta}\,\tau^2(x)+\frac{\Gamma(\alpha)}{\beta}\,r\left(x,\tau(x)\right)-\frac{1}{\beta}p_1\left(x,\tau(x)\right), \tag{3.14}$$

$$f_0(x)=\frac{1}{\beta}\left[f(x)+\Gamma(\alpha)\,\lambda(x)\,\left(-\omega'(x)+\omega'\left(\frac{x}{2}\right)-\varphi'\left(\frac{x}{2}\right)\right)\right]. \tag{3.15}$$

4. Unique solvability of a nonlinear integral equation

By virtue of the class of the function $u(x,y)$, we have

$$\tau(0)=0,\quad \tau(l)=0. \tag{4.1}$$

Considering (3.3), (3.5),(3.6) and due to the boundary conditions (4.1), the nonlinear differential equation (3.13) we reduce to a nonlinear integral equation:

$$\tau(x)+\frac{\gamma}{\beta}\int_0^l G_0(x,t)\,\tau^2(t)dt-\frac{\Gamma(\alpha)}{\beta}\int\limits_0^l G_0(x,t)\,r(t,\tau(t))dt+$$

$$+\frac{1}{\beta}\int\limits_0^l G_0(x,t)\big(p_1(t,\tau(t))-p_2(t,\tau(t))\big)dt=\frac{1}{\beta}q(x). \tag{4.2}$$

where $G_0(x,t)$ is a Greens' function of the problem:

$$\begin{cases}\tau''(x)+\overline{\lambda}(x)\tau'(x)=0,\\ \tau(0)=\tau(l)=0;\end{cases} \tag{4.3}$$

and

$$q(x)=\sqrt{2}\int\limits_0^{l/2} G_0(x,t)\,\psi_1(t)dt-\sqrt{2}\int\limits_{l/2}^{l} G_0(x,t)\,\psi_2(t)dt+$$

$$+\Gamma(\alpha)\int\limits_0^l G_0(x,t)\,\lambda(t)\left(\omega'\Big(\frac{t}{2}\Big)-\omega'(t)-\varphi'\left(\frac{t}{2}\right)\right)dt.$$

Notice that the class of function $\lambda(x)$ (see (2.7)) provides existence and uniqueness of the Greens' function to the problem (4.3).

The unique solvability of the integral equation (4.2) we prove by the method of successive approximations. So, we consider the following successive approximations

$$\tau_n(x)+\frac{\gamma}{\beta}\int\limits_0^l G_0(x,t)\,\tau_{n-1}^2(t)dt-\frac{\Gamma(\alpha)}{\beta}\int_0^l G_0(x,t)\,r(t,\tau_{n-1}(t))dt+$$

$$+\frac{1}{\beta}\int\limits_0^l G_0(x,t)\,\left(p_1(t,\tau_{n-1}(t))-p_2(t,\tau_{n-1}(t))\right)dt=\frac{1}{\beta}q(x),\quad 0\le x\le l \tag{4.4}$$

with zero approximation $\tau_0(x)=\frac{1}{\beta}q(x)$.

Due to the classes of given functions (2.6) and (2.7), we have

$$|\psi_i(x)|\le\psi_{0i},\ \ |\psi(x)|\le\psi_0,\ \ |\Gamma(\alpha)\lambda(x)|\le\lambda_0,\ \ |\lambda'(x)|\le\lambda_{01};$$

$$|G_0(x,t)|\le K_0,\ \ |q(x)|\le q_0,\ \ |\tau_0(x)|\le\frac{1}{\beta}q_0\le q_0,$$

where K_0 and q_0 are known constants.

Theorem 1. *If the all conditions* (2.6), (2.7), (2.8), (2.9) *and* (2.10) *are fulfilled, and*

$$\begin{cases} K_0l\left(\gamma q_0+r_0+p_{01}+p_{02}\right)+1<\beta;\\ K_0l\big(2\gamma q_0+L_0+L_1+L_2\big)<\beta,\end{cases} \tag{4.5}$$

then the integral equation (4.2) *is uniquely solvable.*

Proof. First, we show the boundedness of each term is substantiated by the sequence $\{\tau_n(x)\}$, which is constructed from the approximations (4.4)

$$|\tau_n(x)|\le q_0A_0<q_0, \tag{4.6}$$

for all $n=1,2,3...$, where $A_0=\frac{K_0l(\gamma q_0+r_0+p_{01}+p_{02})+1}{\beta}<1$.

Taking $|\tau_0(x)|\le\frac{1}{\beta}q_0<q_0$ into account, from the recurrent equation (4.4) we find

$$|\tau_1(x)|\le q_0\frac{K_0l(\gamma q_0+r_0+p_{01}+p_{02})+1}{\beta}=q_0A_0.$$

Further, using the method of mathematical induction, we will prove, that

$$|\tau_n(x)|\le q_0A_n<q_0A_0,\ \ n=1,2,3,....$$

With this aim, we assume that $A_{n-1}=\frac{K_0l(\gamma q_0A_{n-2}+r_0+p_{01}+p_{02})A_{n-2}+1}{\beta}$, $n\ge2$, then we obtain

$$A_n=\frac{K_0l(\gamma q_0A_{n-1}+r_0+p_{01}+p_{02})A_{n-1}+1}{\beta}<$$

$$<\frac{K_0l(\gamma q_0A_0+r_0+p_{01}+p_{02})A_0+1}{\beta}=A_1<A_0.$$

Thus, we prove that $|\tau_n(x)|\le q_0A_0<q_0$ for all $n=1,2,3,...$ and $\forall x\in[0,l]$.

Now we prove the absolute and uniform convergence of the functional sequence $\{\tau_n(x)\}$. Taking into account (4.6), from the recurrence equation (4.4) we obtain

$$|\tau_1(x)-\tau_0(x)|\leq q_0A_0,$$

$$|\tau_2(x)-\tau_1(x)|<q_0A_0M,$$

$$...$$

$$|\tau_n(x)-\tau_{n-1}(x)|<q_0A_0M^{n-1},$$

where $M=\frac{K_0l(2\gamma q_0A_0+L_0+L_1+L_2)}{\beta}<1.$
Therefore, by virtue of the conditions of the theorem, we conclude that the functional sequence $\{\tau_n(x)\}$ converges absolutely and uniformly. Theorem 1 is proved.

Using solution of a first boundary value problem for the equation ${}_CD^{\alpha}_{0t}u-u_{xx}=f(x,t)$, in the domain Ω^+, we get [12], [13] :

$$u(x,t)=\frac{1}{\sqrt{\beta}}\int_0^l\tau(\xi)D^{\alpha-1}_{0t}G(x,t,\xi,0)d\xi+\int_0^t\int_0^lG(x,t,\xi,\eta)R(\xi,\eta,u)d\xi d\eta, \tag{4.7}$$

where

$$R(\xi,\eta,u)=\frac{1}{\beta}\left(\gamma u^2+p_1(\xi,\tau(\xi))-f(\xi)\right), \tag{4.8}$$

$$G(x,t,\xi,\eta)=\frac{(t-\eta)^{(\alpha/2)-1}}{2}\sum_{n=-\infty}^{\infty}\left[e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|x-\xi+2nl|}{\sqrt{\beta}(t-\eta)^{\alpha/2}}\right)+e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|x+\xi+2nl|}{\sqrt{\beta}(t-\eta)^{\alpha/2}}\right)\right],$$

$G(x,t,\xi,\eta)$ is Green's function of the first boundary value problem for equation (1.1) in the domain Ω^+, $e^{\beta,\delta}_{\alpha,\mu}$ is Whrite type function

Taking into account (4.8), due to the properties of Green's function $G_\xi(x,t;0,\eta)=G(x,t;l,\eta)=0$, $t\neq\eta$ from the solution (4.7), we have a nonlinear Volterra integral equation of second kind:

$$u(x,t)=\frac{\gamma}{\beta}\int_0^td\eta\int_0^lG(x,t,\xi,\eta)u^2(\xi,\eta)d\xi+\frac{1}{\sqrt{\beta}}\overline{F}(x,t), \tag{4.9}$$

where

$$\overline{F}(x,t)=\int_0^l\tau(\xi)D^{\alpha-1}_{0t}G(x,t,\xi,0)d\xi+\frac{1}{\sqrt{\beta}}\int_0^t\int_0^lG(x,t,\xi,\eta)\left(p_1(\xi,\tau(\xi))-f(\xi)\right)d\xi d\eta.$$

Moreover,

$$|G(x,t,\xi,\eta)|\leq K_1(t-\eta)^{\frac{\alpha}{2}-1},\quad \left|\overline{F}(x,t)\right|\leq f_0. \tag{4.10}$$

Theorem 2. *If the conditions* (2.6) *and*

$$h<\left(\frac{\alpha\sqrt{\beta}(\sqrt{\beta}-1)}{4\gamma K_1f_0}\right)^{\frac{2}{\alpha}} \tag{4.11}$$

are fulfilled, then the integral equation (4.9) *is uniquely solvable.*

Proof. The unique solvability of the integral equation (4.9) will be proven by the method of successive approximations. Let $u_0(x,t)=\frac{\overline{F}(x,t)}{\sqrt{\beta}}$, then taking into account (4.10), we have $|u_0|\leq\frac{f_0}{\sqrt{\beta}}<f_0$. Next, from the recurrence equation

$$u_n(x,t)=\frac{\gamma}{\beta}\int_0^td\eta\int_0^lG(x,t,\xi,\eta)u^2_{n-1}(\xi,\eta)d\xi+\frac{1}{\sqrt{\beta}}\overline{F}(x,t), \tag{4.12}$$

we get

$$|u_1| \le \left|\frac{\gamma}{\beta}\int_0^t d\eta \int_0^l G(x,t,\xi,\eta)u_0^2(\xi,\eta)d\xi\right| + \left|\frac{\overline{F}(x,t)}{\sqrt{\beta}}\right| \le$$

$$\le \frac{2\delta K_1 f_0^2}{\alpha\beta}t^{\frac{\alpha}{2}} + \frac{f_0}{\sqrt{\beta}} \le \frac{f_0[2\gamma K_1 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}]}{\alpha\beta} \le f_0 B_0,$$

where $B_0 = \frac{2\gamma K_1 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}}{\alpha\beta}$.

Taking into account the above from the recurrent equation (4.12), we find

$$|u_2| \le \left|\frac{\gamma}{\beta}\int_0^t d\eta \int_0^l G(x,t,\xi,\eta)u_1^2(\xi,\eta)d\xi\right| + \left|\frac{\overline{F}(x,t)}{\sqrt{\beta}}\right| \le$$

$$\le \frac{2\gamma K_1 f_0^2 B_0^2}{\alpha\beta}t^{\frac{\alpha}{2}} + \frac{f_0}{\sqrt{\beta}} \le \frac{f_0[2\gamma K_1 B_0^2 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}]}{\alpha\beta} \le f_0 B_1,$$

where $B_1 = \frac{2\gamma K_1 B_0^2 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}}{\alpha\beta}$. Moreover, by virtue (4.11), it is easy to show, that $B_0 > B_1 > \frac{1}{\sqrt{\beta}}$. By similar reasoning, we get

$$|u_3| \le \left|\frac{\gamma}{\beta}\int_0^t d\eta \int_0^l G(x,t,\xi,\eta)u_2^2(\xi,\eta)d\xi\right| + \left|\frac{\overline{F}(x,t)}{\sqrt{\beta}}\right| \le$$

$$\le \frac{2\gamma K_1 f_0^2 B_1^2}{\alpha\beta}t^{\frac{\alpha}{2}} + \frac{f_0}{\sqrt{\beta}} \le \frac{f_0[2\gamma K_1 B_1^2 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}]}{\alpha\beta} < f_0 B_2,$$

where $B_2 = \frac{2\gamma K_1 B_1^2 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}}{\alpha\beta}$, $B_0 > B_1 > B_2 > \frac{1}{\sqrt{\beta}}$.

Now, we assume that

$$|u_{n-1}| \le \left|\frac{\gamma}{\beta}\int_0^t d\eta \int_0^l G(x,t,\xi,\eta)u_{n-2}^2(\xi,\eta)d\xi\right| + \left|\frac{\overline{F}(x,t)}{\sqrt{\beta}}\right| \le$$

$$\le \frac{2\gamma K_1 f_0^2 B_{n-3}^2}{\alpha\beta}t^{\frac{\alpha}{2}} + \frac{f_0}{\sqrt{\beta}} \le \frac{f_0[2\gamma K_1 B_{n-3}^2 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}]}{\alpha\beta} < f_0 B_{n-2},$$

where $B_{n-2} = \frac{2\gamma K_1 B_{n-3}^2 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}}{\alpha\beta}$, (and $B_0 > B_1 > B_2 > .... > B_{n-2} > \frac{1}{\sqrt{\beta}}$).

By induction method, one can be see that

$$|u_n| < f_0 B_{n-1} < f_0 B_0 < f_0 < \infty, \tag{4.13}$$

where

$$B_{n-1} = \frac{2\gamma K_1 B_{n-1}^2 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}}{\alpha\beta} < \frac{2\gamma K_1 h^{\frac{\alpha}{2}} f_0 + \alpha\sqrt{\beta}}{\alpha\beta} = B_0.$$

Thus, we observe that all terms of the functional sequence $\{u_n(x,t)\}$, are limited under the conditions of Theorem 2.

Now, let us prove the possibility of convergence of the functional sequence $\{u_n(x,t)\}$. For this purpose, taking into account (4.13) and (4.11), from the recurrence equation (4.12), we get the following:

$$|u_1 - u_0| \le \left|\frac{\gamma}{\beta}\int_0^t d\eta \int_0^l K_1(t-\eta)^{\frac{\alpha}{2}-1}|u_0^2(\xi,\eta)|d\xi\right| \le \frac{2\gamma K_1 f_0^2}{\alpha\beta}t^{\frac{\alpha}{2}} \le f_0 M_0,$$

$$|u_2-u_1|\le\left|\frac{\gamma}{\beta}\int_0^t d\eta\int_0^l K_1(t-\eta)^{\frac{\alpha}{2}-1}|u_1^2-u_0^2|d\xi\right|\le\frac{4\gamma K_1 f_0^2 M_0}{\alpha\beta}t^{\frac{\alpha}{2}}\le f_0 M_0^2,$$

$$|u_3-u_2|\le\left|\frac{\gamma}{\beta}\int_0^t d\eta\int_0^l K_1(t-\eta)^{\frac{\alpha}{2}-1}|u_2^2-u_0^2|d\xi\right|\le\frac{4\gamma K_1 f_0^2 M_0^2}{\alpha\beta}t^{\frac{\alpha}{2}}\le f_0 M_0^3,$$

....

$$|u_n-u_{n-1}|\le\left|\frac{\gamma}{\beta}\int_0^t d\eta\int_0^l K_1(t-\eta)^{\frac{\alpha}{2}-1}|u_{n-1}^2-u_{n-2}^2|d\xi\right|\le f_0 M_0^n,$$

where $M_0=\frac{4\gamma K_1 h^{\frac{\alpha}{2}} f_0}{\alpha\beta}$.

By virtue of the conditions of Theorem 2, we have that $0<M_0<1$, that is, the functional sequence $\{u_n(x,t)\}$ converges absolutely and uniformly. Therefore, the integral equation (4.9) is uniquely solvable. Theorem 2 is proved.

After finding $\tau(x)$ (as a solution to the integral equation (4.2), we find $\nu^-(x)$ from the relation (3.11) in the interval $(0,l)$. Consequently, we restore the solution to Problem in the domain Ω^- as a solution to the Cauchy problem. This gives us the opportunity to restore the solution to Problem in the domain Ω^+ as a solution to the integral equation (4.9).

We formulate the following main theorem:

Theorem 3. *If the all conditions* (2.6)–(2.10)

$$\begin{cases} K_0 l\,(\gamma q_0+r_0+p_{01}+p_{02})+1<\beta; \\ K_0 l\big(2\gamma q_0+L_0+L_1+L_2\big)<\beta, \end{cases}$$

and

$$h<\left(\frac{\alpha\sqrt{\beta}(\sqrt{\beta}-1)}{4\gamma K_1 f_0}\right)^{\frac{2}{\alpha}}$$

are fulfilled, then the inverse problem (1.1)–(2.5) *is uniquely solvable.*

Proof of Theorem 3, follows from Theorem 1 and Theorem 2.

Remark 1. The set of functions and parameters that satisfy all the conditions of Theorem 3 is not empty.

Actually, owing to the class of given functions we can assume, that $m=max\{r_0+p_{01}+p_{01},L_0+L_1+L_1\}$, $\frac{\gamma q_0 l^2}{4}\le 1$, besides, $\gamma q_0=m=1$, $4K_0=l$. Hence, from the first condition of Theorem 3, we get $\beta>\frac{3l^2}{4}+1$. Notice that these values of β exists a value of h satisfying second condition of Theorem 3.

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On the solution of a system of linear Diophantine equations in prime numbers

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Abstract. In the paper, it is proved that the system of linear Diophantine equations consisting of s equations with m unknowns (where $s < m \le 2s$) is solvable in prime numbers with some exceptions. A lower bound for the number of solutions of the system under consideration is also obtained for the exceptional set. The obtained results complement the corresponding results of Wu Fang, M. C. Liu, K. M. Tsang and I. Allakov.

Keywords: system of linear equations, asymptotic formula, power estimate, congruent solvability, positive solvability, Dirichlet character, principal Dirichlet character, Dirichlet $L-$function, exceptional character, exceptional zero, Mangoldt function, minor arcs, major arcs.

MSC (2020): 11D04, 11A41.

1. Introduction

Let $b_1,\ b_2,\dots,b_s$ be natural numbers, $a_{i1},\ a_{i2},\dots,a_{im}$ be integer numbers, $p_1,\ p_2,\dots,\ p_m$ be prime numbers. Consider systems of linear equations

$$b_i = a_{i1}p_1 + a_{i2}p_2 + \cdots + a_{im}p_m, \quad (i = 1,2,\dots,s;\ m > s)\,. \tag{1.1}$$

Let $U_{s,m}(X)$ be the set of $(b_1, b_2, \dots, b_s)$, $1 \le b_1, b_2, \dots, b_s \le X$, for which the system (1.1) is unsolvable in prime numbers and let $E_{s,m}(X) = \text{card}\ U_{s,m}(X)$.

For $m \ge 2s + 1$, Wu Fang [1], studied systems (1.1) under some additional conditions, obtained an asymptotic formula for the number of solutions of system (1.1).

It is known that solvability of system (1.1) is connected with the following two conditions:

a) for any prime number p there are integers $l_1,\dots,l_m$, $1 \le l_1,\dots,l_m \le p-1$, satisfying the system of linear comparisons $a_{i1}l_1 + a_{i2}l_2 + \ldots + a_{im}l_m \equiv b_i \,(modp)\,,\ (i = 1,\dots,s)$.

b) for some positive real numbers $y_1,\dots,y_m$, the equalities $a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{im}y_m = b_i$, hold for $(i = 1, ..., s)$.

Let $W_{s,m}(X)$ be a set of vectors $\vec{b} = (b_1, b_2, \dots, b_s)$, $1 \le b_1, b_2, \dots, b_s \le X$ satisfying conditions a) and b). $R(\vec{b})$ denotes the number of prime solutions of system (1.1) and $B = \max\{3\,|a_{ij}|\}$, $(1 \le i \le s,\ 1 \le j \le m)$.

The works of M.C.Liu K.M.Tsang [2], I.Allakov [3, 4], Hua Loo-Ken (see [5], p. 163), B.Kh.Abrayev [6, 7], B.Kh.Erdonov [8] and others are devoted to the study of the functions $E_{s,m}(X)$, $W_{s,m}(X)$ and $R(\vec{b})$, for different values of s and m.

Note that in the case of $s < m \le 2s$, $(s > 3)$, not only has an asymptotic formula for $R(\vec{b})$ been obtained yet, but in the general case, the existence of solutions to system (1.1) has not been established yet.

In this paper, we consider this general case. Let us set $s = n$, $m = n + k$, $(1 \le k \le n)$ and consider the following system

$$b_i = a_{i1}p_1 + a_{i2}p_2 + ... + a_{i,n+k}p_{n+k}, \quad \left(i = \overline{1,n}\right). \tag{1.2}$$

Thus, system (1.1) is considered for all possible values of s and m. It is known that the solvability of the system (1.2) depends on conditions a) and b). It is easy to see that card$W_{n,n+1}(X) \leq$ card$W_{n,n+k}(X)$. From this and from ([4], Theorem 3.1.1) it follows that the set $W_{n,n+k}(X)$ contains a sufficiently large number of elements.

For convenience, we introduce the following notations:

$$\Delta_{i_1 i_2 \ldots i_n} = det \begin{pmatrix} a_{1i_1} & a_{1i_2} & \ldots & a_{1i_n} \\ a_{2i_1} & a_{2i_2} & \ldots & a_{2i_n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{ni_n} & a_{ni_2} & \ldots & a_{ni_n} \end{pmatrix}, \ 1 \leq i_1, i_2, \ldots, i_n \leq n+k, \ i_1 \neq i_2 \neq \ldots \neq i_n.$$

In addition, we use the notations $\Delta_{12\ldots n} = \Delta$, $\Delta_{k+1,\ldots,n+k} = \bar{\Delta}$ and $\Delta_{i,j}$ is determinant obtained from the determinant $\Delta_{i_1 i_2 \ldots i_n}$ by replacing the elements of the i-th column with the elements of the j-th column.

To avoid trivial cases, we impose the following conditions on the coefficients:

$$\text{all } \Delta_{i_1 \ldots i_n}, \ (1 \leq i_1 < \ldots < i_n \leq n+k) \text{ are nonzero and mutually prime.} \tag{1.3}$$

The main result of the work is the following theorem:

Theorem 1.1. *If $\varepsilon > 0$ is a sufficiently small real number, then:*
a) there exists a sufficiently large number A such that for $X \geq B^A$ the estimate $E_{n,n+k}(X) \leq X^{n-\varepsilon}$ is valid;
b) if $R(\vec{b})$ is the number of representations of the given natural numbers $(b_1, \ldots, b_n)$ in the form of (1.2) for $p_j \leq N$ and $(b_1, \ldots, b_n) \in W_{n,n+k}(X)$, then for all $(b_1, \ldots, b_n) \in W_{n,n+k}(X)$ except for at most $X^{n-\varepsilon}$ values, the estimate $R(\vec{b}) \gg \mathcal{K}^{k-\varepsilon}(\ln \mathcal{K})^{-n-k}$ holds, where $N = 3(n!)^2 B^{2n-1} X$, $\mathcal{K} = 3(n!)^2 B^{2n-1} |\vec{b}| (\sqrt{n})^{-1}$.

2. Notations and the idea of the prove of Theorem 1.1

Let a_{ij} $\left(i = \overline{1,n};\ \ j = \overline{1,n+k}\right)$ be integers satisfying the condition $a_{i1}y_1 + \ldots + a_{i,n+k}y_{n+k} > 0$, $\left(i = \overline{1,n}\right)$, $c_1, c_2, \ldots$ are effectively computable positive constants and δ is a sufficiently small effectively computable positive number. $\mathbb{R}$ is the set of real numbers and let $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n}$.

For any integer $q \geq 1$, we denote by $\sum\limits_{(q)}$ the summation over all $l_1, \ldots, l_{n+k}$ satisfying the conditions $1 \leq l_j \leq q$, $(l_j, q) = 1$, $\sum\limits_{j=1}^{n+k} a_{ij} l_j \equiv b_i \,(\text{mod} q)$ and let $N(q) = \sum\limits_{(q)} 1$.

Let n be a fixed natural number and $X \geq B^{\exp\left(\delta^{-2}\right)}$. Let

$$N := 3(n!)^2 B^{2n-1} X, \quad Q := N^{\delta}, \quad L := N Q^{-\frac{1}{90}}, \ \ T := Q^{\frac{1}{\sqrt{\delta}}}. \tag{2.1}$$

From (2.1), it follows that

$$B \leq Q^{\delta}. \tag{2.2}$$

We denote by $\chi\,(\text{mod} q)$ and $\chi_0\,(\text{mod} q)$ the Dirichlet character and the principal Dirichlet character modulo q, respectively. For arbitrary $y \in \mathbb{R}$ and positive integer q, we define $e(y) = e^{2\pi i y}$ and $e_q(y) = e\left(\frac{y}{q}\right)$,

$$\begin{cases} S(y) := \sum\limits_{L < n \leq N} \Lambda(n)\, e(ny), \quad S_{\chi}(y) := \sum\limits_{L < n \leq N} \Lambda(n)\, \chi(n)\, e(ny), \\ I(y) := \int\limits_{L}^{N} e(xy)\, dx, \quad \tilde{I}(y) := \int\limits_{L}^{N} x^{\tilde{\beta}-1} e(xy)\, dx, \quad I_{\chi}(y) := \int\limits_{L}^{N} e(xy) \sum\limits_{|\gamma| \leq T}{}' x^{\rho-1} dx, \end{cases} \tag{2.3}$$

where $\sum\limits_{|\gamma|\leq T}{}'$ means that the summation is carried out over all zeros $\rho=\beta+i\gamma$ of the function $L(s,\chi)$ in the region $0,5\leq\beta\leq 1-c_1\ln^{-1}T,\ |\gamma|\leq T$ (in particular, $\tilde{\beta}$ is excluded) (see [9], p. 78) and $\Lambda(n)$ is the Mangoldt function. Let $\tau=N^{-1}T^{1/2n}$.
For arbitrary integers $h_1,...,h_n,q$ with the condition

$$1\leq h_1,...,h_n\leq q\leq Q\quad (h_1,...,h_n,q)=1, \tag{2.4}$$

we define $m(h_1,...,h_n,q)=\left\{(x_1,...,x_n)\in\mathbb{R}^n:\ \left|x_i-\frac{h_i}{q}\right|\leq\frac{\tau}{q},\ \ i=\overline{1,n}\right\}$ and $\mathfrak{M}_1,\ \mathfrak{M}_2$ by the following equalities

$$\mathfrak{M}_1=\bigcup_{h_1,...,h_n,q} m(h_1,...,h_n,q),\quad \mathfrak{M}_2=[\tau,\ 1+\tau]^n\backslash\mathfrak{M}_1. \tag{2.5}$$

In what follows, $\mathfrak{M}_1$ is called the major arc, and $\mathfrak{M}_2$ is the minor arc.

It is easy to see that these n-dimensional cubes $m(h_1,...,h_n,q)$ are mutually disjoint (see [3], p. 82) and lie in $[\tau,\ 1+\tau]^n$.

For arbitrary $\vec{b}=(b_1,...,b_n)\in W_{n,n+k}(X)$ and $(x_1,...,x_n)\in\mathbb{R}^n$ we define

$$\bar{x}_b=b_1x_1+...+b_nx_n,\ \ \bar{x}_j=a_{1j}x_1+...+a_{nj}x_n,\quad \left(j=\overline{1,n+k}\right) \tag{2.6}$$

and

$$I\left(\vec{b}\right)=\sum\Lambda(m_1)...\Lambda(m_{n+k}). \tag{2.7}$$

In equality (2.7) the summation is carried out over all m_j that satisfy the conditions $L<m_1,...,m_{n+k}\leq N$ and $\sum\limits_{j=1}^{n+k}a_{ij}m_j=b_i,\quad \left(i=\overline{1,n}\right)$. Using (2.3) and (2.5) we can represent $I\left(\vec{b}\right)$ as [10]:

$$I\left(\vec{b}\right)=\int\limits_{\tau}^{\tau+1}...\int\limits_{\tau}^{\tau+1}e(-\bar{x}_b)\prod_{j=1}^{n+k}S(\bar{x}_j)\,dx_1...dx_n=$$

$$\left(\int\limits_{\mathfrak{M}_1}...\int\ \ +\int\limits_{\mathfrak{M}_2}...\int\ \right)e(-\bar{x}_b)\prod_{j=1}^{n+k}S(\bar{x}_j)\,dx_1...dx_n=I_1\left(\vec{b}\right)+I_2\left(\vec{b}\right). \tag{2.8}$$

Now, if we show that $I(\vec{b})>0$, then we can assert that the system (1.2) is solvable in prime numbers. Since $I(\vec{b})>I_1(\vec{b})-|I_2(\vec{b})|$, the proof of the theorem follows from the following two lemmas.

Lemma 2.1. *For all sets $(b_1,...,b_n)$, $1\leq b_1,...,b_n\leq X$ except for at most $X^nQ^{-k/7(n-1)}$ sets, the estimate $|I_2(\vec{b})|<N^kQ^{-k/6(n-1)}$ holds.*

Lemma 2.2. *For all sets $(b_1,...,b_n)\in W_{n,n+k}(X)$ except for at most $X^nQ^{-\frac{k}{16n(n+2)}}$ sets, the estimate $I_1\left(\vec{b}\right)\gg N^kQ^{-\left(\frac{k}{16n(n+1)}+\frac{k}{14(n-1)}\right)}$ holds.*

We gave the proof of Lemma 2.1 in [11]. Below we prove Lemma 2.2.

First, we simplify the integral over large arcs. For an arbitrary character $\chi(\bmod q)$, we denote

$$C_\chi(m):=\sum_{1\leq l\leq q}\chi(l)e_q(ml),\quad C_q(m)=C_{\chi_0}(m).$$

By analogy with (2.6) we denote:

$$\begin{cases} \bar{h}_j := a_{1j}h_1 + ... + a_{nj}h_n, \quad \bar{h}_b := b_1h_1 + ... + b_nh_n, \\ \bar{\eta}_j := a_{1j}\eta_1 + ... + a_{nj}\eta_n, \quad \bar{\eta}_b := b_1\eta_1 + ... + b_n\eta_n, \quad (j = \overline{1, n+k}), \end{cases} \tag{2.9}$$

then $\bar{x}_j = \frac{\bar{h}_j}{q} + \bar{\eta}_j, \ (j = \overline{1, n+k})$. Using the property of orthogonality of characters [9, 12] and multiplicative properties of the Ramanujan sums, we can represent the sum $S(\bar{x}_j)$ in the following form:

$$S(\bar{x}_j) = \frac{1}{\varphi(q)} \sum_{\chi(\bmod q)} C_{\bar{\chi}}(\bar{h}_j) S_{\chi}(\bar{\eta}_j) + O(\ln^2 N).$$

For $1 \le j \le n+k$, let

$$\begin{cases} G_j(\bar{h}, q, \bar{\eta}) := \sum\limits_{\chi(\bmod q)} C_{\bar{\chi}}(\bar{h}_j) I_{\chi}(\bar{\eta}_j), \\ H_j(\bar{h}, q, \bar{\eta}) := C_q(\bar{h}_j) I(\bar{\eta}_j) - \delta_q C_{\tilde{\chi}\chi_0}(\bar{h}_j) \tilde{I}(\bar{\eta}_j) - G_j(\bar{h}, q, \bar{\eta}), \end{cases} \tag{2.10}$$

where $\delta_q = 1$ or 0 according as $\tilde{r}|q$ or not, where $\tilde{r}$ is a module of exceptional character $\tilde{\chi}$.

Then for all vectors $(b_1, ..., b_n) \in W_{n,n+k}(X)$ except for at most X^nQ^{-1} sets $I_1(\vec{b})$ can be represented as [11]

$$I_1(\vec{b}) = \sum_{q \le Q} \frac{1}{\varphi^{n+k}(q)} \sum_{\bar{h}}{}' e_q(-\bar{h}_b) \int \underset{\mathbb{R}^n}{...} \int e_q(-\bar{\eta}_b) \prod_{j=1}^{n+k} H_j(\bar{h}, q, \bar{\eta}) \, d\eta_1 ... d\eta_n + O(N^kQ^{-1}), \tag{2.11}$$

where $\sum\limits_{\bar{h}}{}'$ denotes the summation over all $h_1, ..., h_n$ satisfying condition (2.4).

3. Singular series and singular integral

For any integer $q \ge 1$ and for any prime p, we denote

$$A(q) := \frac{1}{\varphi^{n+k}(q)} \sum_{\bar{h}}{}' e_q(-\bar{h}_b) \prod_{j=1}^{n+k} C_q(\bar{h}_j), \quad s(p) := 1 + A(p). \tag{3.1}$$

Let $\chi_j(\bmod r_j), \ (j = \overline{1, n+k})$ be primitive characters and $r = [r_1, ..., r_{n+k}]$ be the least common multiple of numbers $r_1, ..., r_{n+k}$. In what follows we will need an estimate of the sum of the form:

$$Z(q) = Z(q, \chi_1, ..., \chi_{n+k}) = \sum_{\bar{h}}{}' e_q(-\bar{h}_b) \prod_{j=1}^{n+k} C_{\chi_j\chi_0}(\bar{h}_j),$$

where q is a multiple of r and χ_0 is modulo q.

The singular series of the problem is studied in [11], and the following lemma is valid for the singular integral.

Lemma 3.1. *For any complex numbers $\rho_j \ (j = \overline{1, n+k})$ satisfying $0 < \operatorname{Re} \rho_j \le 1$, we have*

$$\int \underset{\mathbb{R}^n}{...} \int \left(\prod_{j=1}^{n+k} \int_L^N x_j^{\rho_j - 1} e(\bar{\eta}_j x_j) \, dx_j \right) e(-\bar{\eta}_b) \, d\eta_1 ... d\eta_n = \frac{N^k}{|\bar{\Delta}|} \int \underset{D}{...} \int \prod_{j=1}^{n+k} (Nx_j)^{\rho_j - 1} dx_1 ... dx_k, \tag{3.2}$$

where $x_{k+1},...,x_{n+k}$ *in the integrand on the right hand side are given by*

$$x_i=\bar{\Delta}^{-1}\left(\bar{\Delta}_{i,b}N^{-1}-\bar{\Delta}_{i,1}x_1-\bar{\Delta}_{i,2}x_2-...-\bar{\Delta}_{i,k}x_k\right),\quad \left(i=\overline{k+1,k+n}\right) \tag{3.3}$$

and

$$D=\left\{x_1,...,x_k:\ LN^{-1}\le x_{k+1},...,x_{k+n}\le 1\right\}. \tag{3.4}$$

Furthermore, if $(b_1,...,b_n)\in W_{n,n+k}(X)$ *then*

$$\int\limits_D...\int dx_1...dx_k \gg Q^{-\frac{k}{16n(n+1)}}, \tag{3.5}$$

except for at most $X^nQ^{-\frac{k}{16n(n+2)}}$ *sets* $(b_1,...,b_n)$.

Proof. Equalities (3.2), (3.3), (3.4) can be shown to be valid using a method similar to Lemma 3.6.1 in [3].

Now we will prove (3.5). Depending on the sign of $\bar{\Delta}$, we will split the proof into two cases.

Case 1. Let $\bar{\Delta}>0$. According to (3.3) and (3.4), the multiple integral in (3.5) is equal to the sum of

$$\frac{LN^{-1}\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}+\sum\limits_{1\le j\le k}\bar{\Delta}_{i,j}x_j-\bar{\Delta}_{i,\xi}x_\xi}{-\bar{\Delta}_{i,\xi}}\le x_\xi\le\frac{\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}+\sum\limits_{1\le j\le k}\bar{\Delta}_{i,j}x_j-\bar{\Delta}_{i,\xi}x_\xi}{-\bar{\Delta}_{i,\xi}},$$

where $\left(i=\overline{k+1,k+n},\ \xi=\overline{1,k}\right)$, k dimensional cubes belonging to the space $\left[LN^{-1},\ 1\right]^k$. Here we assume that $-\bar{\Delta}_{i,\xi}$, $\left(i=\overline{k+1,k+n},\ \xi=\overline{1,k}\right)$ are positive. This is always possible, otherwise we can redesignate the indices of the coefficients a_{ij}.

Therefore, we estimate the difference between the upper and lower limits of x_ξ:

$$\frac{\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}}{-\bar{\Delta}_{i,\xi}}-\frac{LN^{-1}\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}}{-\bar{\Delta}_{i,\xi}},\ \left(i=\overline{k+1,k+n},\ \xi=\overline{1,k}\right). \tag{3.6}$$

For $(b_1,...,b_n)\in W_{n,n+k}(X)$, $\left|\bar{\Delta}_{i,b}\right|\le n!B^{n-1}X$ holds and for the second term in (3.6), according to (2.1) and (2.2), $\frac{\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}}{-\bar{\Delta}_{i,\xi}}\le\left(n!B^nQ^{-1/90}+n!B^{n-1}XN^{-1}\right)\left|\bar{\Delta}_{i,\xi}\right|^{-1}\le\left(\left(3n!-1\right)B^n\right)^{-1}$ holds and for the first term, $\frac{LN^{-1}\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}}{-\bar{\Delta}_{i,\xi}}\ge\left(1-n!B^{n-1}XN^{-1}\right)\left|\bar{\Delta}_{i,\xi}\right|^{-1}\ge\frac{1}{n!B^n}-\frac{1}{3(n!)^2B^{2n}}\ge\frac{1}{2n!B^n}\left(2-\frac{2}{3\cdot k\cdot n!B^n}\right)\ge\left(2n!B^n\right)^{-1}$ is suitable. Thus, for each difference in (3.6), the estimate $\gg B^{-n}$ is suitable. If we take into account that there are k such differences, then for the first region, the estimate $\gg B^{-kn}$ holds. Therefore,

$$\int\limits_D...\int dx_1...dx_k\gg B^{-kn}\gg Q^{-\frac{k}{16n(n+1)}}$$

holds for all $(b_1,...,b_n)\in W_{n,n+k}(X)$ and $\delta<\frac{1}{16n^2(n+1)}$.

Case 2. Let $\bar{\Delta}<0$. In this case, the integral $\int\limits_D...\int dx_1...dx_k$ is also equal to the sum of k dimensional cubes formed by the planes

$$\frac{\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}+\sum\limits_{1\le j\le k}\bar{\Delta}_{i,j}x_j-\bar{\Delta}_{i,\xi}x_\xi}{-\bar{\Delta}_{i,\xi}}\le x_\xi\le\frac{LN^{-1}\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}+\sum\limits_{1\le j\le k}\bar{\Delta}_{i,j}x_j-\bar{\Delta}_{i,\xi}x_\xi}{-\bar{\Delta}_{i,\xi}}, \tag{3.7}$$

belonging to the interval $[LN^{-1},\ 1]^k$, where $\left(i=\overline{k+1,k+n},\ \ \xi=\overline{1,k}\right)$. Therefore, we estimate the difference between the upper and lower bounds of x_ξ in (3.7), as

$$\frac{LN^{-1}\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}}{-\bar{\Delta}_{i,\xi}}-\frac{\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}}{-\bar{\Delta}_{i,\xi}},\ \ \left(i=\overline{k+1,k+n},\ \ \xi=\overline{1,k}\right). \tag{3.8}$$

Since $(b_1,...,b_n)\in W_{n,n+k}(X)$, by definition of $W_{n,n+k}(X)$ the system of linear equations $\bar{\Delta}\,y_i=\bar{\Delta}_{i,b}-\bar{\Delta}_{i,1}y_1-...-\Delta_{i,k}y_k,\ \ \left(i=\overline{k+1,k+n}\right)$ has a positive real solution. It follows that $\bar{\Delta}_{i,b}<0$ for all $i=\overline{k+1,k+n}$. For each fixed integer $k\geq 1$ there are at most X^{n-1} values $(b_1,...,b_n)$ satisfying the condition $k=-\bar{\Delta}_{i,b}$ belonging to $[1;\ X]^n$. Therefore, if $k=-\bar{\Delta}_{i,b}\leq\bar{\Delta}_{i,\xi}NQ^{-1/16n(n+1)}$, then there exists a value $(b_1,...,b_n)\in W_{n,n+k}(X)$ not exceeding $\leq\bar{\Delta}_{i,\xi}X^{n-1}NQ^{-1/16n(n+1)},\ \ \left(\xi=\overline{1,k}\right)$, for which the relation $\bar{\Delta}_{i,b}\bar{\Delta}_{i,\xi}^{-1}\leq NQ^{-1/16n(n+1)}$ holds. Thus, for all $(b_1,...,b_n)\in W_{n,n+k}(X)$ except $X^nQ^{-\frac{k}{16n(n+2)}}$ sets of them, the inequality

$$\bar{\Delta}_{i,b}\left(N\bar{\Delta}_{i,\xi}\right)^{-1}>Q^{-1/16n(n+1)},\quad\left(i=\overline{k+1,k+n},\ \ \xi=\overline{1,k}\right), \tag{3.9}$$

holds. Since $\bar{\Delta}<0$, $|\Delta_{i,b}N^{-1}|\leq(3n!B^n)^{-1}$, we have: $\bar{\Delta}-\bar{\Delta}_{i,b}N^{-1}<0$, $\left(i=\overline{k+1,k+n}\right)$. At the same time, according to (3.9) and (2.1), the inequality $\geq Q^{-1/16n(n+1)}-n!B^nQ^{-1/90}>\frac{1}{2}Q^{-1/16n(n+1)}$ is satisfied for the first term in (3.8). Hence,

$$\int\limits_D...\int dx_1...dx_k\geq\left(\frac{1}{2}Q^{-\frac{1}{16n(n+1)}}-LN^{-1}\right)^k\gg Q^{-\frac{k}{16n(n+1)}},$$

except for at most $X^nQ^{-\frac{k}{16n(n+2)}}$ sets $(b_1,...,b_n)\in W_{n,n+k}(X)$. □

4. Proof of Lemma 2.2 and Theorem 1.1

In view of (2.6), when the product $\prod\limits_{j=1}^{n+k}H_j\left(\bar{h},q,\bar{\eta}\right)$ is multiplied out, there are 3^{n+k} terms which belong to three categories:

in group T_1) we include only one member $\prod\limits_{j=1}^{n+k}C_q\left(\bar{h}_j\right)I\left(\bar{h}_j\right)$;

in group T_2) we include those members that contain at least one factor $G_j\left(\bar{h},q,\bar{\eta}\right)$ (their number is equal to $3^{n+k}-2^{n+k}$);

in group T_3) we include the remaining terms (their quantity is equal to $2^{n+k}-1$).

For convenience, we define, for $i=1,2,3$,

$$M_i:=\sum_{q\leq Q}\varphi^{-n-k}(q)\sum_{\bar{h}}{}'e_q\left(-\bar{h}_b\right)\int\limits_{\mathbb{R}^n}...\int\ \{\text{sum of the terms in }T_i\}\,e\left(-\bar{\eta}_b\right)d\eta_1...d\eta_n. \tag{4.1}$$

By (4.1), for all sets $(b_1,...,b_n)\in W_{n,n+k}(X)$ under consideration, except for at most X^nQ^{-1} values of them, equality (2.11) can be rewritten as:

$$I_1\left(\vec{b}\right)=M_1+M_2+M_3+O\left(N^kQ^{-1}\right). \tag{4.2}$$

Let

$$M_0=\frac{N^k}{|\bar{\Delta}|}\prod_p s(p)\int\limits_D...\int\ dx_1...dx_k. \tag{4.3}$$

Then, by virtue of assertion c) of Lemma 5.2 in the work [11] and Lemma 3.1, the estimate

$$M_0 \gg N^k B^{-n} Q^{-\frac{k}{16n(n+1)}} \tag{4.4}$$

holds except for at most $X^n Q^{-\frac{k}{16n(n+2)}}$ sets $(b_1, ..., b_n) \in W_{n,n+k}(X)$.

Lemma 4.1. *For all* $(b_1, ..., b_n) \in W_{n,n+k}(X)$ *the equality* $M_1 = M_0 + O\left(N^k Q^{-4/5}\right)$ *is true.*

Proof. Setting $\rho_j = 1$ in (3.2) and using ([11], Lemma 5.2. d) from (4.1) we find

$$M_1 = \sum_{q\le Q} \frac{1}{\varphi^{n+k}(q)} {\sum_{\bar h}}' e_q\left(-\bar h_b\right) \prod_{j=1}^{n+k} C_q\left(\bar h_j\right) \int \underset{\mathbb{R}^n}{\cdots} \int \left(\prod_{j=1}^{n+k} \int_L^N e\left(\bar\eta_j x\right) dx_j\right) e\left(-\bar\eta_b\right) d\eta_1 ... d\eta_n =$$

$$= \frac{N^k}{|\bar\Delta|} \int \underset{D}{\cdots} \int dx_1 ... dx_k \left(\sum_{q\le Q} A(q)\right) = \frac{N^k}{|\bar\Delta|} \int \underset{D}{\cdots} \int dx_1 ... dx_k \left(\sum_{q=1}^{\infty} A(q)\right) +$$

$$+ O\left(N^k Q^{-1} N^{c_2/\ln\ln N} \ln^{c_3} Q\right). \tag{4.5}$$

By virtue of Lemma 5.2 b), c) in [11] and (3.1) we have $\sum_{q=1}^{\infty} A(q) = \prod_p (1 + A(p)) = \prod_p s(p)$ and it is clear that $N^{c_2/\ln\ln N} \ln^{c_3} Q < Q^{1/5}$. Hence, taking into account equalities (4.3), (4.5) we obtain Lemma 4.1. □

Let $m_1, m_2, ...$be different integers from the set $\{1, ..., n+k\}$. Then we introduce the following definitions:

$$\mathcal{G}(m_1, m_2, ...) := \sum_{(\tilde r)} \tilde\chi\left(l_{m_1}\right) \tilde\chi\left(l_{m_2}\right) ..., \tag{4.6}$$

and

$$\mathcal{P}(m_1, m_2, ...) := \int \underset{D}{\cdots} \int \left[\left(N x_{m_1}\right)^{\tilde\beta - 1} \left(N x_{m_2}\right)^{\tilde\beta - 1} ...\right] dx_1 ... dx_k,$$

where D is defined in (3.4). Clearly,

$$\left|\mathcal{P}(m_1, m_2, ...)\right| \le 1. \tag{4.7}$$

Lemma 4.2. *The following estimates hold:*
a) $\left|\mathcal{G}(m_1, m_2, ...)\right| \le N(\tilde r) \le \varphi^k(\tilde r)$;
b) $\mathcal{G}(m_1, m_2, ...) \ll B^{\frac{1}{2}nk\lambda_2} \tilde r^{-\frac{nk+k}{2n}}$ *except for at most* $X^n \tilde r^{-\frac{k}{n}}$ *sets* $(b_1, ..., b_n) \in [1,\ X]^n$, *where* $\lambda_2 = \frac{(n+k)!}{n!k!}$.

Proof. The proof of assertion a) follows from assertions a) and b) of Lemma 5.1 of [11].
Proof of assertion b). For convenience, consider a specific set of $\{1, 2, ..., n+k\}$, for example $1, 2, ..., n$, that is, $\mathcal{G}(1, ..., n)$. According to (4.6), $\mathcal{G}(1, ..., n)$ can be represented as (see [3], p. 101)

$$\mathcal{G}(1, ..., n) = \frac{1}{\tilde r^n} \sum_{1\le h_1, ..., h_n \le \tilde r} e_{\tilde r}\left(-\bar h_b\right) C_{\tilde\chi}\left(\bar h_1\right) ... C_{\tilde\chi}\left(\bar h_n\right) C_{\tilde r}\left(\bar h_{n+1}\right) ... C_{\tilde r}\left(\bar h_{n+k}\right).$$

Using the Parseval identity (see [13], p. 152), we have

$$\sum_{1\le b_1, ..., b_n \le \tilde r} \left|\mathcal{G}(1, ..., n)\right|^2 = \frac{1}{\tilde r^n} \left| \sum_{1\le h_1, ..., h_n \le \tilde r} C_{\tilde\chi}\left(\bar h_1\right) ... C_{\tilde\chi}\left(\bar h_n\right) C_{\tilde r}\left(\bar h_{n+1}\right) ... C_{\tilde r}\left(\bar h_{n+k}\right) \right|^2.$$

Since $\tilde{\chi}$ is primitive, $C_{\tilde{\chi}}(m) = \tilde{\chi}(m) C_{\tilde{\chi}}(1)$ and $C_{\tilde{\chi}}(1) = \sqrt{\tilde{r}}$. Therefore

$$\sum_{1 \le b_1, ..., b_n \le \tilde{r}} |\mathcal{G}(1, ..., n)|^2 \le \sum_{\substack{1 \le h_1, ..., h_n \le \tilde{r} \\ (h_1, ..., h_n, \tilde{r}) = 1}} \left| C_{\tilde{r}}\left(\bar{h}_{n+1}\right) ... C_{\tilde{r}}\left(\bar{h}_{n+k}\right)\right|^2. \tag{4.8}$$

It is known that the modulus of a quadratic character $\tilde{\chi}$ should take the form $\tilde{r} = \nu_1\nu_2...\nu_\ell$ (where $\nu_1 = 2^t,\ t = \{0, 2, 3\}$ and $\nu_2 < \nu_3 < ... < \nu_\ell$ odd prime numbers). Let

$$U_1 = \left\{ \nu_j |\ \nu_j \nmid \prod_{1 \le i_1 < ... < i_n < n+k} \Delta_{i_1 ... i_n},\ j \ge 2 \right\} \text{ and } U_2 = \{\nu_j |\ \nu_j \notin U_1\}.$$

Set $u_i = \prod_{\nu_j \in U_i} \nu_j$ for $i = 1, 2$ so that $\tilde{r} = u_1 u_2$ and

$$u_2 = \prod_{\nu_j \in U_2} \nu_j = \nu_1 \prod_{\nu_j \in U_2 \setminus \{\nu_1\}} \nu_j \le 8 \left| \prod_{1 \le i_1 < ... < i_n < n+k} \Delta_{i_1 ... i_n} \right| \ll B^{\lambda_2 n}, \text{ where } \lambda_2 = \frac{(n+k)!}{n!k!}.$$

In a similar way to that used in the proof of Lemma 3.5.1 of [3], it is easy to show that

$$\sum_{\substack{1 \le h_1, ..., h_n \le \tilde{r} \\ (h_1, ..., h_n, \tilde{r}) = 1}} \left| C_{\tilde{r}}\left(\bar{h}_{n+1}\right) ... C_{\tilde{r}}\left(\bar{h}_{n+k}\right)\right|^2 = \prod_{j=1}^{\ell} \left\{ \sum_{\substack{1 \le h_1, ..., h_n \le \nu_j \\ (h_1, ..., h_n, \nu_j) = 1}} \left| C_{\nu_j}\left(\bar{h}_{n+1}\right) ... C_{\nu_j}\left(\bar{h}_{n+k}\right)\right|^2 \right\}. \tag{4.9}$$

We fix $\nu_j \in U_1$ and consider the corresponding sum on the right-hand side of equality (4.9)

$$C_{\nu_j}\left(\bar{h}_i\right) = \sum_{l \le \nu_j}{}' e\left(\frac{l\bar{h}_i}{\nu_j}\right) = \begin{cases} \varphi(\nu_j), & \nu_j | \bar{h}_i, \\ -1, & \nu_j \nmid \bar{h}_i, \end{cases} \qquad (i = \overline{n+1, n+k}).$$

There are precisely $(\nu_j - 1)^{n-k}$ sets of $(h_1, ..., h_n)$ for which $\nu_j | \bar{h}_i,\ \ (i = \overline{n+1, n+k})$. Therefore, from (4.9) we find

$$\sum_{\substack{1 \le h_1, ..., h_n \le \nu_j \\ (h_1, ..., h_n, \nu_j) = 1}} \left| C_{\nu_j}\left(\bar{h}_{n+1}\right) ... C_{\nu_j}\left(\bar{h}_{n+k}\right)\right|^2 \le \varphi^{n+k}(\nu_j) \quad \text{for all } \nu_j \in U_1 \tag{4.10}$$

If $\nu_j \in U_2$, then it is obvious that the sum on the left-hand side of inequality (4.10) does not exceed ν_j^{n+2k}.

From (4.8)-(4.10) we obtain

$$\sum_{1 \le b_1, ..., b_n \le \tilde{r}} |\mathcal{G}(1, ..., n)|^2 \ll \prod_{\nu_j \in U_1} \varphi^{n+k}(\nu_j) \prod_{\nu_j \in U_2} \nu_j^{n+2k} \le \tilde{r}^{n+k} u_2^k \ll \tilde{r}^{n+k} B^{nk\lambda_2}.$$

This shows that the number of sets of $(b_1, ..., b_n) \in [1,\ \tilde{r}]^n$ for which $\mathcal{G}(1, ..., n) \gg \tilde{r}^{\frac{nk+k}{2n}} B^{\frac{1}{2}nk\lambda_2}$ does not exceed $\tilde{r}^{\frac{n^2-k}{n}}$. It is clear that $\mathcal{G}(1, ..., n)$ depends on the number of discount classes modulo $\tilde{r}$. So, except for at most $\left(\frac{X}{\tilde{r}}\right)^n \tilde{r}^{\frac{n^2-k}{n}} = X^n \tilde{r}^{-\frac{k}{n}}$ exceptional sets of $(b_1, ..., b_n) \in [1,\ X]^n$, we have

$$\mathcal{G}(1, ..., n) \ll \tilde{r}^{\frac{nk+k}{2n}} B^{\frac{1}{2}nk\lambda_2}. \tag{4.11}$$

For the remaining sets of numbers $m_1, m_2, ...$ from $\{1, 2, ..., n+k\}$ the function $\mathcal{G}(m_1, m_2, ...)$ is estimated similarly and (4.11) remains valid [14]. □

Lemma 4.3. *We have*

$$M_3=\frac{N^k\tilde{r}^n}{\left|\bar{\Delta}\right|\varphi^{n+k}\left(\tilde{r}\right)}\left(\sum_{\substack{q\le Q/\tilde{r}\\(\tilde{r},q)=1}}A\left(q\right)\right)\left[-\sum_{1\le j\le n+k}\mathcal{G}\left(j\right)\mathcal{P}\left(j\right)+\sum_{1\le i<j\le n+k}\mathcal{G}\left(i,j\right)\mathcal{P}\left(i,j\right)-\right.$$

$$\left.-\sum_{1\le i_1<i_2<i_3\le n+k}\mathcal{G}\left(i_1,i_2,i_3\right)\mathcal{P}\left(i_1,i_2,i_3\right)+...+(-1)^{n+k}\mathcal{G}\left(1,...,n+k\right)\mathcal{P}\left(1,..,n+k\right)\right]. \quad (4.12)$$

The proof of this lemma is similar to proof of Lemma 3.7.3 of [3].

In expression (4.12), we consider the following two cases, depending on whether $Q/\tilde{r}$ is "large" or "small".

When $Q/\tilde{r}$ is "large", the series $\sum\limits_{q\le Q/\tilde{r},\ (\tilde{r},q)=1}A\left(q\right)$ in (4.12) is sufficiently long and by Lemma 5.2 d) of [11] we can represent it as:

$$\sum_{q\le Q/\tilde{r},\ (\tilde{r},q)=1}A\left(q\right)=\prod_{p\nmid\tilde{r}}s\left(p\right)+O\left(\tilde{r}Q^{-9/10}\right). \quad (4.13)$$

Denoting the expression in square brackets in (4.12) by $A\left(\mathcal{G},\mathcal{P}\right)$ and using (4.13), we find

$$M_3=\frac{N^k\tilde{r}^n}{\left|\bar{\Delta}\right|\varphi^{n+k}\left(\tilde{r}\right)}\left(\prod_{p\nmid\tilde{r}}s\left(p\right)+O\left(\tilde{r}Q^{-9/10}\right)\right)A\left(\mathcal{G},\mathcal{P}\right).$$

By (4.7) from Lemma 4.2 a) we have

$$M_3=\frac{N^k\tilde{r}^n}{\left|\bar{\Delta}\right|\varphi^{n+k}\left(\tilde{r}\right)}\prod_{p\nmid\tilde{r}}s\left(p\right)A\left(\mathcal{G},\mathcal{P}\right)+O\left(N^k\tilde{r}Q^{-9/10}(\ln\ln Q)^n\right). \quad (4.14)$$

On the other hand, according to Lemma 5.1 e) of work [11], the equality

$$\prod_{p|\tilde{r}}s\left(p\right)=\tilde{r}^n\varphi^{-n-k}\left(\tilde{r}\right)\sum_{(\tilde{r})}1, \quad (4.15)$$

holds. Therefore, from Lemma 4.1 and (4.3) we obtain

$$M_1=\frac{N^k\tilde{r}^n}{\left|\bar{\Delta}\right|\varphi^{n+k}\left(\tilde{r}\right)}\prod_{p\nmid\tilde{r}}s\left(p\right)\sum_{(\tilde{r})}\int\limits_D...\int dx_1...dx_k+O\left(N^kQ^{-4/5}\right). \quad (4.16)$$

From (4.14) and (4.16) it follows that

$$M_1+M_3=\frac{N^k\tilde{r}^n}{\left|\bar{\Delta}\right|\varphi^{n+k}\left(\tilde{r}\right)}\prod_{p\nmid\tilde{r}}s\left(p\right)\sum_{(\tilde{r})}\int\limits_D...\int\prod_{j=1}^{n+k}\left(1-\tilde{\chi}\left(l_j\right)\left(Nx_j\right)^{\tilde{\beta}-1}\right)dx_1...dx_k+O\left(N^k\tilde{r}Q^{-\frac{4}{5}}\right). \quad (4.17)$$

According to the definition of D in equation (3.4), $Nx_j>L$, and from this

$$\prod_{j=1}^{n+k}\left(1-\tilde{\chi}\left(l_j\right)\left(Nx_j\right)^{\tilde{\beta}-1}\right)\ge\left(\left(1-\tilde{\beta}\right)\ln T\right)^{n+k}=\omega^{n+k}, \quad (4.18)$$

follows, where

$$\omega = \begin{cases} \left(1-\tilde{\beta}\right)\ln T, & \text{if } \tilde{\beta} \text{ exists,} \\ 1, & \text{otherwise.} \end{cases}$$

Thus, from (4.18), (4.15) and (4.17) we come to the conclusion

$$M_1 + M_3 \geq \omega^{n+k} M_0 - O\left(N^k \tilde{r} Q^{-4/5}\right). \tag{4.19}$$

In the case where $Q/\tilde{r}$ is small for the sum $\sum\limits_{q\leq Q/\tilde{r},\ (\tilde{r},q)=1} A(q)$ we cannot obtain an estimate like (4.13). In this case, we restrict ourselves to estimating this sum using Lemma 5.2 b) of the work [11], and then for M_3 from (4.12) we obtain the following estimate $M_3 \ll N^k \tilde{r}^n \varphi^{-n-k}(\tilde{r}) A(\mathcal{G},\mathcal{P})$. By virtue of Lemma 4.2 b) and (4.7) we have an estimate $A(\mathcal{G},\mathcal{P}) \ll B^{\frac{1}{2}nk\lambda_2} \tilde{r}^{\frac{nk+k}{2n}}$ except for at most $X^n \tilde{r}^{-\frac{k}{n}}$ sets of $(b_1,...,b_n) \in [1,\ X]^n$. Therefore

$$M_3 \ll \frac{N^k \tilde{r}^n}{\varphi^{n+k}(\tilde{r})} B^{\frac{1}{2}nk\lambda_2} \tilde{r}^{\frac{nk+k}{2n}} (\ln\ln N)^{c_4} \ll N^k \tilde{r}^{\frac{k-nk}{2n}} B^{\frac{1}{2}nk\lambda_2} (\ln\ln N)^{c_5}. \tag{4.20}$$

Lemma 4.4. *For all $(b_1,...,b_n) \in [1,\ X]^n$ the estimate $M_2 \ll M_0 \omega^{n+k} \exp\left(-c_6 \delta^{-\frac{1}{2}}\right)$ holds.*

Proof. The members contained in T_2) are of the form

$$(-1)^m \prod_{j=1}^{l} G_j\left(\bar{h},q,\bar{\eta}\right) \prod_{j=l+1}^{m} \delta_q C_{\tilde{\chi}\chi_0}\left(\bar{h}_j\right) \tilde{I}\left(\bar{\eta}_j\right) \prod_{j=m+1}^{n+k} C_q\left(\bar{h}_j\right) I\left(\bar{\eta}_j\right)$$

or

$$(-1)^m \prod_{j=1}^{l} G_j\left(\bar{h},q,\bar{\eta}\right) \prod_{j=l+1}^{m} C_q\left(\bar{h}_j\right) I\left(\bar{\eta}_j\right) \prod_{j=m+1}^{n+k} \delta_q C_{\tilde{\chi}\chi_0}\left(\bar{h}_j\right) \tilde{I}\left(\bar{\eta}_j\right),$$

where $1 \leq l \leq m \leq n+k$. These expressions are evaluated in the same way, so we will limit ourselves to considering the first of them. We denote its contribution to M_2 by $M_2(m,l)$, then from (4.1) we find [15]

$$M_2(m,l) = (-1)^m \sum_{\substack{q\leq Q,\\ \tilde{r}|q}} \frac{1}{\varphi^{n+k}(q)} \sum_{\bar{h}}{}' e_q(-\bar{h}_b) C_{\tilde{\chi}\chi_0}\left(\bar{h}_{l+1}\right) ... C_{\tilde{\chi}\chi_0}\left(\bar{h}_m\right) C_q\left(\bar{h}_{m+1}\right) ... C_q\left(\bar{h}_{n+k}\right) \times$$

$$\times \sum_{\chi_1} ... \sum_{\chi_l} C_{\tilde{\chi}_1}\left(\bar{h}_1\right) ... C_{\tilde{\chi}_l}\left(\bar{h}_l\right) \int\limits_{\mathbb{R}^n}\!\!...\!\!\int I_{\chi_1}\left(\bar{\eta}_1\right) ... I_{\chi_l}\left(\bar{\eta}_l\right) \tilde{I}_{\chi_l}\left(\bar{\eta}_{l+1}\right) ... \tilde{I}_{\chi_l}\left(\bar{\eta}_m\right) \times$$

$$\times I_{\chi_1}\left(\bar{\eta}_{m+1}\right) ... I_{\chi_l}\left(\bar{\eta}_{n+k}\right) d\eta_1 ... d\eta_n.$$

We denote the integral over $\mathbb{R}^n$ by J, then by (2.3) and (3.2) we have

$$J = \frac{N^k}{|\bar{\Delta}|} \sum_{|\gamma_1|\leq T}{}' ... \sum_{|\gamma_l|\leq T}{}' \int\limits_{D}\!\!...\!\!\int \prod_{j=1}^{l} (Nx_j)^{\rho_j - 1} \prod_{j=l+1}^{m} (Nx_j)^{\tilde{\beta}-1} dx_1 ... dx_k.$$

Therefore,

$$M_2(m,l) = -\frac{N^k}{|\bar{\Delta}|} \int\limits_{D}\!\!...\!\!\int \prod_{j=l+1}^{m} (Nx_j)^{\tilde{\beta}-1} \left(\prod_{j=l+1}^{m} \sum_{r_j\leq Q} \sum_{\chi_j(\bmod r_j)} {}^{*} \sum_{|\gamma_j|\leq T} (Nx_j)^{\rho_j - 1} \right) \times$$

$$\times \sum_{\substack{q\le Q\\ [\tilde r, r_1,...,r_l]|q}} \varphi^{-n-k}(q) Z\left(q; \bar\chi_1, ..., \bar\chi_l, \tilde\chi_{l+1}, ..., \tilde\chi_m, \chi^o_{m+1}, ..., \chi^o_{n+k}\right) dx_1...dx_k,$$

where $\tilde\chi_{l+1} = ... = \tilde\chi_m = \tilde\chi$ and $\chi^o_{m+1} = ... = \chi^o_{n+k} = \chi_0$.

Since $Nx_j \ge L \ge \sqrt{N}$, then in Lemma 4.2 of the work [3], replacing ω^{n+1} by ω^{n+k}, we can calculate the triple sum in brackets, and Lemma 5.3 c) of the work [11] can be applied to estimate the last sum for q and then we get

$$M_2(m,l) \ll M_0\Big(\omega^{n+k}\exp\left(-c_6\delta^{-1/2}\right)\Big)^l \ll M_0\left(\omega^{n+k}\exp\left(-c_6\delta^{-1/2}\right)\right).$$

Collecting the contribution of all such terms, we obtain the statement of the lemma. □

We can now prove Lemma 2.2. We consider the following three cases:

Case 1. There is no exceptional zero $\tilde\beta$. Then M_3 does not exist and, by (4.2), Lemmas 4.1, 4.4 and (4.4),

$$I_1\left(\vec b\right) = M_1 + M_2 + O\left(N^k Q^{-1}\right) \gg N^k B^{-n} Q^{-k/16n(n+1)}$$

except for at most $X^n Q^{-\frac{k}{16n(n+2)}}$ sets $(b_1, ..., b_n) \in W_{n,n+k}(X)$.

Case 2. $\tilde\beta$ exists with $\tilde r \le Q^{\lambda_3}$, $\lambda_3 = k(7(n-1)(n+k))^{-1}$. Then by (4.2),(4.19), (4.3) and Lemma 4.4, we have

$$I_1\left(\vec b\right) \gg M_0\left(1 - \exp\left(-c_6\delta^{-1/2}\right)\right)\omega^{n+k} + O\left(N^k Q^{-\frac{4}{5}+\lambda_3}\right) \tag{4.21}$$

except for at most $X^n Q^{-1}$ sets $(b_1, ..., b_n) \in W_{n,n+k}(X)$. The lower bound for $1-\tilde\beta$ in (9) work [8] together with the fact that $\tilde r \le Q^{\lambda_3}$ implies $\omega = \left(1-\tilde\beta\right)\ln T \gg c_7\Big(\sqrt{\delta}Q^{\lambda_3/2}\ln Q\Big)^{-1}$. With the help of (4.4), we deduce from (4.21) that

$$I_1\left(\vec b\right) \gg N^k\Big(B^n Q^{\frac{k}{16n(n+1)}+\frac{k}{14(n-1)}}\ln^{n+k}Q\Big)^{-1} + O\left(N^k Q^{-\frac{4}{5}+\lambda_3}\right) \gg N^k Q^{-\left(\frac{k}{16n(n+1)}+\frac{k}{14(n-1)}+(n+k)\delta\right)}$$

except for at most $X^n Q^{-k/16n(n+2)}$ sets $(b_1, ..., b_n) \in W_{n,n+k}(X)$.

Case 3. $\tilde\beta$ exists with $\tilde r > Q^{\lambda_3}$. In this case, we apply Lemmas 4.1, 4.4 and (4.20) to (4.2) and deduce that

$$I_1\left(\vec b\right) = M_0\left(1 + O\left(\omega^{n+k}\exp\left(-c_6\delta^{-1/2}\right)\right)\right) + O\left(N^k Q^{-\frac{k^2}{14n(n+k)}}\right)$$

except for at most $X^n \tilde r^{-\frac{k}{n}} \le X^n Q^{-\lambda_4}$, $\lambda_4 = k^2(7n(n-1)(n+k))^{-1}$ sets $(b_1, ..., b_n) \in [1,\ X]^n$. As in the previous two cases, the desired lower bound for $I_1(\vec b)$ follows from (4.4)

$$I_1\left(\vec b\right) \gg N^k B^{-n} Q^{-\frac{k}{16n(n+1)}} - c_8 N^k Q^{-\frac{k^2}{14n(n+k)}} \gg N^k Q^{-\left(\frac{k}{16n(n+1)}+\frac{k}{14(n-1)}+(n+k)\delta\right)}. \tag{4.22}$$

Therefore, based on (4.22) and Lemma 2.1 from (2.8) we have the following estimate:

$$I(\vec b) > I_1(\vec b) - |I_2(\vec b)| > c_9 N^k Q^{-\left(\frac{k}{16n(n+1)}+\frac{k}{14(n-1)}+(n+k)\delta\right)} - N^k Q^{-\frac{k}{6(n-1)}} \gg N^k Q^{-\frac{k}{10(n-1)}}, \tag{4.23}$$

that is, the estimate $I\left(\vec b\right) > 0$ is valid for at most $X^n Q^{-\frac{k}{7(n-1)}} + X^n Q^{-1} + X^n Q^{-\frac{k}{16n(n+1)}} + X^n Q^{-\lambda_4} \le X^{n-\varepsilon}$ sets $(b_1, ..., b_n) \in W_{n,n+k}(X)$.

Thus, we have proved Lemma 2.2 and statement a) of theorem 1.1.

Now we estimate the number of solutions $\vec{b}=(b_1,b_2,...,b_n)$ of system (1.2). It is known that in (2.7) the summation is carried out over all m_j for which the conditions $L<m_1,...,m_{n+k}\le N$ and $\sum\limits_{j=1}^{n+k} a_{ij}m_j=b_i\ \ (i=\overline{1,n})$ are satisfied. Then from (2.7) we have

$$I\left(\vec{b}\right)\le \ln^{n+k}N \sum_{\substack{p_1,..,p_{n+k}\le N\\ \sum\limits_{j=1}^{n+k} a_{ij}p_j=b_i}} 1+\sum_{\ell\ge 2}\sum_{\substack{p_1^\ell,..,p_{n+k}^\ell\le N\\ \sum\limits_{j=1}^{n+k} a_{ij}p_j=b_i}} \ln p_1...\ln p_{n+k}=R\left(\vec{b}\right)\ln^{n+k}N+O\left(N^{\frac{k}{2}}\ln^n N\right).$$

From this inequality and from inequality (4.23) we obtain the estimate $R\left(\vec{b}\right)\gg N^{k-\frac{k\delta}{10(n-1)}}\ln^{-n-k}N$ except for at most $X^{n-\varepsilon}$ sets $(b_1,b_2,...,b_n)\in W_{n,n+k}(X),\ 1\le b_1,b_2,...,b_n\le X$. Considering that $\left|\vec{b}\right|=\sqrt{b_1^2+b_2^2+...+b_n^2}\le\sqrt{n}X$ and $N=3(n!)^2B^{2n-1}X\ge\frac{3(n!)^2B^{2n-1}\left|\vec{b}\right|}{\sqrt{n}}$, we obtain statement b) of the theorem 1.1.

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Third boundary value problem for a third-order inhomogeneous equation with multiple characteristics in three-dimensional space

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Abstract. In this paper, the third boundary value problem for a third-order non-homogeneous equation with multiple characteristics in three-dimensional space is considered. The uniqueness of the solution to the problem is proved by the method of energy integrals, and its existence is proved by the method of separation of variables. The solution is presented in the form of a constructed Green's function. Conditions for the given functions that ensure the regularity of the solution to the problem are found. When substantiating uniform convergence, a difference from zero of the "small denominator" is established.

Keywords: Partial differential equations, third order equation with multiple characteristics, boundary value problem, eigenvalue, eigenfunction, functional series, absolute and uniform convergence.

MSC (2020): 35G15

1. Introduction

The third-order partial differential equations are considered in solving problems in the theory of nonlinear acoustics and in the hydrodynamic theory of space plasma, fluid filtration in porous media [16].

In [14], taking into account the properties of viscosity and thermal conductivity of the gas, it was derived from the Navier-Stokes system that the following third order equation with multiple characteristics containing the second order derivative with respect to time:

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \ \nu = const.$$

This equation describes an axisymmetric flow at $\nu = 1$ and a plane-parallel flow [10] at $\nu = 0$.

The first results on the third-order equation with multiple characteristics were obtained in the works of H. Block [7] and E. Del Vecchio [9]. L. Catabriga in [8] constructed a fundamental solution for equation $D_x^{2n+1} u - D_y^2 u = 0$ in the form of a double improper integral and studied the properties of the potential, solving boundary value problems.

In [11, 12], fundamental solutions of a third-order equation with multiple characteristics containing second derivatives with respect to time, expressed through confluent hypergeometric functions, were constructed, their properties were studied, and estimates were found for $|t| \to \infty$.

In works [6, 13, 15] the features of small denominators that appear when solving differential equations are considered.

1.1. Formulation of the problem.

In the domain $D = \{(x, y, z) : 0 < x < p, 0 < y < q, 0 < z < r\}$, we consider a third-order equation of the form

$$L[u] \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = f(x, y, z) \tag{1.1}$$

where $p, q, r \in R^+$ and for it we study the following problem.

Problem A**.** Find a solution to equation (1.1) in the domain D from class $u(x,y,z) \in C^{3,2,2}_{x,y,z}(D) \cap C^{2,1,1}_{x,y,z}(\overline{D})$, that satisfies the boundary conditions:

$$\begin{cases} \alpha u(x,0,z) + \beta u_y(x,0,z) = 0, \\ \gamma u(x,q,z) + \delta u_y(x,q,z) = 0, \\ u(x,y,0) = u(x,y,r) = 0, \end{cases} \tag{1.2}$$

$$\begin{cases} au(0,y,z) + bu_{xx}(0,y,z) = \psi_1(y,z), \\ cu(p,y,z) + du_{xx}(p,y,z) = \psi_2(y,z), \\ u_x(p,y,z) = \psi_3(y,z), \end{cases} \tag{1.3}$$

where $a,\ b,\ c,\ d,\ \alpha,\ \beta,\ \gamma, \delta \in R\backslash\{0\}$, and $f(x,y,z),\ \psi_i(y,z),\ i = \overline{1,3}$ are given sufficiently smooth functions, and

$$\begin{cases} \alpha \dfrac{\partial^j \psi_i(0,z)}{\partial y^j} + \beta \dfrac{\partial^{j+1}\psi_i(0,z)}{\partial y^{j+1}} = 0,\ \gamma \dfrac{\partial^j \psi_i(q,z)}{\partial y^j} + \delta \dfrac{\partial^{j+1}\psi_i(q,z)}{\partial y^{j+1}} = 0, j = 0,2, \\ \dfrac{\partial^4 \psi_i(y,r)}{\partial y^4} = \dfrac{\partial^4 \psi_i(y,0)}{\partial y^4} = 0, \quad \dfrac{\partial^6 \psi_i(y,r)}{\partial y^4 \partial z^2} = \dfrac{\partial^6 \psi_i(y,0)}{\partial y^4 \partial z^2} = 0, \\ \alpha f(x,0,z) + \beta \dfrac{\partial f(x,0,z)}{\partial y} = 0, \quad \gamma f(x,q,z) + \delta \dfrac{\partial f(x,q,z)}{\partial y} = 0, \\ \dfrac{\partial^2 f(x,y,0)}{\partial y^2} = \dfrac{\partial^2 f(x,y,r)}{\partial y^2} = 0, \end{cases} \quad i = \overline{1,3}. \tag{1.4}$$

Note that for equation (1.1) for $f(x,y,z) = 0,\ a = c = 1,\ b = d = 0$ boundary value problems in finite and infinite domains were studied in [1, 2].

2. Uniqueness of the solution

Theorem 2.1. *If Problem A has a solution, then it is unique when the conditions $ab > 0,\ cd < 0,\ \alpha\beta < 0,\ \gamma\delta > 0$ hold.*

Proof. Assume the opposite, i.e. let Problem A has two solutions $u_1(x,y,z)$ and $u_2(x,y,z)$. Then the function $u(x,y,z) = u_1(x,y,z) - u_2(x,y,z)$ satisfies equation (1.1) with homogeneous boundary conditions. We prove that $u(x,y,z) \equiv 0$ in $\overline{D}$.

In the domain D the identity holds

$$u\,L[u] \equiv \frac{\partial}{\partial x}\left(u\,u_{xx} - \frac{1}{2}\,u_x^2\right) - \frac{\partial}{\partial y}(u\,u_y) + u_y^2 - \frac{\partial}{\partial z}(u\,u_z) + u_z^2 = 0. \tag{2.1}$$

Integrating identity (2.1) over the domain D and taking into account homogeneous boundary conditions, we obtain

$$\frac{1}{2}\int_0^q\int_0^r u_x^2(0,y,z)\,dydz - \frac{d}{c}\int_0^q\int_0^r u_{xx}^2(p,y,z)\,dydz + \frac{b}{a}\int_0^q\int_0^r u_{xx}^2(0,y,z)\,dydz -$$

$$-\frac{\beta}{\alpha}\int_0^p\int_0^r u_y^2(x,0,z)dxdz + \frac{\delta}{\gamma}\int_0^p\int_0^r u_y^2(x,q,z)dxdz + \iiint_D u_y^2(x,y,z)\,dxdydz +$$

$$+\iiint_D u_z^2(x,y,z)\,dxdydz = 0.$$

From the conditions of theorem $ab > 0,\ cd < 0,\ \alpha\beta < 0,\ \gamma\delta > 0$ it follows that $u_y(x, y, z) = 0$ and $u_z(x, y, z) = 0$, i.e. $u(x, y, z) = h(x)$, here $h(x)$ is an arbitrary function satisfying the conditions of the problem. Then, putting into equation (1.1), we have $h'''(x) = 0$. Hence, $h(x) = C_1x^2 + C_2x + C_3$. From condition (1.3), we get

$$\begin{cases} 2bC_1 + aC_3 = 0, \\ (cp^2 + 2d)\, C_1 + cpC_2 + cC_3 = 0, \\ 2pC_1 + C_2 = 0. \end{cases}$$

The basic determinant of this system is equal to

$$\Delta = \begin{vmatrix} 2b & 0 & a \\ cp^2 + 2d & cp & c \\ 2p & 1 & 0 \end{vmatrix} = -acp^2 + 2ad - 2bc.$$

Let $\Delta = 0$, i.e. $-acp^2 + 2ad - 2bc = 0 \ \Rightarrow p^2 = 2\left(\frac{d}{c} - \frac{b}{a}\right)$, taking into account conditions $ab > 0,\ cd < 0$ and $p > 0$, we get $p^2 < 0$, and this contradicts $\Delta = 0$. So $\Delta \neq 0$, then $C_1 = C_2 = C_3 = 0$, from here we have $h(x) = 0$. Consequently, $u(x, y, z) \equiv 0, (x, y, z) \in \overline{D}$. Due to the latter, we obtain $u_1(x, y, z) = u_2(x, y, z)$.

Theorem 2.1 is proved. □

3. Existence of a solution

Theorem 3.1. *If the following conditions hold:*

1) $\dfrac{\partial^7 \psi_i(y, z)}{\partial y^4 \partial z^3} \in C(0 < y < q, 0 < z < r),\ i = \overline{1, 3}$;

2) $\dfrac{\partial^5 f(x, y, z)}{\partial x \partial y^2 \partial z^2} \in C(0 < x < p, 0 < y < q, 0 < z < r)$,

and (1.4), then a solution of Problem A exists.

Proof. We search solution of the problem $u(x, y, z)$ as follows

$$u(x, y, z) = \mathrm{X}(x) \cdot V(y, z).$$

Putting into equation (1.1) and separating the variables, for $V(y, z)$ we have the following problem:

$$\begin{cases} V_{yy} + V_{zz} + \lambda V = 0, \\ \alpha V(0, z) + \beta V_y(0, z) = 0, \\ \gamma V(q, z) + \delta V_y(q, z) = 0, \\ V(y, 0) = V(y, r) = 0, \end{cases} \tag{3.1}$$

where λ is the separation parameter.

Let us find the eigenvalues and eigenfunctions of the problem (3.1). We will search for the solution to the problem (3.1) in the form

$$V(y, z) = Y(y) \cdot Z(z). \tag{3.2}$$

Substituting (3.2) into equation (3.1), and separating the variables, we have the problems

$$\begin{cases} Y'' + \nu Y = 0, \\ \alpha Y(0) + \beta Y'(0) = 0, \\ \gamma Y(q) + \delta Y'(q) = 0, \end{cases} \tag{3.3}$$

$$\left\{ \begin{array}{l} Z'' + \mu Z = 0, \\ Z(0) = Z(r) = 0, \end{array} \right. \tag{3.4}$$

where ν and μ are positive constants related by the relation $\lambda = \nu + \mu$.

Proceeding as in [4], to find the eigenvalues ν in problem (3.3), we obtain the transcendental equation

$$ctg\sqrt{\nu}q = \frac{\alpha\gamma + \delta\beta\nu}{\sqrt{\nu}\left(\gamma\beta - \alpha\delta\right)},$$

it follows that $\sqrt{\nu_n} = \frac{\pi n}{q} + \varepsilon_n$, where $\lim\limits_{n\to\infty} \varepsilon_n = 0$, or $\nu_n = \mathrm{O}\left(n^2\right), \quad n \to \infty$.

The corresponding eigenfunctions have the form

$$Y_n(y) = \left(\alpha \sin\sqrt{\nu_n}y - \beta\sqrt{\nu_n}\cos\sqrt{\nu_n}y\right) A_n,$$

where A_n is an arbitrary constant, and for the problem (3.4) we obtain

$$Z_m\left(z\right) = A_m \sin\sqrt{\mu_m}z,$$

where A_m are arbitrary constants, and $\mu_m = \left(\frac{m\pi}{r}\right)^2$.

It is known [2] that as a solution to the spectral problem (3.3), and (3.4) we take the functions

$$V_{n,m}\left(y,z\right) = \frac{1}{\left\|V_{n,m}\right\|^2} Y_n(y) \sin\frac{m\pi}{r}z, \tag{3.5}$$

where $\left\|V_{n,m}\right\|^2 = \left[\frac{1}{2}\left(\alpha^2 q + \beta^2 q\nu_n - \alpha\beta\right) + \left(\frac{\beta^2\sqrt{\nu_n}}{4} - \frac{\alpha^2}{4\sqrt{\nu_n}}\right)\sin 2\sqrt{\nu_n}q + \frac{\alpha\beta}{2}\cos 2\sqrt{\nu_n}q\right]\frac{r}{2}$.

Now let's expand $f\left(x,y,z\right)$ into a Fourier series of $\left\{V_{n,m}\left(y,z\right)\right\}$:

$$f\left(x,y,z\right) = \sum_{n,m=1}^{+\infty} f_{n,m}\left(x\right) V_{n,m}\left(y,z\right),$$

where $f_{n,m}\left(x\right) = \int\limits_0^q\int\limits_0^r f\left(x,y,z\right) V_{n,m}\left(y,z\right) dydz$.

Integrating function $f_{n,m}\left(x\right)$ by parts and taking into account conditions (1.4), we have the estimate

$$\left|f_{n,m}\left(x\right)\right| \le M\frac{\left|F_{n,m}(x)\right|}{n^2 m^2}, \tag{3.6}$$

where $F_{n,m}(x) = \int\limits_0^q\int\limits_0^r \frac{\partial^4 f\left(x,y,z\right)}{\partial y^2\partial z^2} V_{n,m}\left(y,z\right) dydz$.

In what follows, the maximum value of all found positive known numbers in estimates will be denoted by M.

We search for the solution to Problem A in the form

$$u\left(x,y,z\right) = \sum_{n,m=1}^{+\infty} X_{n,m}\left(x\right) V_{n,m}\left(y,z\right). \tag{3.7}$$

Substituting (3.7) into equation (1.1), taking into account the boundary conditions (1.3), we obtain the following problem:

$$\left\{ \begin{array}{l} X'''_{n,m}\left(x\right) + \lambda_{n,m} X_{n,m}\left(x\right) = f_{n,m}\left(x\right), \\ aX_{n,m}\left(0\right) + bX''_{n,m}\left(0\right) = \psi_{1n,m}, \\ cX_{n,m}\left(p\right) + dX''_{n,m}\left(p\right) = \psi_{2n,m}, \\ X'_{n,m}\left(p\right) = \psi_{3n,m}, \end{array} \right. \tag{3.8}$$

where $\psi_{in,m} = \int\limits_0^q \int\limits_0^r \psi_i(y,z)\, V_{n,m}(y,z)\, dydz, \quad i = \overline{1,3}$.

Applying integration by parts to $\psi_{in,m}$, taking into account condition (1.4), we obtain the estimate

$$|\psi_{in,m}| \le M \frac{|\Psi_{in,m}|}{n^4 m^3}, \tag{3.9}$$

here

$$\Psi_{in,m} = \frac{1}{\|V_{n,m}\|^2} \int\limits_0^q \int\limits_0^r \frac{\partial^7 \psi_i(y,z)}{\partial y^4 \partial z^3} Y_n(y) \cos \frac{m\pi z}{r} dydz.$$

We find the solution to the problem (3.8) by constructing the Green's function, for this purpose using the function

$$U_{n,m}(x) = X_{n,m}(x) - \rho_{n,m}(x), \tag{3.10}$$

let's change the boundary conditions to homogeneous.

Function $\rho_{n,m}(x)$ has the form

$$\rho_{n,m}(x) = \frac{1}{ap^2 + 2b} \left[(x-p)^2 \psi_{1n,m} + \frac{2apx - ax^2 + 2b}{c} \psi_{2n,m} + (x-p)(apx + 2b)\psi_{3n,m} \right]. \tag{3.11}$$

Substituting (3.10), and (3.11) into (3.8) we obtain the problem

$$\begin{cases} U'''_{n,m}(x) + \lambda_{n,m} U_{n,m}(x) = \lambda_{n,m} g_{n,m}(x), \\ aU_{n,m}(0) + bU''_{n,m}(0) = 0, \\ cU_{n,m}(p) + dU''_{n,m}(p) = 0, \\ U'_{n,m}(p) = 0, \end{cases} \tag{3.12}$$

here

$$g_{n,m}(x) = -\frac{1}{ap^2 + 2b} \left[(x-p)^2 \psi_{1n,m} + \frac{2apx - ax^2 + 2b}{c} \psi_{2n,m} + (x-p)(apx + 2b)\psi_{3n,m} \right] + \frac{f_{n,m}(x)}{\lambda_{n,m}}.$$

Taking into account (3.6), (3.9), and $\lambda_{n,m} = \left(\frac{\pi n}{q}\right)^2 + \left(\frac{\pi m}{r}\right)^2 \ge \frac{2\pi^2}{qr} nm$, $n, m \in N$ we have the estimates

$$\begin{gathered} |g_{n,m}(x)| \le M \left(\sum_{i=1}^{3} \frac{|\Psi_{in,m}|}{n^4 m^3} + \frac{|F_{n,m}(x)|}{n^3 m^3} \right), \\ \left| g'_{n,m}(x) \right| \le M \left(\sum_{i=1}^{3} \frac{|\Psi_{in,m}|}{n^4 m^3} + \frac{|F'_{n,m}(x)|}{n^3 m^3} \right). \end{gathered} \tag{3.13}$$

We search for the solution to the problem (3.12) in the form:

$$U_{n,m}(x) = \lambda_{n,m} \int\limits_0^p G_{n,m}(x,\xi) g_{n,m}(\xi)\, d\xi, \tag{3.14}$$

where $G_{n,m}(x,\xi)$ is Green's function of problem (3.12) and has the following form:

$$G_{n,m}(x,\xi) = \begin{cases} G_{1n,m}(x,\xi), \ 0 \le x < \xi, \\ G_{2n,m}(x,\xi), \ \xi < x \le p, \end{cases} \tag{3.15}$$

here

$$G_{1n,m}(x,\xi)=\frac{2}{3k_{n,m}^2\overline{\Delta}}\left[e^{-k_{n,m}\left(\frac{1}{2}\xi+x\right)}\left(\left(\frac{ad}{k_{n,m}^2}-\frac{ac}{k_{n,m}^4}\right)\sin\left(\beta_{n,m}\xi+\frac{\pi}{6}\right)+\left(\frac{bc}{k_{n,m}^2}-bd\right)\cos\beta_{n,m}\xi\right)-\right.$$

$$-e^{k_{n,m}\left(\xi-\frac{3}{2}p-x\right)}\left(\left(\frac{bc}{k_{n,m}^2}-\frac{ad}{k_{n,m}^2}\right)\cos\beta_{n,m}p-\frac{ac}{k_{n,m}^4}\sin\left(\beta_{n,m}p+\frac{\pi}{6}\right)+bd\cos\left(\beta_{n,m}p+\frac{\pi}{3}\right)\right)+$$

$$+e^{\frac{1}{2}k_{n,m}(x-\xi)}\left(\frac{a}{k_{n,m}^2}+b\right)\left(\frac{c}{k_{n,m}^2}-d\right)\sin\left(\beta_{n,m}(\xi-x)+\frac{\pi}{6}\right)-$$

$$-e^{k_{n,m}\left(\xi-\frac{3}{2}p+\frac{1}{2}x\right)}\left(\frac{a}{k_{n,m}^2}+b\right)\left(\frac{c}{k_{n,m}^2}\sin\left(\beta_{n,m}(p-x)+\frac{\pi}{6}\right)+d\cos\beta_{n,m}(p-x)\right)+$$

$$+e^{-\frac{1}{2}k_{n,m}(\xi+3p-x)}\left(\left(\frac{a}{k_{n,m}^2}\sin\left(\beta_{n,m}\xi+\frac{\pi}{6}\right)-b\cos\beta_{n,m}\xi\right)\left(\frac{2c}{k_{n,m}^2}\sin\left(\beta_{n,m}(p-x)+\frac{\pi}{6}\right)+\right.\right.$$

$$\left.+2d\cos\beta_{n,m}(p-x)\right)+\left(\left(\frac{2bc}{k_{n,m}^2}-\frac{2ad}{k_{n,m}^2}\right)\cos\beta_{n,m}p+2bd\cos\left(\beta_{n,m}p+\frac{\pi}{3}\right)-\right.$$

$$\left.\left.\left.-\frac{2ac}{k_{n,m}^4}\sin\left(\beta_{n,m}p+\frac{\pi}{6}\right)\right)\sin\left(\beta_{n,m}(\xi-x)+\frac{\pi}{6}\right)\right)\right],$$

$$G_{2n,m}(x,\xi)=\frac{1}{3k_{n,m}^2\overline{\Delta}}\left(\frac{a}{k_{n,m}^2}+b-2e^{-\frac{3}{2}k_{n,m}\xi}\left(\frac{a}{k_{n,m}^2}\sin\left(\beta_{n,m}\xi+\frac{\pi}{6}\right)-b\cos\beta_{n,m}\xi\right)\right)$$

$$\left[e^{k_{n,m}(\xi-x)}\left(\frac{c}{k_{n,m}^2}-d\right)-2e^{k_{n,m}\left(\xi-\frac{3}{2}p+\frac{1}{2}x\right)}\left(\frac{c}{k_{n,m}^2}\sin\left(\beta_{n,m}(p-x)+\frac{\pi}{6}\right)+d\cos\beta_{n,m}(p-x)\right)\right],$$

$$\overline{\Delta}(n,m)=\left(\frac{a}{k_{n,m}^2}+b\right)\left(\frac{c}{k_{n,m}^2}-d\right)+e^{-\frac{3}{2}k_{n,m}p}\left\{\left(\frac{2bc}{k_{n,m}^2}-\frac{2ad}{k_{n,m}^2}-\frac{ac}{k_{n,m}^4}+bd\right)\cos\beta_{n,m}p-\right.$$

$$\left.-\sqrt{3}\left(\frac{ac}{k_{n,m}^4}+bd\right)\sin\beta_{n,m}p\right\}.$$

Expression $\overline{\Delta}(n,m)$ is a small denominator of a more complex structure than the flat case [3, 4, 5]. In this regard, to justify the existence of a solution to the problem, it is necessary to prove that $\overline{\Delta}(n,m)\neq 0$.To do this, we formulate and prove the following lemma:

Lemma 3.2. *The boundary value problem*

$$\begin{cases} X'''_{n,m}+\lambda_{n,m}X_{n,m}=0,\\ aX_{n,m}(0)+bX''_{n,m}(0)=0,\\ cX_{n,m}(p)+dX''_{n,m}(p)=0,\\ X'_{n,m}(p)=0,\end{cases}\tag{3.16}$$

has only a trivial solution.

Proof. Assume the opposite, let $X_{n,m}(x)\neq 0$. Consider the identity

$$\left(X_{n,m}X''_{n,m}-\frac{1}{2}(X'_{n,m})^2\right)'+\lambda_{n,m}X_{n,m}^2=0,$$

integrating over $(0<x<p)$, and taking into account the boundary conditions, we obtain

$$-\frac{d}{c}(X''_{n,m}(p))^2+\frac{b}{a}(X''_{n,m}(0))^2+\frac{1}{2}X'_n(0)^2+\lambda_{n,m}\int\limits_0^p X_{n,m}^2dx=0.$$

Since $ab>0$, $cd<0$, $\lambda_{n,m}>0$, then $X_{n,m}\equiv 0$. Lemma 3.2 is proved. □

Let there be a number $n*, m*$ such that $\overline{\Delta}\left(n*,m*\right)=0$, then there are constants $C_1^*,\ C_2^*,\ C_3^*$ simultaneously not all equal to zero, satisfying the system

$$\begin{cases} C^*_{1n*,m*}\left(a+bk^2_{n*,m*}\right)+C^*_{2n*,m*}\left(a-\dfrac{1}{2}bk^2_{n*,m*}\right)+C^*_{3n*,m*}\dfrac{\sqrt{3}bk^2_{n*,m*}}{2}=0,\\ C^*_{1n*,m*}e^{-k_{n*,m*}p}\left(c+dk^2_{n*,m*}\right)+C^*_{2n*,m*}e^{\frac{1}{2}k_{n*,m*}p}\left(c\cos\dfrac{\sqrt{3}}{2}k_{n*,m*}p+\right.\\ \left.+dk^2_{n*,m*}\cos\left(\dfrac{\sqrt{3}}{2}k_{n*,m*}p+\dfrac{2\pi}{3}\right)\right)+C^*_{3n*,m*}e^{\frac{1}{2}k_{n*,m*}p}\left(c\sin\dfrac{\sqrt{3}}{2}k_{n*,m*}p+\right.\\ \left.+dk^2_{n*,m*}\sin\left(\dfrac{\sqrt{3}}{2}k_{n*,m*}p+\dfrac{2\pi}{3}\right)\right)=0,\\ -C^*_{1n*,m*}e^{-k_{n,m}p}+C^*_{2n*,m*}e^{\frac{1}{2}k_{n*,m*}p}\cos\left(\dfrac{\sqrt{3}}{2}k_{n*,m*}p+\dfrac{\pi}{3}\right)+C^*_{3n*,m*}e^{\frac{1}{2}k_{n*,m*}p}\times\\ \sin\left(\dfrac{\sqrt{3}}{2}k_{n*,m*}p+\dfrac{\pi}{3}\right)=0. \end{cases}$$

From here we have the function

$$X_{n*,m*}\left(x\right)=C^*_{1n*,m*}e^{-k_{n*,m*}x}+e^{\frac{1}{2}k_{n*,m*}x}\left(C^*_{2n*,m*}\cos\frac{\sqrt{3}}{2}k_{n*,m*}x+C^*_{3n*,m*}\sin\frac{\sqrt{3}}{2}k_{n*,m*}x\right),$$

is a solution to the boundary value problem (3.16), but according to the proven lemma it should be

$$C^*_{1n*,m*}e^{-k_{n*,m*}x}+e^{\frac{1}{2}k_{n*,m*}x}\left(C^*_{2n*,m*}\cos\frac{\sqrt{3}}{2}k_{n*,m*}x+C^*_{3n*,m*}\sin\frac{\sqrt{3}}{2}k_{n*,m*}x\right)\equiv 0,$$

but this is impossible due to the linear independence of the functions

$$e^{-k_{n*,m*}x},\ e^{\frac{1}{2}k_{n*,m*}x}\cos\frac{\sqrt{3}}{2}k_{n*,m*}x,\ e^{\frac{1}{2}k_{n*,m*}x}\sin\frac{\sqrt{3}}{2}k_{n*,m*}x.$$

Hence the function:

$$u*\left(x,y,z\right)=\sum_{n*,m*=1}^{\infty}\left[C^*_{1n*,m*}e^{k_{n*,m*}x}+e^{-\frac{1}{2}k_{n*,m*}x}\left(C^*_{2n*,m*}\cos\frac{\sqrt{3}}{2}k_{n*,m*}x+\right.\right.$$
$$\left.\left.+\ C^*_{3n*,m*}\sin\frac{\sqrt{3}}{2}k_{n*,m*}x\right)\right]V_{n*,m*}\left(y,z\right),$$

is a non-trivial solution to Problem A, and this contradicts the uniqueness theorem.

It is easy to verify that the function defined by formula (3.15) has all the properties of Green's function.

Note that the constructed Green's functions in cases $a=c=0,\ b=d=1$ in [3], $a=c=1,\ b=d=0$ in [4], and $a=d=0,\ b=c=1$ in [5] can be obtained from (3.15).

By (3.7) and (3.10), the solution to the Problem A has the form

$$u\left(x,y,z\right)=\sum_{n,m=1}^{+\infty}\left(U_{n,m}\left(x\right)+\rho_{n,m}\left(x\right)\right)V_{n,m}\left(y,z\right). \tag{3.17}$$

If function $u(x,y,z)$ is defined by series (3.17), and its derivatives u_{xxx}, u_{yy} and u_{zz} converge absolutely and uniformly in $\overline{D}$, then it gives a solution to the Problem A.

Let us prove the absolute and uniform convergence of the series (3.17). From (3.17) we have the estimate

$$|u(x,y,z)| \le M \sum_{n,m=1}^{+\infty} (|U_{n,m}(x)| + |\rho_{n,m}(x)|).$$

Now substituting $G_{n,m}(x,\xi) = -\frac{1}{\lambda_{n,m}} G_{n,m\xi\xi\xi}(x,\xi)$ in (3.14) and integrating, we have

$$U_{n,m}(x) = -g_{n,m}(x) + g_{n,m}(0)\, G_{2n,m\xi\xi}(x,0) - g_{n,m}(p)\, G_{1n,m\xi\xi}(x,p) + \\ + \int_0^p G_{n,m\xi\xi}(x,\xi)\, g_{n,m}{}'(\xi)\, d\xi.$$

Taking into account (3.11), (3.13) and

$$|G_{2n,m\xi\xi}(x,0)| \le M, |G_{1n,m\xi\xi}(x,p)| \le M, |G_{n,m\xi\xi}(x,\xi)| \le M,$$

from the Green's function (3.15), we obtain the estimates

$$|\rho_{n,m}(x)| \le \frac{M}{n^4 m^3}\left(|\Psi_{1n,m}| + |\Psi_{2n,m}| + |\Psi_{3n,m}|\right),$$
$$|U_{n,m}(x)| \le M\left[\sum_{i=1}^{3} \frac{|\Psi_{in,m}|}{n^4 m^3} + \frac{1}{n^3 m^3}\left(|F_{n,m}(0)| + |F_{n,m}(p)| + |F_{n,m}(x)| + |F'_{n,m}(x)|\right)\right].$$

From here

$$|u(x,y,z)| \le M \sum_{n,m=1}^{+\infty}\left[\sum_{i=1}^{3} \frac{|\Psi_{in,m}|}{n^4 m^3} + \frac{1}{n^3 m^3}\left(|F_{n,m}(0)| + |F_{n,m}(p)| + |F_{n,m}(x)| + |F'_{n,m}(x)|\right)\right] < \infty.$$

It follows that the series (3.17) converges absolutely and uniformly.

Now we will prove that the partial derivatives of series (3.17) included in equations (1.1) also converge absolutely and uniformly in the domain $\overline{D}$. To do this, we calculate the partial derivatives of series (3.17) concerning variables y and z up to the second order, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\pi^2}{q^2}\sum_{n,m=1}^{+\infty} n^2\left(U_{n,m}(x) + \rho_{n,m}(x)\right) V_{n,m}(y,z),$$
$$\frac{\partial^2 u}{\partial z^2} = -\frac{\pi^2}{r^2}\sum_{n,m=1}^{+\infty} m^2\left(U_{n,m}(x) + \rho_{n,m}(x)\right) V_{n,m}(y,z).$$

Given the estimate $u(x,y,z)$, we obtain

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \le M \sum_{n,m=1}^{+\infty}\left[\sum_{i=1}^{3} \frac{|\Psi_{in,m}|}{n^2 m^3} + \frac{1}{n m^3}\left(|F_{n,m}(0)| + |F_{n,m}(p)| + |F_{n,m}(x)| + |F'_{n,m}(x)|\right)\right],$$
$$\left|\frac{\partial^2 u}{\partial z^2}\right| \le M \sum_{n,m=1}^{+\infty}\left[\sum_{i=1}^{3} \frac{|\Psi_{in,m}|}{n^4 m} + \frac{1}{n^3 m}\left(|F_{n,m}(0)| + |F_{n,m}(p)| + |F_{n,m}(x)| + |F'_{n,m}(x)|\right)\right].$$

Using the Cauchy-Bunyakovsky and Bessel inequalities, we get

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \le M \sqrt{\sum_{n,m=1}^{+\infty}\left(\frac{1}{n m^3}\right)^2}\left(\sqrt{\sum_{n,m=1}^{+\infty} |F_{n,m}(0)|^2} + \sqrt{\sum_{n,m=1}^{+\infty} |F_{n,m}(p)|^2} + \sqrt{\sum_{n,m=1}^{+\infty} |F_{n,m}(x)|^2} + \right.$$
$$\left. + \sqrt{\sum_{n,m=1}^{+\infty} |F'_{n,m}(x)|^2}\right) \le \overline{M}\left(\left\|\frac{\partial^4 f(0,y,z)}{\partial y^2 \partial z^2}\right\|_{L_2[0<y<q,0<z<r]} + \left\|\frac{\partial^4 f(p,y,z)}{\partial y^2 \partial z^2}\right\|_{L_2[0<y<q,0<z<r]} + \right.$$
$$\left. + \left\|\frac{\partial^4 f(x,y,z)}{\partial y^2 \partial z^2}\right\|_{L_2[0<x<p,0<y<q,0<z<r]} + \left\|\frac{\partial^5 f(x,y,z)}{\partial x \partial y^2 \partial z^2}\right\|_{L_2[0<x<p,0<y<q,0<z<r]}\right) < \infty,$$

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \le M\sqrt{\sum_{n,m=1}^{+\infty}\left(\frac{1}{n^4 m}\right)^2}\left(\sqrt{\sum_{n,m=1}^{+\infty}\left|\Psi_{1n,m}\right|^2}+\sqrt{\sum_{n,m=1}^{+\infty}\left|\Psi_{2n,m}\right|^2}+\sqrt{\sum_{n,m=1}^{+\infty}\left|\Psi_{3n,m}\right|^2}\right)+$$
$$+M\sqrt{\sum_{n,m=1}^{+\infty}\left(\frac{1}{n^3 m}\right)^2}\left(\sqrt{\sum_{n,m=1}^{+\infty}\left|F_{n,m}(0)\right|^2}+\sqrt{\sum_{n,m=1}^{+\infty}\left|F_{n,m}(p)\right|^2}+\sqrt{\sum_{n,m=1}^{+\infty}\left|F_{n,m}(x)\right|^2}+\right.$$
$$\left.+\sqrt{\sum_{n,m=1}^{+\infty}\left|F'_{n,m}(x)\right|^2}\right) \le \overline{M}\left(\sum_{i=1}^{3}\left\|\frac{\partial^7\psi_i(y,z)}{\partial y^4\partial z^3}\right\|_{L_2[0<y<q,0<z<r]}+\left\|\frac{\partial^4 f(0,y,z)}{\partial y^2\partial z^2}\right\|_{L_2[0<y<q,0<z<r]}+\right.$$
$$+\left\|\frac{\partial^4 f(p,y,z)}{\partial y^2\partial z^2}\right\|_{L_2[0<y<q,0<z<r]}+\left\|\frac{\partial^4 f(x,y,z)}{\partial y^2\partial z^2}\right\|_{L_2[0<x<p,0<y<q,0<z<r]}+$$
$$\left.+\left\|\frac{\partial^5 f(x,y,z)}{\partial x\partial y^2\partial z^2}\right\|_{L_2[0<x<p,0<y<q,0<z<r]}\right)<\infty,$$

as

$$\sum_{n,m=1}^{+\infty}\left|\Psi_{in,m}\right|^2 \le \left\|\frac{\partial^7\psi_i(y,z)}{\partial y^4\partial z^3}\right\|^2_{L_2[0<y<q,0<z<r]}, i=\overline{1,3},$$
$$\sum_{n,m=1}^{+\infty}\left|F_{n,m}(x)\right|^2 \le \left\|\frac{\partial^4 f(x,y,z)}{\partial y^2\partial z^2}\right\|^2_{L_2[0<x<p,0<y<q,0<z<r]},$$
$$\sum_{n,m=1}^{+\infty}\left|F'_{n,m}(x)\right|^2 \le \left\|\frac{\partial^5 f(x,y,z)}{\partial x\partial y^2\partial z^2}\right\|^2_{L_2[0<x<p,0<y<q,0<z<r]}, \sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi^2}{6}, \sum_{m=1}^{+\infty}\frac{1}{m^2}=\frac{\pi^2}{6}.$$

Therefore, the series corresponding to the function $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial z^2}$ converges absolutely and uniformly. Absolute and uniform convergence of the third derivative concerning x of the series (3.17) follows from $\left|\frac{\partial^3 u}{\partial x^3}\right| \le \left|\frac{\partial^2 u}{\partial y^2}\right| + \left|\frac{\partial^2 u}{\partial z^2}\right|$ and what was proved above.

Thus, theorem 3.1 is proved. □

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Π-completeness of the space of idempotent probability measures

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Abstract. For a Tychonoff space X, we consider the space $I_\beta(X)$ of idempotent probability measures with compact support. Using a cover γ_X of the origin space X we build an open cover γ_0 of $I_\beta(X)$. Then we show that if γ_X is a disjoint system then γ_0 is a also disjoint system. We obtain that the space of all idempotent probability measures on the Stone-Čech compactification of X is a perfect compactification of $I_\beta(X)$. Finally, applying the Π-completeness criterion we establish that $I_\beta(X)$ is Π-complete if and only if the given Tychonoff space X is Π-complete.

Keywords: Π-complete space; perfect compactification; idempotent probability measure; finite-component cover.

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1. Introduction

The study of topological spaces and their properties has been a central topic in mathematics, with particular interest in understanding completeness properties and compactifications. Among these, the concept of Π-completeness, investigated in [10], has emerged as a significant property that generalizes classical notions of compactness in topology. In [11] it was explored several types of such spaces, which called compact type spaces. Actions by hyperspace functor were studied in [14]. To establish their results the authors of the last paper applied the criterion of Π-completeness of Tychonoff spaces given in [3]. In [8] the hyperspace of a compact Hausdorff space was equipped with a metric different from the well-known Hausdorff metric. We use a notion of density function considered in [1] where it was shown that each idempotent probability measure is determined by a unique density function. Using this property we were able to define the concept of a support of an idempotent probability measure in a completely new way.

It should be noted that idempotent mathematics is increasingly used in various areas of modern research. For example, it plays an important role in artificial intelligence. In [16] some important topological properties of the space of idempotent probability measures on a compact Hausdorff space were obtained. Particularly, normality properties considered in [12] was proved for the functor I of idempotent probability measures in the category $\mathcal{C}omp$ of compact Hausdorff spaces and their continuous maps. Using a construction suggested in [4] the author of [7] has extended this functor on the category $\mathcal{T}ych$ of Tychonoff spaces and their continuous maps. Other types of extensions of the functor I were studied in [9] and [13].

In this paper, we continue the investigation of Π-completeness of topological spaces. We show the Π-completeness of the space $I_\beta(X)$ of idempotent probability measures with compact support is equivalent to Π-completeness of a given Tychonoff space X. To get this result we use the Π-completeness criterion of Tychonoff spaces. To make this criterion to be met firstly we showed that the space $I(\beta X)$ of all probability measures on the Stone-Čech compactification of X is a perfect compactification of $I_\beta(X)$ of idempotent probability measures with compact support. It is worth recalling that the set systems play a key role in studying of classes of compact type space. For a given system γ_X of subsets of a Tychonoff space X we build a set system γ_0 in $I_\beta(X)$. We show that the openness of γ_X implies the openness of γ_0. Moreover, we obtain that if γ_X is a disjoint system then γ_0 is also a disjoint system.

In the present paper, we also give a construction of a system of sets in the space of idempotent probability measures which generates the topology τ_p of point-wise convergence in it. In construction of this system we use the density function of the idempotent probability measures.

Note that in [15] it was introduced a set system in the space $I(X)$ of idempotent probability measures on a given compact Hausdorff space X which generates a topology τ_Z in $I(X)$. The last topology thinner than the topology of point-wise convergence, i.e., $\tau_p \subset \tau_Z$.

2. Preliminaries

In the present paper, by a space we mean a topological T_1-space, and by a map a continuous map. Recall that a Tychonoff space is [5] a space X such that every closed subset F in X and any point $x \in X \setminus F$ are functionally separated, i.e., there exists a continuous function $\varphi\colon X \to \mathbb{R}$ for which $\varphi(x) = 0$ and $\varphi(y) = 1$ in $y \in F$.

A collection ω of subsets of a set X is said to be *star countable* (respectively, *star-finite*) if each element of ω intersects at most a countable set(respectively, finite) of elements of ω. A collection ω of subsets of a set X *refines* a collection Ω of subsets of X if for each element $A \in \omega$ there is an element $B \in \Omega$ such that $A \subset B$. They also say that ω is a *refinement* of Ω. For a point $x \in X$ and a natural number n the inequality $Kp(x, \omega) \leq n$ means that no more than n elements of ω contain x ([2], p. 270), and $Kp\,\omega \leq n$ means that $Kp(x, \omega) \leq n$ for every $x \in X$.

A finite sequence of subsets $M_0, \ldots, M_s$ of a set X is [10] a *chain* connecting sets M_0 and M_s, if $M_{i-1} \cap M_i \neq \varnothing$ for all $i = 1, \ldots, s$. A collection ω of subsets of a set X is said to be *connected* if for any pair of sets M, $M' \subset X$ there exists a chain ω connecting the sets M and M'. The maximal connected sub-collections of ω are called *components* of ω. A star-finite open cover of a space X is said to be *a finite-component cover* if the number of elements of each component is finite.

For a collection $\omega = \{O_\alpha\colon\ \alpha \in A\}$ of subsets of a space X we put $[\omega] = [\omega]_X = \{[O_\alpha]_X\colon\ \alpha \in A\}$. For a space X, its subspace W and a point $x \in X \setminus W$ we say that an open in X cover λ of the space W pricks out the point x in X if $x \notin \cup[\lambda]_X$ [11].

For a Tychonoff space X let βX be its the Stone-Čech compactification (i.e., the maximal compact extension).

Definition 2.1. [11] A Tychonoff space X is said to be Π-complete if for every point $x \in \beta X \setminus X$ there exists a finite component cover ω of X which pricks out the point x in βX.

Let υX be a compact extension of a Tychonoff space X. If $H \subset X$ is an open set in X, then by $O(H)$ (or by $O_{\upsilon X}(H)$) we denote the maximal (by inclusion) open set in υX satisfying $O_{\upsilon X}(H) \cap X = H$. It is easy to see that

$$O_{\upsilon X}(H) = \bigcup_{\substack{\Gamma \in \tau_{\upsilon X},\\ \Gamma \cap X = H}} \Gamma$$

where $\tau_{\upsilon X}$ is the topology of the space υX.

A compactification υX of a Tychonoff space X is called *perfect* with *respect to an open set* H in X if the equality $[Fr_X H]_{\upsilon X} = Fr_{\upsilon X} O_{\upsilon X}(H)$ holds. If υX is perfect for every open set in X, then *it is called a perfect compactification* of the space X ([2], p. 232). For a topological space X and its subset A, a set $Fr_X A = [A]_X \cap [X \setminus A]_X = [A]_X \setminus Int_X A$ is called a boundary of A.

A compactification υX of space X is perfect if and only if for any two disjoint open sets U_1 and U_2 in X the equality $O(U_1 \cup U_2) = O(U_1) \cup O(U_2)$ holds [2]. The Stone-Čech compactification βX of a Tychonoff space X is a perfect compactification of X. The equality $O(U_1 \cup U_2) = O(U_1) \cup O(U_2)$ holds for every pair of open sets U_1 and U_2 in X if and only if X is normal, and the compactification υX coincides with the Stone-Čech compactification βX, i.e., $\upsilon X \cong \beta X$.

The following criterion plays a key role in investigating the class of Π-complete spaces.

Theorem 2.2. [3] *A Tychonoff space X is* Π*-complete if and only if for every $x \in bX \setminus X$ of an arbitrary perfect compactification bX there exists an open $bX \setminus X$ cover ω of X with $Kp\,\omega = 1$, pricking out x in bX (i.e., $x \notin \cup[\omega]_{bX}$).*

Since the Stone-Čech compactification βX of a Tychonoff space X is a perfect compactification of X, then Theorem 2.2 implies the following assertion.

Corollary 2.3. *A Tychonoff space X is* Π*-complete if and only if for every $x \in \beta X \setminus X$ there exists a cover ω of X with $Kp\,\omega = 1$, pricking out x in βX.*

Note that every compact Hausdorff space is a Π-complete space. The square of the Sorgenfrey line (that is the set of real numbers equipped with the topology generated by sets $[a,\, b)$, here $-\infty < a < b < +\infty$) is Π-complete, but it is not a paracompact space (hence, it is not a compact Hausdorff space). The space $T(\omega_1)$ of all ordinal numbers less than the first uncountable ordinal number ω_1 is a normal space but it is not Π-complete.

Let us list some known properties of Π-complete spaces.

(1) A closed subset of a Π-complete space is Π-complete ([10], p. 19).

(2) If $f\colon X \to Y$ is a perfect map in a Π-complete space Y then X is also Π-complete ([10], p. 26).

(3) A Π-complete space is complete in Dieudonné sense ([10], p. 18.)

The author of [16] observed the functor $I\colon \mathfrak{Comp} \to \mathfrak{Comp}$ and showed that it is normal. Then in [7] using the construction suggested by A.Ch. Chigogidze [4], it was obtained an extension $I_\beta\colon \mathfrak{Tych} \to \mathfrak{Tych}$. Here, the sign $\mathfrak{Comp}$ means the category of compact Hausdorff spaces and their continuous maps, and $\mathfrak{Tych}$ the category of Tychonoff spaces and their continuous maps.

In [16] on a compact Hausdorff space X an idempotent probability measure is defined as a functional $\mu\colon C(X) \to \mathbb{R}$ that meets the following conditions:

1) $\mu(c_X) = c$ for every constant function $c_X\colon X \to \mathbb{R}$, $c \in \mathbb{R}$. Here $c_X(x) = c$;

2) $\mu(c \odot \varphi) = c \odot \mu(\varphi)$, $c \in \mathbb{R}$, $\varphi \in C(X)$. Here $c \odot \varphi = c + \varphi$;

3) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$, $\varphi,\, \psi \in C(X)$. Here $\varphi \oplus \psi = \max\{\varphi,\, \psi\}$.

A set of all idempotent probability measures in X is denoted by $I(X)$. It is endowed with the topology τ_p of pointwise convergence. For $\mu \in I(X)$ sets

$$\langle \mu;\, \varphi_1,\, \ldots,\, \varphi_n;\, \theta\rangle = \{\nu \in I(X)\colon\ |\nu(\varphi_i) - \mu(\varphi_i)| < \theta,\, i = 1,\, \ldots,\, n\}$$

forms a base of $I(X)$ at μ. Here $\varphi,\, \ldots,\, \varphi_n \in C(X)$, $\theta > 0$.

Note that a function $f\colon X \to [-\infty,\, +\infty)$ is said to be an *upper semi-continuous* if for each $x \in X$ and for every real number r that satisfies $f(x) < r$, there exists an open neighborhood $U \subset X$ of x such that $f(x') < r$ for all $x' \in U$.

Now we consider a compact Hausdorff space X, and put

$$USC_0(X) = \Big\{f\colon X \to [-\infty,\, 0]\,\Big|\, f \text{ is an upper semi-continuous function such that there exists } x \in X \text{ with } f(x) = 0\Big\}.$$

For every idempotent probability measure $\nu \in I(X)$ there exists [1] a unique upper semi-continuous function $\lambda \in USC_0(X)$ such that $\nu = \bigoplus_{x \in X} \lambda(x) \odot \delta_x$.

Consequently [15],

$$I(X) = \left\{ \underset{x\in X}{\oplus} \lambda(x) \odot \delta_x : \; \lambda \in USC_0(X) \right\}.$$

A set

$$\operatorname{supp} \mu = \{x \in X : \; \lambda(x) > -\infty\}$$

we will call [15] the support of an idempotent probability measure $\mu = \underset{x\in X}{\oplus} \lambda(x) \odot \delta_x$.

3. A perfect compactification and Π-completeness of the space of idempotent probability measures

For an element $\alpha \in \mathbb{R}_{\max}$ and a subset $A \subset I(X)$ we put $\alpha \odot A = \{\alpha \odot \mu : \mu \in A\}$. Note that if $\alpha < 0$, then the set $\alpha \odot A$ is not a subset of $I(X)$ but it is a subset of the parallel space $\alpha \odot I(X)$. Let

$$\alpha \odot A \oplus \beta \odot B = \{\alpha \odot \mu \oplus \beta \odot \nu : \mu \in A, \, \nu \in B\}$$

for some $\alpha, \, \beta \in \mathbb{R}_{\max}$ and $A, \, B \subset I(X)$. Evidently, $\alpha \odot A \oplus \beta \odot B \subset I(X)$ if and only if $\alpha \oplus \beta = 0$.

Now we note the following remarkable property.

Remark 3.1. For every pair of $\alpha, \, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$, we have

$$\alpha \odot I(X) \oplus \beta \odot I(X) = I(X).$$

This property can be demonstrated by direct verification. So, for $\mu, \, \nu \in I(X)$, and elements $\alpha, \, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$, a max-plus combination $\alpha \odot \mu \oplus \beta \odot \nu$ is an idempotent probability measure on X.

Proposition 3.2. *Let $\langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle$ and $\langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle$ be neighborhoods of idempotent probability measures μ and ν, respectively. Then a set*

$$\langle \alpha \odot \mu \oplus \beta \odot \nu; \langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle; \, \langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle\rangle_{\oplus} =$$
$$= \{\alpha \odot \mu' \oplus \beta \odot \nu' \in I(X) : \; \mu' \in \langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle \;\; \textit{and} \;\; \nu' \in \langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle\}$$

is open in $I(X)$.

Proof. Denote $\Xi = \langle \alpha \odot \mu \oplus \beta \odot \nu; \langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle; \, \langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle\rangle_{\oplus}$. It is easy to note that

$$\Xi = \alpha \odot \langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle \bigoplus \beta \odot \langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle.$$

Three cases are possible.

1) $-\infty \le \alpha < \beta = 0$. Then $\Xi \cap I(X) = \langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle$; in this case the set $\alpha \odot \langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle$ is a subset of the “parallel” space $\alpha \odot I(X)$ as an open subset.

2) $-\infty \le \beta < \alpha = 0$. Then $\Xi \cap I(X) = \langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle$, and $\beta \odot \langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle$ is open in the “parallel” space $\beta \odot I(X)$.

So, in the above two cases, in general, the set $\alpha \odot \langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle$ is open in the space $\alpha \odot I(X)$, and $\beta \odot \langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle$ is in $\beta \odot I(X)$. Consequently, the set Ξ is open in $\alpha \odot I(X) \oplus \beta \odot I(X) = I(X)$.

3) $\alpha = \beta = 0$. Then $\Xi \cap I(X) = \langle \mu; \, \varphi_1, \, \ldots, \, \varphi_n; \, \theta\rangle \bigoplus \langle \nu; \, \psi_1, \, \ldots, \, \psi_k; \, \delta\rangle$ is open $I(X) \oplus I(X) = I(X)$. Proposition 3.2 is proved.

□

Remark 3.3. Consider any finite set of elements $\alpha_i \in \mathbb{R}_{\max}$ with $\overset{n}{\underset{i=1}{\oplus}} \alpha_i = 0$. Then the equality in Remark 3.1 can be rewritten as

$$\overset{n}{\underset{i=1}{\oplus}} \alpha_i \odot I(X) = I(X).$$

Consequently, Proposition 3.2 can be reformulated for any finite summands.

In the case of sets $\langle \mu_i;\, \varphi_i;\, \theta\rangle$, $i = 1, \ldots, n$, for simplicity we will use the notation $\langle \overset{n}{\underset{i=1}{\oplus}} \alpha_i \odot \mu_i;\, \varphi_1, \ldots, \varphi_n;\, \theta\rangle_\oplus$ instead of $\langle \overset{n}{\underset{i=1}{\oplus}} \alpha_i \odot \mu_i;\, \langle \mu_1;\, \varphi_1;\, \theta\rangle; \ldots, \langle \mu_n;\, \varphi_n;\, \theta\rangle\rangle_\oplus$.

Since $USC_0(X)$ is a subspace in $\mathbb{R}^X_{\max}$, its neighborhood system at an upper semi-continuous function $\lambda \in USC_0(X)$ has a shape

$$(\lambda;\, x_1, \ldots, x_n;\, \theta) = \{\gamma \in USC_0(X)\colon\ |\gamma(x_i) - \lambda(x_i)| < \theta,\, i = 1, \ldots, n\}.$$

Here $x_i \in X$, $i = 1, \ldots, n$; $\theta > 0$.

For points $x_i \in X$ and their open neighborhoods Ux_i, $i = 1, \ldots, n$, it is easy to see that the set

$$\begin{aligned}&(\lambda;\, Ux_1, \ldots, Ux_n;\, \theta) =\\ &\qquad = \{\gamma \in USC_0(X)\colon\ \text{there is } y_i \in Ux_i \text{ such that } |\gamma(y_i) - \lambda(x_i)| < \theta,\, i = 1, \ldots, n\}\end{aligned}$$

is also open as well. Indeed, take any $\gamma \in (\lambda;\, Ux_1, \ldots, Ux_n;\, \theta)$. For $i = 1, \ldots, n$, let $|\gamma(y_i) - \lambda(x_i)| = a_i < \theta$ for some $y_i \in Ux_i$. Put $a = \max\{a_1, \ldots, a_n\}$. Then for each $\xi \in (\gamma;\, y_1, \ldots, y_n; \theta - a)$ we have

$$|\xi(y_i) - \lambda(x_i)| = |\xi(y_i) - \gamma(y_i) + \gamma(y_i) - \lambda(x_i)| \le |\xi(y_i) - \gamma(y_i)| + |\gamma(y_i) - \lambda(x_i)| < \theta - a + a = \theta,$$

i.e., $\xi \in (\lambda;\, Ux_1, \ldots, Ux_n;\, \theta)$. So, $(\gamma;\, y_1, \ldots, y_n; \theta - a) \subset (\lambda;\, Ux_1, \ldots, Ux_n;\, \theta)$ which shows the openness of $(\lambda;\, Ux_1, \ldots, Ux_n;\, \theta)$.

Now let us consider sets of the following type

$$\begin{aligned}\langle \mu;\, U_1, \ldots, U_n;\, \theta\rangle = &\{\nu \in I(X)\colon\ \operatorname{supp}\nu \subset \overset{n}{\underset{i=1}{\cup}} U_i,\ \text{and there is a couple of points } x \in \operatorname{supp}\mu \cap U_i,\\ &y \in \operatorname{supp}\nu \cap U_i \text{ such that } |\chi_\nu(y) - \chi_\mu(x)| < \theta,\, i = 1, \ldots, n\}. \end{aligned} \tag{3.1}$$

Note that

$$\begin{aligned}&\langle \mu; U_1, \ldots, U_n;\, \theta\rangle =\\ &\quad = \{\nu \in I(X)\colon\ \exists x \in \operatorname{supp}\mu \cap U_i,\, \exists y \in \operatorname{supp}\nu \cap U_i,\, |\chi_\nu(y) - \chi_\mu(x)| < \theta,\, i = 1, \ldots, n\} \cap\\ &\qquad \cap \left\{\nu \in I(X)\colon\ \operatorname{supp}\nu \subset \overset{n}{\underset{i=1}{\cup}} U_i\right\}.\end{aligned}$$

Denote

$$A = \{\nu \in I(X)\colon\ \exists x \in \operatorname{supp}\mu \cap U_i,\, \exists y \in \operatorname{supp}\nu \cap U_i,\, |\chi_\nu(y) - \chi_\mu(x)| < \theta\}$$

and

$$B = \left\{\nu \in I(X)\colon\ \operatorname{supp}\nu \subset \overset{n}{\underset{i=1}{\cup}} U_i\right\}.$$

Then $\langle \mu;\, U_1, \ldots, U_n;\, \theta\rangle = A \cap B$.

Proposition 3.4. *For open sets $U_1, \ldots, U_n$ in a compact Hausdorff space X, every $\mu \in I(X)$ and $\theta > 0$, the set A is open in $I(X)$ with respect to the topology of pointwise convergence.*

Proof. Suppose that throughout the Proof, i takes values $1, \ldots, n$. Let

$$a_{i\mu} = \sup\{\chi_\mu(x) \colon x \in \operatorname{supp}\mu \cap U_i\}$$

(the existence of the supremum follows from the well-known analogue of the Weierstrass theorem for upper semicontinuous functions). There is $x_{i\mu} \in \operatorname{supp}\mu \cap U_i$ such that $a_{i\mu} < \chi_\mu(x_{i\mu}) + \frac{\theta}{2}$. By the definition $\chi_\mu(x_{i\mu}) \le a_{i\mu} \le 0$. Since X is a compact Hausdorff space there exists a continuous function $\varphi_i \colon X \to \mathbb{R}$ such that $\varphi_i(x_{i\mu}) = \sup\{\varphi_i(x) \colon x \in U_i\} = 0$, and

$$\varphi_i^{-1}\left((-\theta; 0]\right) = U_i.$$

It is easy to see that

$$\mu^i = \bigoplus_{x \in \operatorname{supp}\mu \cap U_i} (-a_{i\mu}) \odot \chi_\mu(x) \odot \delta_x = -a_{i\mu} \odot \bigoplus_{x \in \operatorname{supp}\mu \cap U_i} \chi_\mu(x) \odot \delta_x$$

is an idempotent probability measure on X. Moreover, $\mu = \bigoplus\limits_{i=1}^{n} a_{i\mu} \odot \mu^i$. Then Proposition 3.2 and Remark 3.3 imply that

$$\langle \mu; \varphi_1, \ldots, \varphi_n; \tfrac{\theta}{2}\rangle_\oplus = \left\{ \bigoplus_{i=1}^{n} a_{i\mu} \odot \nu^i \colon \nu^i \in \langle \mu^i; \varphi_i; \tfrac{\theta}{2}\rangle, \ i = 1, \ldots, n \right\}$$

is an open set in $I(X)$.

Consider an arbitrary idempotent probability measure $\nu \in \langle \mu; \varphi_1, \ldots, \varphi_n; \frac{\theta}{2}\rangle_\oplus$. Then $\nu = \bigoplus\limits_{i=1}^{n} a_{i\mu} \odot \nu^i$ and $\nu^i \in \langle \mu^i; \varphi_i; \frac{\theta}{2}\rangle$, $i = 1, \ldots, n$.

Let $\mu^i(\varphi_i) = -a_{i\mu} + \chi_\mu(x_{i\mu}) + \varphi(x_{i\mu})$ and $\nu^i(\varphi_i) = -a_{i\mu} + \chi_\nu(y_{i\nu}) + \varphi(y_{i\nu})$ for some $x_{i\mu}$, $y_{i\nu} \in X$. Suppose $y_{i\nu} \notin U_i$. Then $\varphi_i(y_{i\nu}) \le -\theta$. Since $-a_{i\mu} + \chi_\nu(y_{i\nu}) \le 0$ (otherwise, ν^i is not an idempotent probability measure), we have

$$-a_{i\mu} + \chi_\nu(y_{i\nu}) + \varphi_i(y_{i\nu}) \le -\theta.$$

On the other side,

$$a_{i\mu} - \chi_\mu(x_{i\mu}) = a_{i\mu} - \chi_\mu(x_{i\mu}) - \varphi_i(x_{i\mu}) < \tfrac{\theta}{2}.$$

Adding the last two inequalities, one has

$$\nu^i(\varphi_i) - \mu^i(\varphi_i) < -\tfrac{\theta}{2}.$$

The obtained contradiction implies $y_{i\nu} \in U_i$. Then $-\frac{\theta}{2} < \varphi_i(y_{i\nu}) \le 0$. Using $\varphi_i(x_{i\mu}) = 0$, we rewrite this inequality as follows

$$0 \le \varphi_i(x_{i\mu}) - \varphi_i(y_{i\nu}) < \tfrac{\theta}{2}.$$

The relation $\nu^i \in \langle \mu^i; \varphi_i; \frac{\theta}{2}\rangle$ gives

$$-\tfrac{\theta}{2} < \nu^i(\varphi_i) - \mu^i(\varphi_i) < \tfrac{\theta}{2},$$
$$-\tfrac{\theta}{2} < \chi_\nu(y_{i\nu}) - \chi_\mu(x_{i\mu}) + \varphi_i(y_{i\nu}) - \varphi_i(x_{i\mu}) < \tfrac{\theta}{2}.$$

Adding the inequalities $0 \le \varphi_i(x_{i\mu}) - \varphi_i(y_{i\nu}) < \frac{\theta}{2}$ to the last one, we have

$$-\tfrac{\theta}{2} < \chi_\nu(y_{i\nu}) - \chi_\mu(x_{i\mu}) < \theta.$$

So, $\nu \in A$ that means the openness of A. Proposition 3.4 is proved.

□

Proposition 3.5. *A system*

$$\mathcal{B}(\mu) = \{\langle \mu;\, U_1,\, \ldots,\, U_k;\, \theta\rangle\colon\ U_i \text{ is open in } X,\ i = 1,\, \ldots,\, n;\ \theta > 0\}$$

forms a base of some topology in $I(X)$ at the point μ.

The Proof of this proposition repeats the proof of the similar statement from [15]. The generated topology denote by τ_Z.

Now the set B is open in $I(X)$. Or, equivalently, the set

$$I(X) \setminus B = \left\{\nu \in I(X)\colon\ \operatorname{supp} \nu \cap \left(X \setminus \bigcup_{i=1}^{n} U_i\right) = \varnothing\right\}.$$

is closed in $I(X)$.

Consider an arbitrary net $\{\nu_\alpha\colon\ \alpha \in \mathfrak{A}\} \subset I(X) \setminus B$. Since $I(X)$ is compact this net converges to some $\nu_0 \in I(X)$, i.e., there exists a limit $\nu_0 = \lim_{\alpha\in\mathfrak{A}} \nu_\alpha \in I(X)$. We have to show that $\operatorname{supp} \nu_0 \cap \left(X \setminus \bigcup_{i=1}^{n} U_i\right) = \varnothing$. The net $\{\nu_\alpha\colon\ \alpha \in \mathfrak{A}\}$ generates the net $\{\operatorname{supp} \nu_\alpha\colon\ \alpha \in \mathfrak{A}\}$ of nonempty compact subsets $\operatorname{supp} \nu_\alpha$ of the compact Hausdorff space X. Then $\{\operatorname{supp} \nu_\alpha\}$ has (see, for example [6]) a limit F_0 with respect to the Vietoris topology in the hyperspace $\exp X$ of X which does not meet with $X \setminus \bigcup_{i=1}^{n} U_i$, i.e., $F_0 \cap \left(X \setminus \bigcup_{i=1}^{n} U_i\right) = \varnothing$. It is easy to see $F_0 = \operatorname{supp} \nu_0$. Consequently, $\operatorname{supp} \nu_0 \cap \left(X \setminus \bigcup_{i=1}^{n} U_i\right) = \varnothing$. So, $I(X) \setminus B$ is closed in $I(X)$ which implies the openness of B. Hence, the set

$$\langle \mu;\, U_1,\, \ldots,\, U_n;\, \theta\rangle = A \cap B$$

is open as an intersection of two open sets A and B.

Proposisitons 3.4 and 3.5 imply the following statement.

Corollary 3.6. *The topology τ_Z is thinner than the topology τ_p of point-wise convergence on $I(X)$.*

For a Tychonoff space X, following [7], consider the set

$$I_\beta(X) = \{\mu \in I(\beta X)\colon\ \operatorname{supp}\, \mu \subset X\}.$$

Since the support $\operatorname{supp}\, \mu$ of $\mu \in I_\beta(X)$ is compact, μ is called an idempotent measure with compact support. $I_\beta(X)$ is considered as a subspace of $I(\beta X)$. The set $I_\beta(X)$ equips with the topology τ_Z.

Consider subsets $M_1,\, \ldots,\, M_n$ of a Tychonoff space X, numbers $\varepsilon_1,\, \ldots,\, \varepsilon_n$, with $-\infty < \varepsilon_i < 0$, and construct the following set

$$\langle M_1,\, \ldots,\, M_n;\, \varepsilon_1,\, \ldots,\, \varepsilon_n\rangle = \Big\{\mu \in I_\beta(X)\colon\ \operatorname{supp} \mu \subset \bigcup_{i=1}^{n} M_i, \text{ and there is } x \in S_i(\mu) \text{ such that } \chi_\mu(x) > \varepsilon_i,\ i = 1, \ldots, n\Big\}, \tag{3.2}$$

here $S_i(\mu) \equiv \operatorname{supp} \mu \cap M_i$, $i = 1,\, \ldots,\, n$.

The following propositions are proved quite easily.

Lemma 3.7. *For open subsets $U_1,\, \ldots, U_n$ in a Tychonoff space X, and $\varepsilon_1,\, \ldots,\, \varepsilon_n$ with $-\infty < \varepsilon_i < 0$, the set of the form (3.2) is open in $I(X)$ with respect to the topology τ_Z.*

Lemma 3.8. *For compact subsets $F_1, \ldots, F_n$ in X, and $\varepsilon_1, \ldots, \varepsilon_n$ with $-\infty < \varepsilon_i < 0$, the set of the form (3.2) is closed in $I(X)$ with respect to the topology τ_Z with inequality $>$ replaced by $\geq$.*

Now we will establish necessary and sufficient condition for a non-empty intersection of sets of the form (3.2).

Lemma 3.9. *Let $U_1, \ldots, U_n, V_1, \ldots, V_k$ be nonempty subsets in a Tychonoff space X, $-\infty < \varepsilon_i < 0$, $i = 1, \ldots, n$, $\infty < \delta_j < 0$, $j = 1, \ldots, k$. Then*

$$\langle U_1, \ldots, U_n; \varepsilon_1, \ldots, \varepsilon_n\rangle \cap \langle V_1, \ldots, V_k; \delta_1, \ldots, \delta_k\rangle \neq \varnothing$$

if and only if

($i_\cap$) *for every $i \in \{1, \ldots, n\}$ there exists $j = j(i) \in \{1, \ldots, k\}$ such that $U_i \cap V_{j(i)} \neq \varnothing$,*

and

($j_\cap$) *for every $j \in \{1, \ldots, k\}$ there exists $i = i(j) \in \{1, \ldots, n\}$ such that $U_{i(j)} \cap V_j \neq \varnothing$.*

Proof. We have $\operatorname{supp}\mu \subset \bigcup\limits_{i=1}^{n} U_i \cap \bigcup\limits_{i=1}^{k} V_j = \bigcup\limits_{\substack{i=1,\\ j=1}}^{\substack{k,\\ n}} U_i \cap V_j$. Let both conditions ($i_\cap$) and ($j_\cap$) hold. Take a point $x_{ij} \in U_i \cap V_j$ (we will not consider cases with empty intersections) and build an idempotent probability measure $\mu = \bigoplus\limits_{ij} \lambda_{ij} \odot \delta_{x_{ij}}$ here $\lambda_{ij} \geq \max\{\varepsilon_i, \delta_j\}$ and $\bigoplus\limits_{ij} \lambda_{ij} = 0$. Then $\mu \in \langle U_1, \ldots, U_n; \varepsilon_1, \ldots, \varepsilon_n\rangle \cap \langle V_1, \ldots, V_k; \delta_1, \ldots, \delta_k\rangle$.

Now suppose that $\langle U_1, \ldots, U_n; \varepsilon_1, \ldots, \varepsilon_n\rangle \cap \langle V_1, \ldots, V_k; \delta_1, \ldots, \delta_k\rangle \neq \varnothing$, and let $\mu \in \langle U_1, \ldots, U_n; \varepsilon_1, \ldots, \varepsilon_n\rangle \cap \langle V_1, \ldots, V_k; \delta_1, \ldots, \delta_k\rangle$. Assume that one of the conditions ($i_\cap$) or ($j_\cap$) is not satisfied. Let ($i_\cap$) be not true for certainty. Then there exists $i_0 \in \{1, \ldots, n\}$ with $U_{i_0} \cap V_j = \varnothing$ for all $j \in \{1, \ldots, k\}$. The last assertion implies $U_{i_0} \cap \left(\bigcup\limits_{j=1}^{k} V_i\right) = \varnothing$. On the other hand $\operatorname{supp}\mu \subset \left(\bigcup\limits_{j=1}^{k} V_i\right)$ and $\operatorname{supp}\mu \cap U_{i_0} \neq \varnothing$. The obtained contradiction completes the proof of Lemma 3.9.

□

Corollary 3.10. *Let $U_1, \ldots, U_n, V_1, \ldots, V_k$ be nonempty subsets in a Tychonoff space X. Then*

$$\langle U_1, \ldots, U_n; \varepsilon_1, \ldots, \varepsilon_n\rangle \subset \langle V_1, \ldots, V_k; \delta_1, \ldots, \delta_k\rangle$$

if and only if

($i_\subset$) *for every $i \in \{1, \ldots, n\}$ there exists $j = j(i) \in \{1, \ldots, k\}$ such that $U_i \subset V_{j(i)}$, $\varepsilon_i \geq \delta_{j(i)}$,*

and

($j_\subset$) *for every $j \in \{1, \ldots, k\}$ there exists $i = i(j) \in \{1, \ldots, n\}$ such that $U_{i(j)} \subset V_j$, $\varepsilon_{i(j)} \geq \delta_j$.*

In the case $\varepsilon_1 = \ldots = \varepsilon_n = \varepsilon$ we will use the notation $\langle U_1, \ldots, U_n; \varepsilon\rangle$ instead of $\langle U_1, \ldots, U_n; \varepsilon_1, \ldots, \varepsilon_n\rangle$.

For a finite system of sets $U_1, \ldots, U_n$ in a Tychonoff space X we put

$$\langle U_1, \ldots, U_n\rangle = \bigcup_{\varepsilon > -\infty} \langle U_1, \ldots, U_n; \varepsilon\rangle.$$

According to the construction we have

$$\langle U_1,\,\ldots,\,U_n\rangle = \Big\{\mu\in I(X)\colon\ \operatorname{supp}\mu\subset\bigcup_{i=1}^{n}U_i,\ S_i(\mu)\equiv\operatorname{supp}\mu\cap U_i\neq\varnothing,$$
$$\text{and there exists } x\in S_i(\mu) \text{ with } \chi_\mu(x)>-\infty,\ i=1,\ldots,n\Big\}.$$

On the other side since $\chi_\mu(x)>-\infty$ holds for any $x\in\operatorname{supp}\mu$ by the definition the following equality takes place

$$\langle U_1,\,\ldots,\,U_n\rangle = \Big\{\mu\in I(X)\colon\ \operatorname{supp}\mu\subset\bigcup_{i=1}^{n}U_i,\ \operatorname{supp}\mu\cap U_i\neq\varnothing,\ i=1,\ldots,n\Big\}.$$

Lemma 3.11. *Let $U_1,\,\ldots,\,U_n$ be open sets in a Tychonoff space X. Then*

$$[\langle U_1,\,\ldots,\,U_n\rangle]_{I(\beta X)} = \langle [U_1]_{\beta X},\,\ldots,\,[U_n]_{\beta X}\rangle.$$

Proof. Corollary 3.10 implies $\langle U_1,\,\ldots,\,U_n\rangle\subset\langle [U_1]_{\beta X},\,\ldots,\,[U_n]_{\beta X}\rangle$. By virtue of Lemma 3.8, $\langle [U_1]_{\beta X},\,\ldots,\,[U_n]_{\beta X}\rangle$ is closed in $I(X)$. Hence,

$$[\langle U_1,\,\ldots,\,U_n\rangle]_{\beta X}\subset\langle [U_1]_{\beta X},\,\ldots,\,[U_n]_{\beta X}\rangle.$$

Theorem 3.3 in [6] implies the inverse inclusion, i.e.,

$$\langle [U_1]_{\beta X},\,\ldots,\,[U_n]_{\beta X}\rangle\subset[\langle U_1,\,\ldots,\,U_n\rangle]_{I(\beta X)}.$$

Lemma 3.11 is proved.

□

Lemma 3.12. *For an open cover $\gamma_X=\{U_\alpha\colon\ \alpha\in\mathfrak{A}\}$ of a Tychonoff space X the system*

$$\gamma_0=\{\langle U_1,\,\ldots,\,U_n\rangle\colon U_i\in\gamma,\ i=1,\,\ldots,\,n;\ n\in\mathbb{N}\}$$

forms an open cover of $I_\beta(X)$.

Proof. At first we will show that for any finite system of open in X sets $U_1,\,\ldots,\,U_n$ the set $\langle U_1,\,\ldots,\,U_n\rangle$ is an open set in $I_\beta(X)$ with respect to the topology of pointwise convergence.

Take any $\mu\in\langle U_1,\,\ldots,\,U_n\rangle$. We claim that for an arbitrary positive θ the open neighbourhood $\langle\mu;\,U_1,\,\ldots,\,U_n;\,\theta\rangle$ of μ is a subset of $\langle U_1,\,\ldots,\,U_n\rangle$. In fact, for every $\nu\in\langle\mu;\,U_1,\,\ldots,\,U_n;\,\theta\rangle$ we have $\operatorname{supp}\nu\subset\bigcup_{i=1}^{n}U_i$, $\operatorname{supp}\nu\cap U_i\neq\varnothing$, and $\chi_\mu(y)-\theta<\chi_\nu(x)(<\chi_\mu(y)+\theta)$ at $x\in\operatorname{supp}\nu$, $y\in\operatorname{supp}\mu$. Since $\chi_\mu(y)>-\infty$, $y\in\mu$, then $\chi_\nu(x)>-\infty$. Consequently, $\nu\in\langle U_1,\,\ldots,\,U_n\rangle$, i.e., $\langle\mu;\,U_1,\,\ldots,\,U_n;\,\theta\rangle\subset\langle U_1,\,\ldots,\,U_n\rangle$. Thus, $\langle U_1,\,\ldots,\,U_n\rangle$ is an open set in $I(X)$.

It remains to verify that

$$\bigcup\gamma_0\supset I_\beta(X).$$

Let μ be an arbitrary element of $I_\beta(X)$. Consider a cover $\{U_\alpha\colon\ \alpha\in\mathfrak{A}_\mu\}\subset\gamma_X$ of the compact set $\operatorname{supp}\mu$. The compactness of the support provides the existence of a finite open subcover $\{U_1,\,\ldots,\,U_n\}$ such that

$$S_i(\mu)=\operatorname{supp}(\mu)\cap U_i\neq\varnothing.$$

consequently, $\mu\in\langle U_1,U_2,\,\ldots,\,U_n\rangle\in\gamma_0$. So, we conclude that every element of $I_\beta(X)$ is contained in some element of γ_0. We have completed the proof of Lemma 3.12.

□

Proposition 3.13. *If an open cover $\gamma_X = \{U_\alpha \colon \alpha \in \mathfrak{A}\}$ of a Tychonoff space X has a property $Kp\,\gamma_X = 1$ then the cover*

$$\gamma_0 = \{\langle U_1, \ldots, U_n\rangle \colon U_i \in \gamma,\ i = 1, \ldots, n;\ n \in \mathbb{N}\}$$

of $I_\beta(X)$ also has the property $Kp\,\gamma_0 = 1$.

Proof. Let $\langle G_1, \ldots, G_k\rangle$ be an element of γ_0. Since $Kp\,\gamma_X = 1$, Lemma 3.9 implies $\langle U_1, \ldots, U_n\rangle \cap \langle G_1, \ldots, G_k\rangle \neq \varnothing$ if and only if $k = n$ and for every $i \in \{1, \ldots, n\}$ the equality $G_i = U_j$ holds for some unique $j \in \{1, \ldots, k\}$. In other words $\langle G_1, \ldots, G_k\rangle \cap \langle U_1, \ldots, U_n\rangle \neq \varnothing$ if and only if $\{G_1, \ldots, G_k\} = \{U_1, \ldots, U_n\}$. Hence, $Kp\,\gamma_0 = 1$.

Consider any $\mu \in I_\beta(X)$. There is a subfamily $\gamma_X(\mu) \subset \gamma_X$ such that $\operatorname{supp}\mu \subset \bigcup\limits_{U \in \gamma_X(\mu)} U$. Since $\operatorname{supp}\mu$ is compact and γ_X is an open cover of X, there exists a finite subcollection $\{U_1, \ldots, U_n\} \subset \gamma_X$ such that $\operatorname{supp}\mu \subset \bigcup\limits_{i=1}^{n} U_i$. Moreover, $\operatorname{supp}\mu \cap U_i \neq \varnothing$, so we have $\mu \in \langle U_1, \ldots, U_n\rangle$. Thus, μ is covered by an element of γ_0, proving that γ_0 is a cover of $I_\beta(X)$. Proposition 3.13 is proved.

□

To prove the next result we need the following statement.

Lemma 3.14. [14] *Let γX be a compact extension of a space X and, V, W be disjoint open sets in γX. Let $V \cap X = V_\gamma$, and $W \cap X = W_\gamma$. Then the following equality is true:*

$$[X \setminus V_\gamma]_{\gamma X} \cap [X \setminus W_\gamma]_{\gamma X} = [X \setminus (V_\gamma \cup W_\gamma)]_{\gamma X}$$

Obviously, $I(\beta X)$ is one of compact extensions of $I_\beta(X)$. In the following result we show that $I(\beta X)$ is a perfect extension of $I_\beta(X)$.

Theorem 3.15. *For a Tychonoff space X the space $I(\beta X)$ of all idempotent probability measures on βX is a perfect compactification of the space $I_\beta(X)$ of idempotent probability measures with compact support.*

Proof. It is sufficient to consider basic open sets in $I_\beta(X)$. Let U_1 and U_2 be disjoint open sets in X. Since βX is the perfect compactification of X we have $O_{\beta X}(U_1 \cup U_2) = O_{\beta X}(U_1) \cup O_{\beta X}(U_2)$. Consider open sets

$$\langle U_i\rangle = \{\mu \in I_\beta(X) \colon \operatorname{supp}\mu \subset U_i\}, \qquad i = 1, 2$$

in $I_\beta(X)$. It is clear that $\langle U_1\rangle \cap \langle U_2\rangle = \varnothing$. We will show that

$$O_{I(\beta X)}(\langle U_1\rangle \cup \langle U_2\rangle) = O_{I(\beta X)}(\langle U_1\rangle) \cup O_{I(\beta X)}(\langle U_2\rangle).$$

The inclusion $\supset$ straightly follows from the definition of sets of the type $O(H)$. Therefore, we have to show the inverse inclusion. Let $\mu' \in I(\beta X)$ be an idempotent probability measure such that $\mu' \notin O_{I(\beta X)}(\langle U_1\rangle) \cup O_{I(\beta X)}(\langle U_2\rangle)$. Then $\mu' \in I(\beta X) \setminus O_{I(\beta X)}(\langle U_i\rangle)$, $i = 1, 2$. From [2] (see, p. 234) we have

$$I(\beta X) \setminus O_{I(\beta X)}(\langle U_i\rangle) = \big[I_\beta(X) \setminus \langle U_i\rangle\big]_{I(\beta X)}, \qquad i = 1, 2.$$

Hence, $\mu' \in [I_\beta(X) \setminus \langle U_i\rangle]_{I(\beta X)}$, $i = 1, 2$. Since $\langle U_1\rangle \cap \langle U_2\rangle = \varnothing$ by Lemma 3.14 we have

$$[I_\beta(X) \setminus \langle U_1\rangle]_{I(\beta X)} \cap [I_\beta(X) \setminus \langle U_2\rangle]_{I(\beta X)} = [I_\beta(X) \setminus (\langle U_1\rangle \cup \langle U_2\rangle)]_{I(\beta X)}.$$

Thus, $\mu' \in [I_\beta(X) \setminus (\langle U_1\rangle \cup \langle U_2\rangle)]_{I(\beta X)}$, i.e., $\mu' \in I(\beta X) \setminus O_{I(\beta X)}\langle(U_1\rangle \cup \langle U_2\rangle)$. Consequently, $\mu' \notin O_{I(\beta X)}(\langle U_1\rangle \cup \langle U_2\rangle)$. Thus, we have established that the inclusion $O_{I(\beta X)}(\langle U_1\rangle \cup \langle U_2\rangle) \subset O_{I(\beta X)}(\langle U_1\rangle) \cup O_{I(\beta X)}(\langle U_2\rangle)$ is also true. Theorem 3.15 is proved.

□

Now we can state the main result of the paper.

Theorem 3.16. *For a Tychonoff space X, the space $I_\beta(X)$ of idempotent probability measures with compact support is* Π*-complete if and only if X is* Π*-complete.*

Proof. If $I_\beta(X)$ is Π-complete then Property 1 of Π-complete spaces (see, Preliminaries) implies the Π-completeness of the closed subset $X \subset I_\beta(X)$.

Let now X be a Π-complete space and $\mu \in I(\beta X) \setminus I_\beta(X)$. Due to Theorem 3.15 $I(\beta X)$ is a perfect compactification of $I_\beta(X)$. Hence, we have to show the existence of an open cover Ω of $I_\beta(X)$ with $Kp\,\Omega = 1$, that pricks out μ in $I(\beta X)$. We have $\operatorname{supp}\mu \not\subset X$. Owing to Corollary 2.3 for every point $x \in \operatorname{supp}\mu \setminus X \subset \beta X \setminus X$ there exists an open cover ω_x with $Kp\,\omega_x = 1$, pricks out x in βX, i.e., $x \notin \cup[\omega_x]_{\beta X}$. Fix a point $x_0 \in \operatorname{supp}\mu \setminus X$. Then $\operatorname{supp}\mu \not\subset [U]_{\beta X}$ for every $U \in \omega_{x_0}$. Hence $\mu \notin \langle [U_1]_{\beta X}, \ldots, [U_n]_{\beta X} \rangle$ for any n-tuple $\{U_1 \ldots, U_n\} \subset \omega_{x_0}$. Now the equality in Lemma 3.11 gives

$$\mu \notin \cup[\Omega]_{I(\beta X)} = \cup\{[\langle U_1, \ldots, U_n \rangle]_{I(\beta X)} \colon U_i \in \omega_{x_0}, i = 1, \ldots, n; n \in \mathbb{N}\}.$$

Finally, applying Theorem 2.2 and Proposition 3.13 we complete the proof. Theorem 3.16 is proved.

□

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On geometry of orbits of vector fields

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Abstract. As is known, a vector field is one of the basic concepts of geometry. In problems of the qualitative theory of differential equations, control theory, foliation theory, as well as in problems of mechanics, vector fields play an important role. This paper studies the geometry of orbits of two vector fields in four-dimensional space. The geometry of the singular foliation generated by the orbits of these vector fields is described. It is shown that the regular leaf of this foliation is a two-dimensional surface of non-zero normal curvature and nonzero Gauss torsion.

Keywords: Vector field,orbit of a family of vector fields, normal curvature of a surface,Gauss torsion of a surface.

MSC (2020): 37C10, 57R30.

1. Introduction

Let M be a smooth manifold of dimension n.

First of all let us recall notion of singular foliation [14].

Definition 1.1. A subset L of M is said to be a $k-leaf$ of M if there exists a differentiable structure σ on L such that

(i) (L,σ) is a connected k-dimensional immersed submanifold of M, and

(ii) if N is an arbitrary locally connected topological space,and $f: N \to M$ is a continuous function such that $f(N) \subset L$, then $f: N \to (L,\sigma)$ is continuous.

It follows from the properties of immersions that if $f: N \to M$ is a differentiable mapping of manifolds such that $f(N) \subset L$, then $f: N \to (L,\sigma)$ is also differentiable.In particular, σ is the unique differentiable structure on L which makes L into an immersed k-dimensional submanifold of M.

Since M is paracompact,every connected immersed submanifold of M is separable,and so the dimensional k of a leaf L is uniquely determined.

Definition 1.2. We say that $\mathbf{F}$ is a singular C^q-foliation of M if $\mathbf{F}$ is partition of M into C^q-leaves of M such that,for every $x \epsilon M$, there exists a local C^q-chart ψ of M with the following properties:

(a) The domain of ψ is of the from $U \times W$, where U is an open neighborhood of 0 in R^k, W is an open neighborhood of 0 in R^{n-k}, and k is the dimension of the leaf through x.

(b) $\psi(0,0) = x$.

(c) If L is a leaf of $\mathbf{F}$, then $L \cap \psi(U \times W) = \psi(U \times l)$, where $l = \{w \epsilon W : \psi(0,w) \epsilon L\}$.

A leaf dimension of which is maximal is called regular otherwise it is called singular.

It is known that orbits of a family of vector fields generate singular foliation [13],[14].There are many investigations which devoted to the topology and geometry of singular foliations [1, 7].

In this paper we study the geometry of the singular foliation which generated by orbits of two vector fields.

Let $V(M)$ is the set of all smooth (class C^∞) vector fields defined on M. The set $V(M)$ is a linear space over the field of real numbers and Lie algebra in which a binary operation is Lie bracket $[X,Y]$ of vector fields $X, Y \in V(M)$.

Let us consider a set $D \subset V(M)$ and denote the smallest Lie subalgebra containing D by $A(D)$. The family D may contain finitely or infinitely many smooth vector fields. For a point $x \in M$, by $t \to X^t(x)$ we denote integral curve of the vector field X passing through the point x for $t = 0$. The map $t \to X^t(x)$ is defined in some region $I(x)$, which in the general case depends not only from the field X, but also from the starting point x.

Definition 1.3. The orbit $L(x)$ of a family D of vector fields through a point x is the set of points y in M such that there exist real numbers $t_1, t_2, ..., t_k$ and vector fields $X_{i_1}, X_{i_2}, ..., X_{i_k}$ in D (where k is an arbitrary positive integer) such that

$$y = X_{i_k}^{t_k}(X_{i_{k-1}}^{t_{k-1}}(...(X_{i_1}^{t_1})...)).$$

Numerous investigations have been devoted to the study of geometry of orbits of vector fields [1]-[14].

The fundamental result in study of orbits is Sussmann theorem [13], which asserts that every orbit of smooth vector fields with Sussmann topology has differential structure with respect to which it is a immersed submanifold of M.

Recall that a mapping P that takes each point $x \in M$ to some subspace $P(x) \subset T_xM$ is called a distribution. If dim $P(x) = k$ for all $x \in M$, then P is called a k-dimensional distribution. A distribution P is said to be smooth if, for each point $x \in M$, there exists a neighborhood $U(x)$ of the point and smooth vector fields $X_1, X_2, ..., X_m$ defined on $U(x)$ such that the vectors

$$X_1(y), X_2(y), ..., X_m(y)$$

form a basis of the subspace $P(y)$ for each $y \in U(x)$.

A family D of smooth vector fields naturally generates the smooth distribution that takes each point $x \in M$ to the subspace $P(x)$ of the tangent space T_xM spanned by the set

$$D(x) = \{X(x) : X \in D\}.$$

Obviously, the dimension of the subspace $P(x)$ can vary from point to point.

A distribution P is said to be completely integrable if, for each point $x \in M$, there exists a connected submanifold N_x of the manifold M such that $T_yN_x = P(y)$ for all $y \in N_x$.

The submanifold N_x is callled an integral submanifold of the distribution P. For a vector field X, we write $X \in P$ if $X(x) \in P(x)$ for all $x \in M$.

A distribution P is said to be involutive if the inclusion $X, Y \in P$ implies that $[X, Y] \in P$, where $[X, Y]$ is the Lie bracket of the fields X and Y.

The Frobenius theorem [7] provides a necessary and suffcient condition for the completely integrability of a distribution of constant dimension.

Theorem 1.4. *(Frobenius) A distribution P on a manifold M is completely integrable if and only if it is involutive.*

Let $A(D)$ be the smallest Lie algebra containing the set D. By setting $A_x(D) = \{X(x) : X \in A(D)\}$, we obtain an involutive distribution $P_D : x \to Ax(D)$. If the dimension $dimA_x(D)$ is independent of x, then the distribution $P_D : x \to Ax(D)$ is completely integrable by the Frobenius theorem.

If the dimension $dimA_x(D)$ depends on x, then, as examples show, the distribution $P_D : x \to A_x(D)$ is not necessarily completely integrable.

The Frobenius theorem generalized by Hermann to distributions of variable dimension provides a necessary and sufficient condition for the complete integrability of distributions which is finitely generated [7].

Definition 1.5. A system of vector fields

$$D = \{X_1, X_2, ..., X_k\}$$

on M is in involution if there exist smooth real-valued functions $f_{ij}^l(x), x \in M,\ i, j, l = 1, ..., k$ such that for each (i, j) it takes

$$[X_i, X_j] = \sum_{l=1}^{k} f_{ij}^l(x) X_l.$$

Theorem 1.6. *The system*

$$D = \{X_1, X_2, ..., X_k\}$$

of smooth vector fields on M generates completely integrable distribution if and only if it is in involution.

2. Geometry of orbits of vector fields

Let us consider a family of $D = \{X, Y\}$ vector fields on four-dimensional Euclidean space E^4 with the cartesian coordinates x_1, x_2, t, u.

$$X = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, Y = (x_1^2 - x_2^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2} - 4x_1u\frac{\partial}{\partial u}. \tag{2.1}$$

Theorem 2.1. *Orbits of the family of vector fields (2.1) generate singular foliation singular leaves of which are a point and one dimensional submanifolds,every regular leaf of which is a surface nonzero normal curvature and nonzero Gauss torsion*

Proof. Let us recall some characteristics of two dimensional surface F in four-dimensional Euclidean space E^4.

Consider on the surface F at the point x some direction given by the nonzero vector ξ.

The vector ξ and the normal plane N of the surface at the point x define a hyperplane $E^3(x, \xi N)$ in E^4 which intersects the surface F along some curve γ. The curve γ is called the normal section of the surface F at the point x along the direction ξ. By its construction, the γ curve is a three-dimensional curve. Curvature $k_N(x, \xi)$ and torsion $\chi_N(x, \xi)$ of the curve γ at the point x are called, respectively, the normal curvature and the normal torsion of the surface at the point x in the direction ξ.

Geometry of two dimensional surfaces four-dimensional Euclidean space E^4 is a very important part of differential geometry and studied by many authors [4],[5],[6],[12].

Let S be the set of two-dimensional surfaces in the space E^4 whose normal torsion is equal to zero at any point in any direction. It is known that two-dimensional hyperplane surfaces belong to the set S, but do not exhaust it. Thus, the two-dimensional torus $S^1 \times S^1$ on the hypersphere S^3 in E^4 belongs to the set S, but is not hyperplane. The description of hyperplane two-dimensional surfaces in the set S is given in [6].

First of all we calculate Lie brocket and find $[X, Y] = 0$. It follows from Hermann theorem the family D is completely integrable.

Vector field

$$X = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \tag{2.2}$$

generate following one parametrical group of transformations

$$(x_1, x_2, t, u) \to (x_1, x_2, te^s, ue^s) : s \in R$$

We find the invariant functions of these transformations. It is known that [11, p. 117] a smooth function $f : M \to R$ is an invariant function of the transformation group G, acting on M if and only if $Xf = 0$ for each infinitesimal generator X of the group G.

Using this criterion, we find that the functions

$$F^1(t, x_1, x_2, u) = \frac{t}{x_2^2 u}, F^2(t, x_1, x_2, u) = \frac{x_2}{x_1^2 + x_2^2} \tag{2.3}$$

are invariant functions which follows from the following equalities

$$X(F_1) = X(F_2) = 0, Y(F_1) = Y(F_2) = 0. \tag{2.4}$$

This invariant functions give us a family of two dimensional surfaces

$$\begin{cases} \frac{t}{x_2^2 u} = C_1 \\ \frac{x_2}{x_1^2+x_2^2} = C_2 \end{cases} \tag{2.5}$$

which are can be parameterised by the following equations

$$\begin{cases} t = t \\ x_1 = \dfrac{1}{2C_2} \cos\tau \\ x_2 = \dfrac{1}{2C_2}(1 + \sin\tau) \\ u = \dfrac{4C_2^2 t}{C_1(1 + \sin\tau)^2}, C_1 > 0, \end{cases} \tag{2.6}$$

where C_1, C_2 are constants.

For given C_1, C_2 let us denote by F^C component of connectedness of the surface which is defined by system of equations (2.5). For definiteness, we will assume that $C_1 > 0$.

If $p^0(x_1^0, x_2^0, t^0, u^0) \in F^C$, it follows from equalities (2.4) the orbit $L(p^0)$ of a family D of vector fields through the point p^0 is contained in the surface F^C.

If the point p^0 is the origin of a coordinate system, then it is a fixed point for vector fields, and in this case the orbit $L(p^0)$ is the point itself.

If $C_1 = 0$ then $t^0 = 0$ and $u^0 = 0$. It follows the vector field X vanish at the point p^0. In this case the orbit $L(p^0)$ is a integral line of the the vector field Y.

Let $C_1 > 0$ for definiteness. It follows that $t^0 > 0$, $u^0 > 0$.

We assume that $x_1^0 \neq 0$. In this case minimal subalgebra $A(D)$ of the algebra of all vector FIELDS which contains D is a two dimensional. In particularly, vectors $X(p_o)$, $Y(p_o)$ are linearly independent In this case it follows from [13] the orbit $L(p^0)$ is a two dimensional surface in F^C.

Now one can check the metric characteristics of the surface F^C. In order to find NORMAL curvature we will use formulas from the paper [5]. The normal curvature of the surface at the point x in the direction $\xi = \{\xi_1, \xi_2\}$ is calculated by the following formula:

$$k_N(x, \xi) = \left(\left(\frac{\sum\limits_{i=1}^{2} b_{ij}\xi_i\xi_j}{\sum\limits_{i=1}^{2} g_{ij}\xi_i\xi_j} \right)^2 + \left(\frac{\sum\limits_{i=1}^{2} c_{ij}\xi_i\xi_j}{\sum\limits_{i=1}^{2} g_{ij}\xi_i\xi_j} \right)^2 \right)^{\frac{1}{2}}. \tag{2.7}$$

where b_{ij}, c_{ij} are coefficients of the second quadratic forms, g_{ij} are coefficients of the first quadratic form. These coefficients are calculated by formulas

$$(\vec{r}_{tt}, \vec{n}_1) = b_{11}, (\vec{r}_{t\tau}, \vec{n}_1) = b_{12}, (\vec{r}_{\tau\tau}, \vec{n}_1) = b_{22}$$

$$(\vec{r}_{tt}, \vec{n}_2) = c_{11}, (\vec{r}_{t\tau}, \vec{n}_2) = c_{12}, (\vec{r}_{\tau\tau}, \vec{n}_2) = c_{22} \tag{2.8}$$

$$g_{11} = \vec{r}_t^{\,2}, g_{12} = (\vec{r}_t, \vec{r}_\tau), g_{22} = \vec{r}_\tau^{\,2},$$

where n_1, n_2 are normal unit vector. By calculating we have

$$\vec{r}_t = \{1, 0, 0, \frac{4C_2^2 t}{C_1(1+sin\tau)^2}\}, \vec{r}_\tau = \{0, -\frac{1}{2C_2}sin\tau, \frac{1}{2C_2}cos\tau, -\frac{8C_2^2 t cos\tau}{C_1(1+sin\tau)^3}\}$$

$$\vec{n}_1 = \{0, cos\tau, sin\tau, 0\}, \vec{n}_2 = \{-\frac{2C_2}{C_1(1+sin\tau)^2}, -\frac{4C_2^2 sin2\tau}{C_1(1+sin\tau)^3}, \frac{8C_2^2 t cos^2\tau}{C_1(1+sin\tau)^3}, \frac{1}{2C_2}\}$$

$$\vec{r}_{tt} = \{0,0,0,0\}, \vec{r}_{t\tau} = \{0,0,0,-\frac{8C_2^2 cos\tau}{C_1(1+sin\tau)^3}\}, \vec{r}_{\tau\tau} = \{0, -\frac{1}{2C_2}cos\tau, -\frac{1}{2C_2}sin\tau, \frac{8C_2^2 t(3-2sin\tau)}{C_1(1+sin\tau)^3}\}$$

Also we have

$$b_{11} = b_{12} = 0, b_{22} = -\frac{1}{2C_2}, c_{11} = 0, c_{12} = -\frac{4C_2 cos\tau}{C_1(1+sin\tau)^3}, c_{22} = \frac{4C_2 t(3-2sin\tau)}{C_1(1+sin\tau)^3} \tag{2.9}$$

$$g_{11} = 1 + \frac{16C_2^4}{C_1^2(1+sin\tau)^4}, g_{12} = -\frac{32C_2^4 t cos\tau}{C_1^2(1+sin\tau)^5}, g_{22} = \frac{1}{4C_2^2} + \frac{64C_2^4 t^2 cos^2\tau}{C_1^2(1+sin\tau)^6} \tag{2.10}$$

As result for the normal curvature of the surface at the point (t, τ) in the direction $\xi = \{\xi_1, \xi_2\}$ we have

$$k_N(x,\xi) = \frac{\left(\frac{\xi_2^4}{4C_2^2} + \left(\frac{4C_2(t(3-2sin\tau)\xi_2^2 - 2cos\tau\xi_1\xi_2)}{C_1(1+sin\tau)^3}\right)^2\right)^{\frac{1}{2}}}{(1+\frac{16C_2^4}{C_1^2(1+sin\tau)^4})\xi_1^2 + \frac{64C_2^4 t cos\tau}{C_1(1+sin\tau)^5}\xi_1\xi_2 + (\frac{1}{4C_2^2} + \frac{64C_2^4 t^2 cos^2\tau}{C_1^2(1+sin\tau)^6})\xi_2^2} \tag{2.11}$$

Let us recall the notion of Gaussian torsion of two dimensional surface in four dimensional Euclidian space. The Gaussian torsion χ_G is an invariant of the extrinsic geometry of the surface. If a and b are the semiaxes of an ellipse of normal curvature, then $\chi_G = \pm 2ab$, where the sign is taken to be plus in the case when under a rotation of the tangent vector in the positive direction the corresponding point on the ellipse moves in the positive direction in accordance with the orientation in the normal plane, and minus if this point moves in the negative direction.In order to find Gaussian torsion we will use formulas from the paper [2].

Let us denote by **h** the vector of the dimension 10 with components

$$h^{11}, h^{12}, h^{13}, h^{14}, h^{22}, h^{23}, h^{24}, h^{33}, h^{34}, h^{44} \tag{2.12}$$

which are calculated by following formulas

$$h^{ir} = \delta_{ir} - \frac{1}{\Delta}\langle \eta_i, \eta_r \rangle.$$

where δ_{ir} is Kronecker symbol, bracket $\langle\cdot,\cdot\rangle$ is inner product and

$$\Delta = |gradF^1|^2 \cdot |gradF^2|^2 - \langle gradF^1, gradF^2\rangle^2.$$

Note that Δ is the length of bivector

$$[gradF^1|, |gradF^2]$$

and $\Delta > 0$ due to the regularity of the surface F^C. Vectors η_i are defined by following formulas

$$\eta_i = (F_i^1 gradF^2 - F_i^2 gradF^1),$$

where $i = 1, 2, 3, 4$.

We also use notation for partial derivatives

$$\frac{\partial F^i}{\partial x_k} = F_k^i.$$

and also use renumbering $x_1 = t$, $x_2 = x_1$, $x_3 = x_2$, $x_4 = u$.

We also need (6×1) matrix $\mathbf{q}$ with components q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}, where

$$q_{ij} = \varepsilon^{ijkl}\frac{1}{\sqrt{\Delta}}\begin{vmatrix} F_k^1 & F_l^1 \\ F_k^2 & F_l^2 \end{vmatrix}$$

where ε^{ijkl} is Kronecker symbol.

We also introduce (10×6) matrix $\mathbf{B}$

$$\mathbf{B} = \frac{1}{\sqrt{\Delta}}\begin{pmatrix} (1112) & (1113) & (1114) & (1123) & (1124) & (1134) \\ (1212) & (1213) & (1214) & (1223) & (1224) & (1234) \\ (1312) & (1313) & (1314) & (1323) & (1324) & (1334) \\ (1412) & (1413) & (1414) & (1423) & (1424) & (1434) \\ (2212) & (2213) & (2214) & (2223) & (2224) & (2234) \\ (2312) & (2313) & (2314) & (2323) & (2324) & (2334) \\ (2412) & (2413) & (2414) & (2423) & (2424) & (2434) \\ (3312) & (3313) & (3314) & (3323) & (3324) & (3334) \\ (3412) & (3413) & (3414) & (3423) & (3424) & (3434) \\ (4412) & (4413) & (4414) & (4423) & (4424) & (4434) \end{pmatrix}$$

with elements

$$(ijkl) = \begin{vmatrix} F_{ik}^1 & F_{il}^1 \\ F_{jk}^2 & F_{jl}^2 \end{vmatrix} + \begin{vmatrix} F_{jk}^1 & F_{jl}^1 \\ F_{ik}^2 & F_{il}^2 \end{vmatrix}.$$

Now we ready to write the formula for the Gauss torsion $\chi_{\mathbf{G}}$

$$\chi_{\mathbf{G}} = \langle \mathbf{h}, \mathbf{B}\mathbf{q}\rangle.$$

Now, using the above formulas, let's move on to calculating torsion.

$$gradF^1 = \{\frac{1}{x_2^2 u}, 0, -\frac{2t}{x_2^3 u}, -\frac{t}{x_2^2 u^2}\}, gradF^2 = \{0, -\frac{2x_1x_2}{(x_1^2+x_2^2)^2}, \frac{x_1^2 - x_2^2}{(x_1^2+x_2^2)^2}, 0\}$$

$$\Delta = \frac{(u^2+t^2)(x_1^2+x_2^2)^2 + 16t^2u^2x_1^2}{x_2^4 u^4 (x_1^2+x_2^2)^4}$$

$$h^{11}=1-\frac{1}{\Delta x_2^4u^2(x_1^2+x_2^2)^2},\qquad h^{12}=\frac{4tx_1(x_1^2-x_2^2)}{\Delta x_2^4u^2(x_1^2+x_2^2)^4},\qquad h^{13}=\frac{8tx_1}{\Delta x_2^3u^2(x_1^2+x_2^2)^4}$$

$$h^{14}=\frac{t(x_1^2-x_2^2)^2-4x_1^2x_2^2}{\Delta x_2^4u^3(x_1^2+x_2^2)^4},\quad h^{22}=1-\frac{4x_1^2(u^2(x_2^2+4t^2)+t^2x_2^2)}{\Delta x_2^4u^4(x_1^2+x_2^2)^4},\quad h^{23}=\frac{2x_1x_2(x_1^2-x_2^2)(u^2+t^2)}{\Delta x_2^4u^4(x_1^2+x_2^2)^4}$$

$$h^{24}=\frac{4t^2x_1(x_2^2-x_1^2)}{\Delta x_2^4u^3(x_1^2+x_2^2)^4},\qquad h^{33}=1-\frac{(x_1^2-x_2^2)^2(u^2+t^2)+16t^2x_1^2u^2}{\Delta x_2^4u^4(x_1^2+x_2^2)^4},\qquad h^{34}=-\frac{8t^2x_1^2}{\Delta x_2^3u^3(x_1^2+x_2^2)^4},$$

$$h^{44}=\frac{t^2}{\Delta x_2^4u^4(x_1^2+x_2^2)^4}.$$

For the matrix B we have

$$(1223)=\frac{4(3x_1^2-x_2^2)}{x_2^2u(x_1^2+x_2^2)^3},\quad (1224)=\frac{2(3x_1^2-x_2^2)}{x_2u^2(x_1^2+x_2^2)^3},\quad (1234)=\frac{2x_1(3x_2^2-x_1^2)}{x_2^2u^2(x_1^2+x_2^2)^3},$$

$$(1323)=\frac{4x_1(3x_2^2-x_1^2)}{x_2^3u(x_1^2+x_2^2)^3},\quad (1324)=\frac{2x_1(3x_2^2-x_1^2)}{x_2^2u^2(x_1^2+x_2^2)^3},\quad (1334)=\frac{2(x_2^2-3x_1^2)}{x_2u^2(x_1^2+x_2^2)^3},$$

$$(2312)=\frac{4(x_2^2-3x_1^2)}{x_2^2u(x_1^2+x_2^2)^3},\quad (2313)=\frac{4x_1(x_1^2-3x_2^2)}{x_2^3u(x_1^2+x_2^2)^3},\quad (2323)=\frac{12t(x_2^2-3x_1^2)}{x_2^3u(x_1^2+x_2^2)^3},$$

$$(2324)=\frac{4t(x_2^2-3x_1^2)}{x_2^2u^2(x_1^2+x_2^2)^3},\quad (2334)=\frac{4tx_1(x_1^2-3x_2^2)}{x_2^3u^2(x_1^2+x_2^2)^3},\quad (2412)=\frac{2(x_2^2-3x_1^2)}{x_2^2u^2(x_1^2+x_2^2)^3},$$

$$(2413)=\frac{2x_1(x_1^2-3x_2^2)}{x_2u^2(x_1^2+x_2^2)^3},\quad (2423)=\frac{4t(x_2^2-3x_1^2)}{x_2^2u^2(x_1^2+x_2^2)^3},\quad (2424)=\frac{2t(x_2^2-3x_1^2)}{x_2^2u^3(x_1^2+x_2^2)^3},$$

$$(2434)=\frac{4tx_1(x_1^2-3x_2^2)}{x_2^2u^3(x_1^2+x_2^2)^3},\quad (3312)=\frac{8x_1(x_1^2-3x_2^2)}{x_2^3u(x_1^2+x_2^2)^3},\quad (3313)=\frac{8x_2(3x_1^2-x_2^2)}{x_2^2u(x_1^2+x_2^2)^3},$$

$$(3323)=\frac{24x_1t(x_1^2-3x_2^2)}{x_2^4u(x_1^2+x_2^2)^3},\quad (3324)=\frac{4x_1t(x_1^2-3x_2^2)}{x_2^3u^2(x_1^2+x_2^2)^3},\quad (3334)=\frac{8t(3x_1^2-x_2^2)}{x_2^2u^2(x_1^2+x_2^2)^3},$$

$$(3412)=\frac{2x_1(x_1^2-3x_2^2)}{x_2^2u^2(x_1^2+x_2^2)^3},\quad (3413)=\frac{2(3x_1^2-x_2^2)}{x_2u^2(x_1^2+x_2^2)^3},\quad (3423)=\frac{4tx_1(x_1^2-3x_2^2)}{x_2^3u^2(x_1^2+x_2^2)^3},$$

$$(3424)=\frac{4tx_1(x_1^2-3x_2^2)}{x_2^2u^3(x_1^2+x_2^2)^3},\quad (3434)=\frac{4t(3x_1^2-x_2^2)}{x_2u^3(x_1^2+x_2^2)^3}.$$

$$\mathbf{B}=\frac{1}{\sqrt{\Delta}}\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&(1223)&(1224)&(1234)\\ 0&0&0&(1323)&(1324)&(1334)\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ (2312)&(2313)&0&(2323)&(2324)&(2334)\\ (2412)&(2413)&0&(2423)&(2424)&(2434)\\ (3312)&(3313)&0&(3323)&(3324)&(3334)\\ (3412)&(3413)&0&(3423)&(3424)&(3434)\\ 0&0&0&0&0&0 \end{pmatrix}$$

For the vector q

$$\mathbf{q}=\frac{2}{\sqrt{\Delta}x_2u(x_1^2+x_2^2)^2}\begin{pmatrix} \dfrac{t(x_1^2-x_2^2)}{x_2u}\\ \dfrac{2x_1t}{u}\\ -\dfrac{4x_1t}{x_2}\\ 0\\ \dfrac{x_2^2-x_1^2}{x_2}\\ -2x_1 \end{pmatrix}$$

It follows for the Gauss torsion we have following

$$\chi_{\mathbf{G}} = h^{23}((2312)q_1 + (2313)q_2 + (2324)q_5 + (2334)q_6) + \\ + h^{24}((2412)q_1 + (2413)q_2 + (2424)q_5 + (2434)q_6)) + h^{33}((2312)q_1 + (3313)q_2 + (3324)q_5 + (3334)q_6) + \\ + h^{24}((3412)q_1 + (3413)q_2 + (3424)q_5 + (3434)q_6) + h^{12}((1224)q_5 + (1234)q_6).$$

This formula shows at regular points Gauss torsion is not equal zero. It is known that in this case the orbit is a not hyperplane surface i.e. it is not contained in a hyperplane [2, 4]. □

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On a linear inverse problem with nonlocal boundary conditions of periodic type for a three-dimensional equation of mixed type of the second kind of the fourth order in a parallelepiped

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Abstract. This article discusses the correctness of one linear inverse problem with nonlocal boundary conditions of periodic type for a three-dimensional equation of mixed type of the second kind of the fourth order in a parallelepiped. For this problem, the existence and uniqueness theorems for a generalized solution to one linear inverse problem with nonlocal boundary conditions of periodic type in a certain class of integrals functions are proved using the methods of Fourier, "ε-regularization", a priori estimates, approximating sequences and contracting mappings.

Keywords: three-dimensional equation of mixed type of the second kind of the fourth order, linear inverse problem with nonlocal boundary conditions of periodic type, correctness of the problem, Fourier method, methods of " $\varepsilon-$regularization", a priori estimates, sequence of approximations and Fourier transforms.

MSC (2020): 35R30, 35M10, 35A01, 35A02.

1. Introduction

In the process of studying nonlocal problems, a close relationship was revealed between problems with nonlocal boundary conditions and inverse problems. By now, inverse problems for classical equations such as parabolic, elliptic and hyperbolic types of the second order have been studied quite well [1], [17], [24], [25]. Linear inverse problems for equations of mixed type of the second order in the plane have been studied in the works of A.G. Megrabov , K.B. Sabitov and their students [21], [23].

For three-dimensional equations of mixed type of both first and second kind of the second order, they were studied in the works of S.Z. Dzhamalov, and for multidimensional equations of mixed type of the second order of both first and second kind with local and non-local conditions in bounded domains, they were studied and developed in the works of S.Z. Dzhamalov, R.R. Ashurov and S.Z. Dzhamalov, S.G. Pyatkov and S.Z. Dzhamalov, R.R. Ashurov, A.I.Kozhanov [11], [12], [13], [14],[28]. For high-order mixed-type equations, such problems have not been studied in practice. We will try to partially fill this gap in this paper. In this paper, for the study of unique solvability of inverse problems for a three-dimensional equation of mixed type of the second kind, fourth order in a rectangle, a new method is proposed. Which is based on reducing the inverse problem to direct non-local boundary value problems of periodic type for a family of loaded differential equations of mixed type of the second kind, fourth order .

Let us recall that a loaded equation is usually called an equation with partial derivatives that contains in the coefficients or on the right-hand side the values of certain functionals from the solution of the equation [20], [25]. In the domain

$$G = (0,1) \times (0,T) \times (0,\ell) = Q \times (0,\ell) =$$

$$= \{(x,t,y) : 0 < x < 1; 0 < t < T < +\infty; 0 < y < \ell\}.$$

Let's consider equations of mixed type of the second kind of the fourth order.

$$Lu = Pu + Mu + Nu = \psi(x,t,y)\,, \tag{1.1}$$

where

$$Pu = \sum_{i=0}^{4} K_i(x,t) D_t^i u, \quad Mu = au_{xxxx} - bu_{xxtt} - cu_{xx}, \quad Nu = u_{yyyy}.$$

Here, $K_4(x,t) = K_4(t)$, $K_4(0) = K_4(T) = 0$; a, b, c are constants greater than 0, and $D_t^i u = \frac{\partial^i u}{\partial t^i}$, where $i = 0, 1, 2, 3, 4$, and $D_t^0 u = u$.
Equation (1.1) refers to mixed-type equations of the second kind, since no restrictions are imposed on the sign of the function $K_4(t)$ with respect to the variable t inside the segment $[0, T]$ [5], [6], [7], [11].
In the future, we will assume that $\psi(x,t,y) = g(x,t,y) + h(x,t) \cdot f(x,t,y)$, where $g(x,t,y)$ and $f(x,t,y)$ are given functions, and the function $h(x,t)$ is subject to definition.
Linear inverse problem. Find functions $\{u(x,t,y), h(x,t)\}$, satisfying equation (1.1) in the domain G, such that the function $u(x,t,y)$ satisfies the following nonlocal boundary conditions of periodic type:

$$\gamma D_t^q \left. u\right|_{t=0} = D_t^q \left. u\right|_{t=T}; \quad q = 0, 1, 2 \tag{1.2}$$

$$D_x^p \left. u\right|_{x=0} = D_x^p \left. u\right|_{x=1}; \quad p = 0, 1, 2, 3 \tag{1.3}$$

$$D_y^p \left. u\right|_{y=0} = D_x^p \left. u\right|_{y=\ell}; \quad p = 0, 1, 2, 3 \tag{1.4}$$

and an additional condition.

$$u(x,t,\ell_0) = \varphi_0(x,t), \quad \text{where} \quad 0 < \ell_0 < \ell < +\infty \tag{1.5}$$

and together with the functions $h(x,t)$ belongs to the class

$$U = \left\{ (u,h) \mid u \in H_2^{4,3}(G),\ h \in W_2^4(Q) \right\}.$$

Here, $H_2^{4,3}(G)$ denotes the anisotropic Sobolev space with the norm

$$|||u|||^2_{H_2^{4,3}(G)} = \sum_{k=1}^{\infty} (1 + \mu_k^4)^3 \left\| u_k \right\|^2_{W_2^4(Q)}, \quad \text{(A)}.$$

where $W_2^4(Q)$ denotes the Sobolev space, and $u_k(x,t)$ for $k = 1, 2, 3, \ldots$ represents the Fourier coefficients of the function $u(x,t,y)$ with respect to the eigenfunctions $Y_k(y)$ and the eigenvalues μ_k, ($\mu_1 < \mu_2 < \mu_3 < \ldots$) of the fourth-order spectral problem with periodic boundary conditions, given by:

$$Y_k^{(IV)} = \mu_k^4 Y_k,$$
$$D_y^p Y_k\big|_{y=0} = D_y^p Y_k\big|_{y=\ell}, \quad p = 0, 1, 2, 3.$$

It is known that the system of eigenfunctions $\{Y_k(y)\}$ forms an orthonormal basis in the space $L_2(0, \ell)$, [2], [16], [26].

Definition 1.1. A regular solution of problem (1.1)-(1.5) will be called a function $u(x,t,y) \in U$, satisfying equation (1.1) almost everywhere in the domain G and with boundary conditions (1.2)-(1.5).

We will prove the unique solvability of problem (1.1)-(1.5) using the Fourier method, i.e. to find the solution to problem (1.1)-(1.5), we use the Fourier method for the variable y, for problem (1.1)-(1.5). Namely, we seek the solution to problem (1.1)-(1.5) in the form

$$u(x,t,y) = \sum_{k=1}^{\infty} u_k(x,t) Y_k(y)$$

where the functions $Y_k(y)$ form an orthonormal basis in the space $L_2[0,\ell]$, as defined earlier [2],[26] and the functions $u_k(x,t)$, $k=1,2,3,...$ are to be determined.

We will require the following conditions to be met .

Let all the coefficients of equation (1.1) be sufficiently smooth functions in the domain $\overline{Q}$, and let the following conditions be satisfied regarding the coefficients, the right-hand side, and the given function $\varphi_0(x,t)$;

Condition 1: ***Non-local condition***

$$K_{4t}(0)=K_{4t}(T);\ K_i(x,0)=K_i(x,T);\quad i=0,2,3,\quad \forall x\in[0,1],$$

$$\gamma g(x,0,z)=g(x,T,z),\quad \gamma f(x,0,z)=f(x,T,z).$$

$$K_1(x,t)>0,\quad K_3(x,t)>0$$

are sufficiently large functions. In addition, let the following conditions be met for the coefficients of equation (1.1):

$$-(2K_3+(2j-3)K_{4t}+3\lambda K_4)\geq\delta_3>0,\quad j=0,1,2;$$

$$2K_1-K_{2t}+\lambda K_2\geq\delta_2>0,\quad \lambda K_0-K_{0t}\geq\delta_1>0,$$

for any $(x,t)\in\overline{Q}$, where $\lambda=\frac{2}{T}\ln|\gamma|>0,\quad |\gamma|>1.$

***Smoothness*:**

$$g(x,t,l_0)=g_0(x,t)\in C^{0,1}_{x,t}(\bar{Q}),\quad f(x,t,l_0)=f_0(x,t)\in C^{0,1}_{x,t}(\bar{Q}),$$

$$|f_0(x,t)|\geq\eta>0,\quad f\in H_2^{3,3}(G),\quad g\in H_2^{2,3}(G).$$

Condition 2:

$$\varphi_0(x,t)\in W_2^5(Q);\quad \gamma D_t^q\,\varphi_0|_{t=0}=D_t^q\varphi_0|_{t=T},\quad q=0,1,2,3,4;$$

$$D_x^p\,\varphi_0|_{x=0}=D_x^p\varphi_0|_{x=1},\quad p=0,1,2,3.$$

To further investigate the inverse problem, we need the following notations and embedding theorems.

Theorem 1.2. *(Sobolev embeddings). There is a continuous embedding*

$$W_2^{\alpha+2}(Q)\subset C^\alpha(\bar{Q}),$$

where $\alpha=0,1,2,...,$ *i.e.*

$$\|\vartheta\|^2_{C^\alpha(\bar{Q})}\leq c_{\alpha+2}\,\|\vartheta\|^2_{W_2^{\alpha+2}(\bar{Q})},$$

where $c_{\alpha+2}$ *are positive constants* [16], [23].

Let us denote the constant number by

$$c_1=\sum_{k=1}^{\infty}\frac{\mu_k^4}{(1+\mu_k^2)^3}.$$

When obtaining various a priori estimates, we often use the Cauchy inequality with σ [16]:

$$\forall u,\vartheta\geq0;\quad \forall\sigma>0;\quad 2u\cdot\vartheta\leq\sigma u^2+\sigma^{-1}\vartheta^2$$

Now, in order to formulate the main result, it is necessary to perform some construction formalities. Let us consider the traces of equation (1.1) for $y = \ell_0$:

$$Lu|_{y=\ell_0} = P\varphi_0 + M\varphi_0 + \sum_{k=1}^{\infty} \mu_k^4 u_k Y_k(\ell_0) = g_0(x,t) + h(x,t) f_0(x,t)$$

Now, taking into account the additional condition (1.5) and the fact that $f_0 \neq 0$, we define the unknown function $h(x,t)$ in the form of an integral:

$$h(x,t) = \frac{1}{f_0(x,t)} \left[\Phi_0 + \sum_{k=1}^{\infty} \mu_k^4 u_k(x,t) Y_k(\ell_0) \right]$$

Where

$$\Phi_0 = L_0\varphi_0 - g_0, \quad g(x,t,\ell_0) = g_0(x,t),$$

$$L_0\varphi_0 = P\varphi_0 + M\varphi_0,$$

$$P\varphi_0 = \sum_{i=0}^{4} k_i(x,t) D_t^i \varphi_{i0},$$

$$M\varphi_0 = aD_x^4\varphi_0 - bD_{x,t}^{2,2}\varphi_0 - cD_x^2\varphi_0$$

and to determine the functions $u_k(x,t)$ in the domain $Q = (0,1) \times (0,T)$ we obtain an infinite number of loaded systems of fourth-order mixed-type equations:

$$\begin{aligned} \Im u_k = Pu_k + Mu_k + \mu_k^4 u_k = g_k(x,t) + \\ + \frac{f_k\,(x,t)}{f_0(x,t)} \left[\Phi_0 + \sum_{m=1}^{\infty} \mu_m^4 u_m(x,t) Y_m(\ell_0) \right] = F(u_k), \end{aligned} \tag{1.6}$$

with nonlocal boundary conditions of periodic type:

$$\gamma D_t^q\, u_k|_{t=0} = D_t^q u_k|_{t=T}\,; \quad q = 0,1,2, \tag{1.7}$$

$$D_x^p u_k|_{x=0} = D_x^p\, u_k|_{x=1}\,, \quad p = 0,1,2,3. \tag{1.8}$$

The main result is

Theorem 1.3. *Let the above listed conditions 1 and 2 be satisfied for the coefficients of equation* (1.1)*, and let* $q = M \, |||f|||^2_{H_2^{3,3}(G)} < 1$*, where* $M = 124\,\lambda^4 m\ \delta_0^{-2} e^{2\lambda T} \eta^{-2} \|f_0\|^2_{C^{0,1}_{x,t}(\bar{Q})}$, $\delta_0 = \min\{\delta_3, \lambda\{a,b,c\}, \delta_2, \delta_1\}$*, where* $m = 10c_1c_2c_3$ *is the value defined above. Then there exists a unique solution to the linear inverse problem* (1.1)-(1.5) *from the specified class U.*

Proof. We will prove Theorem 1.3 using the following scheme:

(1) Let us show that the solution to problem (1.1)-(1.4), $u(x,t,z) \in U$, satisfies the additional condition (1.5).

(2) To prove the unique solvability of problem (1.6)-(1.8), we first investigate the unique solvability of the auxiliary problem, that is, we analyze the solvability of the family of nonlinear loaded differential equations of the fifth order with a small parameter (2.1)-(2.3).

(3) Then, using this auxiliary problem (2.1)-(2.3), we study the unique solvability of the family of semi-nonlinear loaded equations of mixed type of the second kind, fourth order (1.6)-(1.8).

(4) Using the unique solvability of problem (1.6)-(1.8), we will show the unique solvability of the linear inverse problem (1.1)-(1.5).

Now, let's start implementing this scheme. Let there be a solution to the problem from (1.1)–(1.4). First, we show that the function $u(x,t,z)$ satisfies the additional condition (1.5), i.e., $u(x,t,\ell_0)=\varphi_0(x,t)$. Let us assume the opposite. Let $u(x,t,\ell_0)=\sum\limits_{k=1}^{\infty}u_k(x,t)Y_k(\ell_0)=\vartheta(x,t)\neq\varphi_0(x,t)$, and consider a new function $z(x,t)=\vartheta(x,t)-\varphi_0(x,t)$ in the domain Q. Then, for the function $z(x,t)=u(x,t,\ell_0)-\varphi_0(x,t)=\sum\limits_{k=1}^{\infty}u_k(x,t)Y_k(\ell_0)-\varphi_0(x,t)$ in the domain Q from (1.6)–(1.8), multiplying (1.6)–(1.8) by $Y_k(\ell_0)$ and summing over k from 1 to ∞, we obtain the following identity:

$$\sum_{k=1}^{\infty}\Im(u_kY_k(\ell_0))=P\left(\sum_{k=1}^{\infty}u_kY_k(\ell_0)\right)+M\left(\sum_{k=1}^{\infty}u_kY_k(\ell_0)\right)+\sum_{k=1}^{\infty}\mu_k^4u_kY_k(\ell_0) \tag{1.9}$$

$$=\sum_{k=1}^{\infty}g_kY_k(\ell_0)+\frac{\sum\limits_{k=1}^{\infty}f_kY_k(\ell_0)}{f_0(x,t)}\left[\Phi_0+\sum_{m=1}^{\infty}\mu_m^4u_mY_m(\ell_0)\right]$$

□

From here, taking into account the boundary conditions (1.3) and (1.4), we obtain the following problem:

$$L_0\,z=\sum_{i=0}^{4}K_i\,D_t^i\,z+M\,z=0. \tag{1.10}$$

with nonlocal boundary conditions of the periodic type:

$$\gamma D_t^q\,z|_{t=0}=D_t^q\,z|_{t=T}\,,\quad q=0,1,2, \tag{1.11}$$

$$D_x^pz|_{x=0}=D_x^p\,z|_{x=1}\,,\quad p=0,1,2,3. \tag{1.12}$$

Now we will prove the uniqueness of the solution to the problem (1.10)-(1.12). To do this, consider the identity

$$2(L_0z,e^{-\lambda t}z_t)_0=0.$$

Integrating the identity by parts, taking into account the conditions of Theorem 1.3 and the boundary conditions (1.11) and (1.12), we obtain the inequality

$$\|z\|_2\leq 0,$$

from which it follows that $z(x,t)=0$. This shows that the problem (1.10)-(1.12) has a unique solution, which implies that the solution to the problem (1.1)-(1.4) satisfies the additional condition (1.5), i.e.

$$u(x,t,\ell_0)=\varphi_0(x,t).$$

2. A Family of Loaded Differential Equations of the Fifth Order with a Small Parameter.

We will prove the solvability of problem (1.6)–(1.8) using the methods of "ε-regularization", successive approximations, and a priori estimates. Specifically, we consider the domain $Q = (0,1)\times(0,T)$ for a family of loaded differential equations of the fifth order with a small parameter [5], [6], [7], [11].

$$\begin{aligned} \Im_\varepsilon u_k &= -\varepsilon \tfrac{\partial \Delta^2 u_{k,\varepsilon}}{\partial t} + Pu_{k,\varepsilon} + Mu_{k,\varepsilon} + \mu_k^4 u_{k,\varepsilon} \\ &= g_k(x,t) + \tfrac{f_k(x,t)}{f_0(x,t)}\left[\Phi_0 + \sum_{m=1}^{\infty} \mu_m^4 u_{m,\varepsilon}(x,t) Y_m(\ell_0)\right] = F(u_{k,\varepsilon}), \end{aligned} \tag{2.1}$$

with nonlocal boundary conditions of periodic type:

$$\gamma D_t^q\, u_{k,\varepsilon}\big|_{t=0} = D_t^q u_{k,\varepsilon}\big|_{t=T}, \quad q = 0,1,2,3,4, \tag{2.2}$$

$$D_x^p u_{k,\varepsilon}\big|_{x=0} = D_x^p\, u_{k,\varepsilon}\big|_{x=1}, \quad p = 0,1,2,3. \tag{2.3}$$

where ε is a small positive number. In the future, when proving Theorem 1.3 and the correctness of problem (2.1)–(2.3), we will need the following notations and auxiliary lemmas. Let us define the spaces of vector functions $W_l(Q) = \{\vartheta_k : \vartheta_k \in W_2^l(Q)\}$, where $l = 0,1,2,3,4,5$ and $k = 1,2,3,\ldots$ with a finite norm. The norm is defined as:

$$\langle \vartheta_k \rangle_l^2 \equiv \sum_{k=1}^{\infty} (1+\mu_k^4)^3 \, \|\vartheta_k\|^2_{W_2^l(Q)}\,; i = 0,1,2,3,4,5 \tag{C}$$

It is obvious that the space $W_l(Q)$, with the given norm (C), is a Banach space. From the definition of the spaces $W_2^l(Q)$ for $l = 0,1,2,3,4,5$, it follows that

$$W_5(Q) \subset W_4(Q) \subset W_3(Q) \subset W_2(Q) \subset W_1(Q) \subset W_0(Q).$$

Below, we will denote by $W(Q)$ the class of vector functions $\{\vartheta_k(x,t)\}_{k=1}^{\infty}$ such that $\{\vartheta_k(x,t)\}_{k=1}^{\infty} \in W_4(Q)$, and $\left\{\frac{\partial}{\partial t}\Delta^2 \vartheta_k\right\}_{k=1}^{\infty} \in W_0(Q)$, satisfying the corresponding conditions (2.2) and (2.3).

Definition 2.1. The solution of problem (2.1)-(2.3) will be called a vector-function $\{u_{s,\varepsilon}(x,t)\} \in W(Q)$ satisfying equation (2.1) almost everywhere in the domain Q.

Now we will prove the solvability of problem (2.1)-(2.3) in the domain Q using the methods of successive approximations

$$\begin{aligned} \Im_\varepsilon u_{k,\varepsilon}^{(l)} &= -\varepsilon \tfrac{\partial \Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t} + Pu_{k,\varepsilon}^{(l)} + Mu_{k,\varepsilon}^{(l)} + \mu_k^4 u_{k,\varepsilon}^{(l)} \\ &= g_k(x,t) + \tfrac{f_k(x,t)}{f_0(x,t)}\left[\Phi_0 + \sum_{m=1}^{\infty} \mu_m^4 u_{m,\varepsilon}^{(l-1)}(x,t) Y_m(\ell_0)\right] = F(u_{k,\varepsilon}^{(l-1)}), \end{aligned} \tag{2.4}$$

with nonlocal boundary conditions of periodic type:

$$\gamma D_t^q\, u_{k,\varepsilon}^{(l)}\Big|_{t=0} = D_t^q u_{k,\varepsilon}^{(l)}\Big|_{t=T}, \quad q = 0,1,2,3,4, \tag{2.5}$$

$$D_x^p\, u_{k,\varepsilon}^{(l)}\Big|_{x=0} = D_x^p u_{k,\varepsilon}^{(l)}\Big|_{x=1}, \quad p = 0,1,2,3. \tag{2.6}$$

Lemma 2.2. *Let all the conditions of Theorem 1.3 be satisfied, then the following estimates are valid for solving problem* (2.4)-(2.6)*:*

$$I).\quad \tfrac{\varepsilon}{\delta_0}(\left\langle u_{k,\varepsilon ttt}^{(l)}\right\rangle_0^2+\left\langle u_{k,\varepsilon ttx}^{(l)}\right\rangle_0^2+\left\langle u_{k,\varepsilon ttt}^{(l)}\right\rangle_0^2)+\left\langle u_{k,\varepsilon}^{(l)}\right\rangle_2^2\le A_1=$$
$$=248\delta_0^{-2}e^{2\lambda T}\left[\langle g_k\rangle_0^2+2\eta^{-2}c_2\langle f_k\rangle_2^2\left(T_0^2\|\varphi_0\|_{W_2^4(Q)}^2+\|g_0\|_0^2\right)\right]$$

$$II).\quad \tfrac{\varepsilon}{\delta_0}\left\langle\frac{\partial\Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t}\right\rangle_0^2+\left\langle u_{k,\varepsilon}^{(l)}\right\rangle_4^2\le A_2\equiv$$
$$\equiv 1860\lambda^4\delta_0^{-2}e^{2\lambda T}\cdot[\langle g_k\rangle_1^2+2\eta^{-2}c_3\|f_0\|_{C_{x,t}^{0,1}(\bar Q)}^2\langle f_k\rangle_3^2[T_1^2\|\varphi_0\|_5^2+\|g_0\|_1^2]]$$

through $\delta_0,\lambda,\eta,c_2,c_3,T_i(i=0,1)-$ *denoting some positive constant numbers that are defined higher, independent of the parameters* l,ε,k.

Proof of Lemma 2.2. Let us consider the identity

$$\left|-2\left(\Im u_{k,\varepsilon}^{(l)},e^{-\lambda t}u_{k,\varepsilon t}^{(l)}\right)_0\right|=\left|-2\left(F\left(u_{k,\varepsilon}^{(l-1)}\right),e^{-\lambda t}u_{k,\varepsilon t}^{(l)}\right)_0\right| \tag{2.7}$$

where $\lambda>0$, we will choose the constant later. Using the conditions of Theorem 1.3 and the boundary conditions (2.5), (2.6), by integrating by parts identity (2.7) and applying the Cauchy inequalities with σ [12], from identity (2.7) it is easy to obtain the following inequality

$$\begin{aligned}&\left|-2\int_Q e^{-\lambda t}\Im_\varepsilon u_{k,\varepsilon}^{(l)}\cdot u_{k,\varepsilon t}^{(l)}\,dx\,dt\right|\ge\varepsilon\cdot e^{-\lambda T}\left(\left\|u_{k,\varepsilon ttt}^{(l)}\right\|_0\cdot\left\|u_{k,\varepsilon ttx}^{(l)}\right\|_0\cdot\left\|u_{k,\varepsilon txx}^{(l)}\right\|_0\right)+\\&+\int_Q e^{-\lambda t}\Big\{-(2K_3-3K_{4t}+3\lambda K_4)u_{k,\varepsilon tt}^{2(l)}+\lambda a u_{k,\varepsilon xx}^{2(l)}+\lambda b u_{k,\varepsilon xt}^{2(l)}+\lambda c u_{k,\varepsilon x}^{2(l)}+\\&\quad+(2K_1-K_{2t}+\lambda K_2)u_{k,\varepsilon t}^{2(l)}+(\lambda K_0-K_{0t})u_{k,\varepsilon}^{2(l)}+\mu_k^4u_{k,\varepsilon}^{2(l)}\Big\}\,dx\,dt-\\&-2\sigma\left\|u_{k,\varepsilon tt}^{(l)}\right\|_0^2-5\lambda^4K\sigma^{-1}\left\|u_{k,\varepsilon t}^{(l)}\right\|_0^2-\\&-\int_{\partial Q}e^{-\lambda t}\Big\{2K_4u_{k,\varepsilon ttt}^{(l)}u_{k,\varepsilon t}^{(l)}-2(K_{4t}-\lambda K_4)u_{k,\varepsilon tt}^{(l)}u_{k,\varepsilon t}^{(l)}-K_4u_{k,\varepsilon tt}^{2(l)}\Big\}e_t\,ds-\\&-\int_{\partial Q}e^{-\lambda t}\Big\{2au_{k,\varepsilon xxx}^{(l)}u_{k,\varepsilon t}^{(l)}-2au_{k,\varepsilon xx}^{(l)}u_{k,\varepsilon tx}^{(l)}-2bu_{k,\varepsilon xxt}^{(l)}u_{k,\varepsilon t}^{(l)}\Big\}e_x\,ds\end{aligned} \tag{2.8}$$

where, is $\lambda-const>0$, $\overrightarrow{e}=(e_x=(\overrightarrow{e},x);e_t=(\overrightarrow{e},t))-$the unit vector of the inner normal to the boundary ∂Q. The conditions of Theorem 1.3 ensure that the integral over the domain is not negative Q. Taking into account the nonlocal boundary conditions of the periodic (2.5), (2.6) and the conditions of Theorem 1.3 and $\gamma^2=e^{\lambda T}$ we obtain that the boundary integrals vanish. Consequently, from inequality (2.8), choosing the coefficients $\delta_3-2\sigma\ge\delta_{30}>0$, $\delta_2-5\lambda^4\sigma^{-1}K\ge\delta_{20}>0$, where $K=\max\left\{\|K_4\|_{C^2(Q)}^2,\ \|K_3\|_{C^1(Q)}^2\right\}$, we get the following inequality from the bottom left

$$\begin{aligned}&\left|-2\int_Q e^{-\lambda t}\Im_\varepsilon u_{k,\varepsilon}^{(l)}\cdot u_{k,\varepsilon t}^{(l)}\,dx\,dt\right|\ge\varepsilon\cdot e^{-\lambda T}\left(\left\|u_{k,\varepsilon ttt}^{(l)}\right\|_0^2+\left\|u_{k,\varepsilon ttx}^{(l)}\right\|_0^2+\left\|u_{k,\varepsilon txx}^{(l)}\right\|_0^2\right)+\\&+\int_Q e^{-\lambda t}\left(\delta_{30}u_{k,\varepsilon tt}^{2(l)}+\lambda au_{k,\varepsilon xx}^{2(l)}+\lambda bu_{k,\varepsilon xt}^{2(l)}+\lambda cu_{k,\varepsilon x}^{2(l)}+\right.\\&\left.+\delta_{20}u_{k,\varepsilon t}^{2(l)}+\left(\delta_1+\mu_1^2\right)u_{k,\varepsilon}^{2(l)}\right)dx\,dt\end{aligned} \tag{2.9}$$

Now denoting through $\delta_0 = \min\{\delta_{03}, \lambda a, \lambda b, \lambda c, \delta_{02}, \delta_1 + \mu_1^2\}$ we obtain from (2.9), for solving problem (2.4)-(2.6), the first a priori estimate from below

$$\left|-2\int_Q e^{-\lambda t}\Im_\varepsilon u_{k,\varepsilon}^{(l)} u_{k,\varepsilon t}^{(l)}dxdt\right| \geq \varepsilon e^{-\lambda T}\left(\left\|u_{k,\varepsilon ttt}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon ttx}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon txx}^{(l)}\right\|_0^2\right) + \delta_0 e^{-\lambda T}\left\|u_{k,\varepsilon}^{(l)}\right\|_2^2 \tag{2.10}$$

Now, applying the Cauchy inequality to σ identity (2.7), from the top right we obtain

$$\begin{aligned}
&\left|-2\left(F(u_{k,\varepsilon}^{(l-1)}), e^{-\lambda t}u_{k,\varepsilon t}^{(l)}\right)_0\right| \leq \\
&\leq \left|\left(g_k + \frac{f_k(x,t)}{f_0(x,t)}\left[\Phi_0 + \sum_{k=1}^{\infty}\mu_k^4 u_{k,\varepsilon}^{(l-1)}Y_k(\ell_0)\right], e^{-\lambda t}u_{k,\varepsilon t}^{(l)}\right)_0\right| \leq \\
&\leq 3\sigma^{-1}\left\|\hat{u}_{k,\varepsilon}^{(\theta)}\right\|_1^2 + 2\sigma\left[\|g_k\|_0^2 + 2\eta^{-2}\|f_k\|_{C(\bar{Q})}^2\left(T_0^2\|\varphi_0\|_{W_2^4(Q)}^2 + \|g_0\|_0^2\right)\right] + \\
&\quad + 2c_1\sigma\eta^{-2}\|f_k\|_{C(\bar{Q})}^2\sum_{k=1}^{\infty}(1+\mu_k^4)^3\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_1^2
\end{aligned} \tag{2.11}$$

where $T_0 = 2\max\left\{\|K_i(x,t)\|_{C(\bar{Q})} \mid i = 0,1,2,3,4; \lambda\max\{a,b,c\}\right\}$; $c_1 = \sum_{k=1}^{\infty}\frac{\mu_k^8}{(1+\mu_k^4)^3}$. Combining inequalities (2.10) and (2.11), we obtain

$$\begin{aligned}
&\varepsilon\Big(\left\|u_{k,\varepsilon ttt}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon ttx}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon txx}^{(l)}\right\|_0^2\Big) + \left(\delta_0 - 3e^{\lambda T}\sigma^{-1}\right)\left\|\hat{u}_{k,\varepsilon}^{(l)}\right\|_2^2 \leq \\
&\leq 2\sigma e^{\lambda T}\Big[\|g_k\|_0^2 + 2\eta^{-2}\|f_k\|_{C(\bar{Q})}^2\left(T_0^2\|\varphi_0\|_{W_2^4(Q)}^2 + \|g_0\|_0^2\right)\Big] + \\
&+2c_1\sigma\eta^{-2}e^{\lambda T}\|f_k\|_{C(\bar{Q})}^2\sum_{k=1}^{\infty}(1+\mu_k^4)^3\left\|u_{k,\varepsilon}^{(l-1)}\right\|_2^2.
\end{aligned} \tag{2.12}$$

Applying the Sobolev embedding theorem $\|f_k\|_{C(\bar{Q})}^2 \leq c_2\|f_k\|_{W_2^2(\bar{Q})}^2$ [16], [23] to inequality (2.13), we obtain

$$\begin{aligned}
&\varepsilon\left(\left\|u_{k,\varepsilon ttt}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon ttx}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon txx}^{(l)}\right\|_0^2\right) + \left(\delta_0 - 3e^{\lambda T}\sigma^{-1}\right)\left\|\hat{u}_{k,\varepsilon}^{(l)}\right\|_2^2 \leq \\
&\leq 2\sigma e^{\lambda T}\left[\|g_k\|_0^2 + 2\eta^{-2}c_2\|f_k\|_{W_2^2(\bar{Q})}^2\left(T_0^2\|\varphi_0\|_{W_2^4(Q)}^2 + \|g_0\|_0^2\right)\right] + \\
&+2c_1c_2\sigma\eta^{-2}e^{\lambda T}\|f_k\|_{W_2^2(\bar{Q})}^2\sum_{k=1}^{\infty}\left(1+\mu_k^4\right)^3\left\|u_{k,\varepsilon}^{(l-1)}\right\|_2^2.
\end{aligned} \tag{2.13}$$

Taking into account the conditions of the Theorem 1.3 and $\sigma^{-1} = \frac{\delta_0}{62e^{\lambda T}}$, $\delta_0 - 3e^{\lambda T}\sigma^{-1} > \delta_0 - 31e^{\lambda T}\sigma^{-1} = \frac{\delta_0}{2} > 0$, dividing inequality (2.13) by $\delta_0 > 0$, multiplying by $(1+\mu_k^4)^3$, and summing over k from 1 to ∞, we obtain the first recurrence formula

$$\begin{aligned}
&\frac{\varepsilon}{\delta_0}(\left\langle u_{k,\varepsilon ttt}^{(l)}\right\rangle_0^2 + \left\langle u_{k,\varepsilon ttx}^{(l)}\right\rangle_0^2 + \left\langle u_{k,\varepsilon txx}^{(l)}\right\rangle_0^2) + \left\langle u_{k,\varepsilon}^{(l)}\right\rangle_2^2 \leq \\
&+124\delta_0^{-2}e^{2\lambda T}\left[\langle g_k\rangle_0^2 + 2\eta^{-2}c_2\langle f_k\rangle_2^2\left(T_0^2\|\varphi_0\|_{W_2^4(Q)}^2 + \|g_0\|_0^2\right)\right] + \\
&+124c_1c_2\delta_0^{-2}\eta^{-2}e^{2\lambda T}\langle f_k\rangle_2^2\left\langle u_{k,\varepsilon}^{(l-1)}\right\rangle_2^2.
\end{aligned} \tag{2.14}$$

We introduce the notation

$$124\delta_0^{-2}e^{2\lambda T}\left[\langle g_k\rangle_0^2 + 2\eta^{-2}c_2\langle f_k\rangle_2^2\left(T_0^2\|\varphi_0\|_{W_2^4(Q)}^2 + \|g_0\|_0^2\right)\right] = A_1$$

and using conditions of Theorem 1.3 $124c_1c_2\delta_0^{-2}\eta^{-2}e^{2\lambda T}\left\langle f_k\right\rangle_2^2 < q = M\left\langle f_k\right\rangle_3^2 < 1$ from the recurrent formula (2.14), we obtain the validity of the estimate I). For this, we take function $\left\{u_\varepsilon^{(-1)}\right\} \equiv \{0\}$ as an "initial approximation". Hence

$$\frac{\varepsilon}{\delta_0}(\left\langle u_{k,\varepsilon ttt}^{(0)}\right\rangle_0^2 + \left\langle u_{k,\varepsilon ttx}^{(0)}\right\rangle_0^2 + \left\langle u_{k,\varepsilon txx}^{(0)}\right\rangle_0^2) + \left\langle u_{k,\varepsilon}^{(0)}\right\rangle_2^2 \leq$$

$$\leq 124\delta_0^{-2}e^{2\lambda T}\left[\left\langle g_k\right\rangle_0^2 + 2\eta^{-2}c_2\left\langle f_k\right\rangle_2^2\left(T_0^2\left\|\varphi_0\right\|_{W_2^4(Q)}^2 + \left\|g_0\right\|_0^2\right)\right] \equiv A_1$$

Continuing this process, by induction we obtain the first a priori estimate for any function $\left\{u_{k,\varepsilon}^{(l)}\right\}$,
$\forall l \geq 0$

$$\frac{\varepsilon}{\delta_0}(\left\langle u_{k,\varepsilon ttt}^{(l)}\right\rangle_0^2 + \left\langle u_{k,\varepsilon ttx}^{(l)}\right\rangle_0^2 + \left\langle u_{k,\varepsilon txx}^{(l)}\right\rangle_0^2) + \left\langle u_{k,\varepsilon}^{(l)}\right\rangle_2^2 \leq A_1\cdot\sum_{k=0}^{l}q^k < \frac{A_1}{1-q} \tag{2.15}$$

Now we prove the validity of estimate II). To do this, consider the following identity:

$$\left|-2\int_Q \Im_\varepsilon u_{k,\varepsilon}^{(l)}\, e^{-\lambda t}Pu_{k,\varepsilon}^{(l)}\,dxdt\right| = \left|-2\int_Q F\left(u_{k,\varepsilon}^{(l-1)}\right)e^{-\lambda t}Pu_{k,\varepsilon}^{(l)}dxdt\right|, \tag{2.16}$$

where $Pu_{k,\varepsilon}^{(l)} \equiv \frac{\partial\Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t} - 2\lambda\frac{\partial^2\Delta u_{k,\varepsilon}^{(l)}}{\partial t^2} + 3\lambda^2\frac{\partial\Delta u_{k,\varepsilon}^{(l)}}{\partial t} - \frac{\lambda u_{k,\varepsilon tt}^{(l)}}{2} + \frac{\lambda^2 u_{k,\varepsilon}^{(l)}}{16}$.

$$\frac{\partial\Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t} = \frac{\partial}{\partial t}\left(u_{k,\varepsilon tttt}^{(l)} + 2u_{k,\varepsilon ttxx}^{(l)} + u_{k,\varepsilon xxxx}^{(l)}\right)$$

Integrating (2.16) by parts, taking into account the conditions of Theorem 1.3 and boundary conditions (2.4), (2.5) we obtain the following inequality

$$\begin{aligned}
&\left|-2\int_Q \Im_\varepsilon u_{k,\varepsilon}^{(l)}\cdot e^{-\lambda t}Pu_{k,\varepsilon}^{(l)}\,dxdt\right| \geq \varepsilon\left\|\frac{\partial\Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t}\right\|_0^2 + \\
&+\int_Q e^{-\lambda\cdot t}\{-(2K_3 + K_{4t} + 3\lambda K_4)u_{k,\varepsilon tttt}^{2(l)} - \\
&-(2K_3 - K_{4t} + 3\lambda K_4)u_{k,\varepsilon ttxx}^{2(l)} - (2K_3 + K_{4t} + 3\lambda K_4)u_{k,\varepsilon tttx}^{2(l)} + \\
&+\lambda\, a u_{k,\varepsilon xxxx}^{2(l)} + \lambda\, b u_{k,\varepsilon xxtt}^{2(l)} + \lambda\, c u_{k,\varepsilon xxxt}^{2(l)}\}\,dx\,dt + \\
&+\rho\cdot\left\|u_{k,\varepsilon}^{(l)}\right\|_3^2 - N_1\cdot\sigma\cdot(\left\|u_{k,\varepsilon tttt}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon ttxx}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon tttx}^{(l)}\right\|_0^2) - \\
&-N_2\cdot\sigma\cdot(\left\|u_{k,\varepsilon xxxx}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon ttxx}^{(l)}\right\|_0^2 + \left\|u_{k,\varepsilon txxx}^{(l)}\right\|_0^2) - \\
&-c(\sigma^{-1},\lambda,K)\left\|u_{k,\varepsilon}^{(l)}\right\|_3^2 + \int_{\partial Q} e^{-\lambda\cdot t}B\,(u_{k,\varepsilon}^{(l)}(s),K_i(s))\,ds \quad i=\overline{0,4}
\end{aligned} \tag{2.17}$$

where $\rho,\ N_i\ (i=1,2)$ are positive numbers depending on the norm of a function $K_i(x,t)$ for $i=\overline{0,3}$ in the space $C^3(\overline{Q})$,

$$K = \max\left\{\left\|K_4(t)\right\|_{C^3[0,T]}, \left\|K_i(x,t)\right\|_{C^2(\overline{Q})}\right\},$$

$\sigma, c(\sigma^{-1})$ are coefficients of the Cauchy inequality with $\sigma\in[12]$, and $B(u_{k,\varepsilon}^{(l)}(s),K_i(s))$ are functions depending on the traces of the functions $u_{k,\varepsilon}^{(l)}(x,t)$ and $K_i(x,t)$ on the boundary of the domain Q.

Let

$$\delta_{30} = \min\left\{\delta_3, \lambda a, \lambda b, \lambda c, \delta_2, \delta_1 + \mu_1^2\right\},$$

and denote by $N = \max\{N_1, N_2\}$.

Taking into account the condition of Theorem 1.3 and the boundary conditions (2.5), (2.6), and $\gamma^2 = e^{\lambda T}$, we obtain that in (2.17), the boundary integrals will vanish. Now, choosing σ, ρ such that: $\delta_{30} - N\sigma \geq \frac{\delta_{30}}{2} > 0$, $\rho - c(\sigma^{-1}, \lambda, K) \geq \rho_0 > 0$ and $\delta_0 = \min\{\frac{\delta_{30}}{2}, \rho_0\} > 0$, from inequality (2.17), we obtain the required second estimate:

$$\left| -2\int_Q \Im_\varepsilon u_{k,\varepsilon}^{(l)} \cdot P u_{k,\varepsilon}^{(l)}\, dx\, dt \right| \geq \varepsilon e^{-\lambda T} \left\| \frac{\partial \Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t} \right\|_0^2 + \delta_0 e^{-\lambda T} \left\| u_{k,\varepsilon}^{(l)} \right\|_4^2. \tag{2.18}$$

Now, applying the Cauchy inequality σ to the right-hand side of identity (2.16), it is easy to obtain the following inequality on the upper right:

$$\left| -2\int_Q F\left(u_{k,\varepsilon}^{(l-1)}\right) \cdot e^{-\lambda t} P u_{k,\varepsilon}^{(l)} dxdt \right| =$$

$$= \left| -2(g_k + \frac{f_k}{f_0}\,[\Phi_0 + \sum_{k=1}^{\infty} \mu_k^4\, u_{k,\varepsilon}^{(l-1)}\, Y_k(\ell_0)], e^{-\lambda t} P u_{k,\varepsilon}^{(l)} dxdt)_0 \right| \leq$$

$$\leq 31\sigma^{-1} e^{\lambda T} \left\| u_{k,\varepsilon}^{(l)} \right\|_4^2 +$$

$$+15\sigma\lambda^4 e^{\lambda T}\left(\|g_k\|_1^2 + 2\eta^{-2} c_1 \|f_0\|^2_{C^{0,1}_{x,t}(\bar{Q})} \|f_k\|^2_{C^1(\bar{Q})}\left(T_1^2\|\varphi_0\|_5^2 + \|g_0\|_1^2\right)\right) +$$

$$+ 6\sigma\lambda^4\eta^{-2} e^{\lambda T} c_1 \|f_0\|^2_{C^{0,1}_{x,t}(\bar{Q})} \|f_k\|^2_{C^1(\bar{Q})} \sum_{k=1}^{\infty} (1+\mu_k^4)^3 \left\| u_{k,\varepsilon}^{(l-1)} \right\|_4^2, \tag{2.19}$$

where $T_1 = 2\max\left\{\|K_i(x,t)\|_{C^1(\bar{Q})}\, (i = \overline{0,4}); \lambda\{a,b,c\}\right\}$.
Combining inequalities (2.17) and (2.18), we obtain

$$\begin{aligned}
&\varepsilon \left\| \frac{\partial \Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t} \right\|_0^2 + \left(\delta_0 - 31 e^{\lambda T}\sigma^{-1}\right) \left\| u_{k,\varepsilon}^{(l)} \right\|_4^2 \quad \leq \\
&\leq 15\sigma\lambda^4 e^{\lambda T} \cdot [\|g_k\|_1^2 + 2\eta^{-2} \|f_0\|^2_{C^{0,1}_{x,t}(\bar{Q})} \|f_k\|^2_{C^1(\bar{Q})} [T_1^2 \|\varphi_0\|_5^2 + \|g_0\|_1^2]] \\
&+6\sigma\lambda^4\eta^{-2} e^{\lambda T} c_1 \|f_0\|^2_{C^{0,1}_{x,t}(\bar{Q})} \|f_k\|^2_{C^1(\bar{Q})} \sum_{k=1}^{\infty} (1+\mu_k^4)^3 \cdot \left\| u_{k,\varepsilon}^{(l-1)} \right\|_4^2.
\end{aligned} \tag{2.20}$$

Applying Sobolev embedding theorems $\|f_k\|^2_{C^1(\bar{Q})} \leq c_3 \|f_k\|^2_{W_2^3(\bar{Q})}$to inequality (2.19), we obtain

$$\begin{aligned}
&\varepsilon \left\| \frac{\partial \Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t} \right\|_0^2 + \left(\delta_0 - 31 e^{\lambda T}\sigma^{-1}\right) \left\| u_{k,\varepsilon}^{(l)} \right\|_4^2 \quad \leq \\
&\leq 15\sigma\lambda^4 e^{\lambda T} \cdot [\|g_k\|_1^2 + 2\eta^{-2} c_3 \|f_0\|^2_{C^{0,1}_{x,t}(\bar{Q})} \|f_k\|_3^2 [T_1^2 \|\varphi_0\|_5^2 + \|g_0\|_1^2]] \\
&+6\sigma\lambda^4\eta^{-1} e^{\lambda T} c_1 c_3 \|f_0\|^2_{C^{0,1}_{x,t}(\bar{Q})} \|f_k\|_3^2 \sum_{k=1}^{\infty} (1+\mu_k^4)^3 \cdot \left\| u_{k,\varepsilon}^{(l-1)} \right\|_4^2.
\end{aligned} \tag{2.21}$$

According to the conditions $\delta_0 - 31 e^{\lambda T}\sigma^{-1} \geq \frac{\delta_0}{2} > 0$ of Theorem 1.3, dividing the inequality (2.21) by δ, then multiplying by $(1+\mu_k^4)^3$, and summing over k from 1 to ∞, we obtain the second recurrence formula:

$$\frac{\varepsilon}{\delta_0}\left\langle \frac{\partial \Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t}\right\rangle_0^2 + \left\langle u_{k,\varepsilon}^{(l)}\right\rangle_4^2 \quad \leq$$
$$\leq 1860\delta_0^{-2}\lambda^4 e^{2\lambda T}\cdot\left[\langle g_k\rangle_1^2 + 2\eta^{-2}c_3\left\|f_0\right\|^2_{C^{0,1}_{x,t}(\bar{Q})}\langle f_k\rangle_3^2\left[T_1^2\left\|\varphi_0\right\|_5^2+\left\|g_0\right\|_1^2\right]\right]+ \tag{2.22}$$
$$+372\delta_0^{-2}\lambda^4\eta^{-2}e^{2\lambda T}c_1c_3\left\|f_0\right\|^2_{C^{0,1}_{x,t}(\bar{Q})}\langle f_k\rangle_3^2\left\langle u_{k,\varepsilon}^{(l-1)}\right\rangle_4^2.$$

Let us introduce the notation $1860\delta_0^{-2}\lambda^4 e^{2\lambda T}\cdot[\langle g_k\rangle_1^2+2\eta^{-2}c_3\left\|f_0\right\|^2_{C^{0,1}_{x,t}(\bar{Q})}\langle f_k\rangle_3^2[T_1^2\left\|\varphi_0\right\|_5^2+\left\|g_0\right\|_1^2]]\equiv A_2$. Taking into account the conditions of Theorem 1.3,

$$372\delta_0^{-2}\lambda^4\eta^{-2}e^{2\lambda T}c_1c_3\left\|f_0\right\|^2_{C^{0,1}_{x,t}(\bar{Q})}\langle f_k\rangle_3^2 < q = M\langle f_k\rangle_3^2 < 1$$

from the recurrence formula (2.22), we obtain the validity of the second a priori estimate. Indeed, for this, as an “initial approximation” we take the function $\left\{u_{k,\varepsilon}^{(-1)}\right\}\equiv\{0\}$. Then, for the “zero approximation” we have

$$\frac{\varepsilon}{\delta_0}\left\langle \frac{\partial \Delta^2 u_{k,\varepsilon}^{(0)}}{\partial t}\right\rangle_0^2 + \left\langle u_{k,\varepsilon}^{(0)}\right\rangle_2^2 \leq A_2.$$

Continuing this process, we obtain by induction the second a priori estimate for any function $\left\{u_{k,\varepsilon}^{(l)}\right\}, \quad \forall l\geq 0,$

$$\frac{\varepsilon}{\delta_0}\left\langle \frac{\partial \Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t}\right\rangle_0^2 + \left\langle u_{k,\varepsilon}^{(l)}\right\rangle_2^2 \leq A_2\cdot\sum_{k=0}^{l}q^k \leq \frac{A_2}{1-q}.$$

From here, as in the proof of estimate I, estimate II is easily obtained.

Lemma 2.2 is proven.

Let us introduce a new function from the space $W(Q,\mathbb{R})$ by the formula

$$\vartheta_{k,\varepsilon}^{(l)} = u_{k,\varepsilon}^{(l)} - u_{k,\varepsilon}^{(l-1)}, \quad \varepsilon>0, \quad l=0,1,2,3,\ldots$$

Then the following lemma holds.

Lemma 2.3. *Let all the conditions of Theorem 1.3 be satisfied. Then for the function*

$$\left\{\vartheta_{k,\varepsilon}^{(l)}\right\}\in W(Q,\mathbb{R}),$$

the following estimates hold:

$$III)\quad \frac{\varepsilon}{\delta_0}\left(\left\langle u_{k,\varepsilon ttt}^{(l)}\right\rangle_0^2 + \left\langle u_{k,\varepsilon ttx}^{(l)}\right\rangle_0^2 + \left\langle u_{k,\varepsilon ttt}^{(l)}\right\rangle_0^2\right) + \left\langle u_{k,\varepsilon}^{(l)}\right\rangle_2^2 \leq A_1 q^{(l)},$$

$$IV)\quad \frac{\varepsilon}{\delta_0}\left\langle \frac{\partial \Delta^2 u_{k,\varepsilon}^{(l)}}{\partial t}\right\rangle_0^2 + \left\langle u_{k,\varepsilon}^{(l)}\right\rangle_4^2 \leq A_2 q^{(l)}.$$

Proof of Lemma 2.3. From (2.4)-(2.6), for the function vector $\left\{\vartheta_{k,\varepsilon}^{(l)}\right\}\in W(Q,\mathbb{R})$ we obtain the following problem:

$$\Im_\varepsilon \vartheta_{k,\varepsilon}^{(l)} = -\varepsilon \frac{\partial \Delta^2 \vartheta_{k,\varepsilon}^{(l)}}{\partial t} + L_0 \vartheta_{k,\varepsilon}^{(l)} + \mu_k^4 \vartheta_{k,\varepsilon}^{(l)} = \frac{f_k\,(x,t)}{f_0(x,t)} \sum_{k=1}^{\infty} \mu_k^4 \vartheta_{k,\varepsilon}^{(l-1)} Y_k(\ell_0) \equiv F(\vartheta_{k,\varepsilon}^{(l-1)}), \qquad (2.23)$$

with non-local boundary conditions of periodic type:

$$\gamma D_t^q\, \vartheta_{k,\varepsilon}^{(l)}\Big|_{t=0} = D_t^q\, \vartheta_{k,\varepsilon}^{(l)}\Big|_{t=T}, \quad q = 0,1,2,3,4, \qquad (2.24)$$

$$D_x^p \vartheta_{k,\varepsilon}^{(l)}\Big|_{x=0} = D_x^p \vartheta_{k,\varepsilon}^{(l)}\Big|_{x=1}, \quad p = 0,1,2,3. \qquad (2.25)$$

Here, $\varepsilon > 0$, $\quad l = 0,1,2,\ldots$
Therefore, as in the proof of Lemma 2.2,

$$\left\{ \vartheta_{k,\varepsilon}^{(l)} \right\} = \left\{ u_{k,\varepsilon}^{(l)} \right\} - \left\{ u_{k,\varepsilon}^{(l-1)} \right\} \in W(Q,\mathbb{R})$$

we obtain the third recurrence formula for the function:

$$\frac{\varepsilon}{\delta_0} \left(\left\langle \vartheta_{k,\varepsilon ttt}^{(l)} \right\rangle_0^2 + \left\langle \vartheta_{k,\varepsilon ttx}^{(l)} \right\rangle_0^2 + \left\langle \vartheta_{k,\varepsilon ttt}^{(l)} \right\rangle_0^2 \right) + \left\langle \vartheta_{k,\varepsilon}^{(l)} \right\rangle_2^2 \le q \left\langle \vartheta_{k,\varepsilon}^{(l-1)} \right\rangle_2^2. \qquad (2.26)$$

Then, repeating the reasoning of Lemma 2.2, from (2.26) we obtain the a priori estimate III). The proof of IV) follows similarly. **Lemma 2.3 is proved.**

Theorem 2.4. *Let all the conditions of Theorem 1.3 be satisfied. Then problem* (2.4)-(2.6) *is uniquely solvable in* $W(Q,R)$.

Proof. We prove Theorem 2.4 using the method of contraction mappings [9]. Let $\Im_\varepsilon$ be the operator corresponding to the differential expression (2.4) and conditions (2.2), (2.3). We denote by $\Im_\varepsilon^{-1}$ the formal inverse operator. The existence of $\Im_\varepsilon^{-1}$ follows from [28]. We define the operator $W(Q,R)$ in the space:

$$u_{k,\varepsilon}^{(l)} = \Im_\varepsilon^{-1} F(u_{k,\varepsilon}^{(l-1)}) \equiv \Re u_{k,\varepsilon}^{(l-1)}.$$

1. Let us show that the operator $\Re$ maps spaces $W(Q,R)$ into themselves.
Let $\left\{ u_{k,\varepsilon}^{(l-1)} \right\} \in W(Q,R)$, then for the solution of problem (2.4)-(2.6), the statement of Lemma 2.2 is valid, i.e., estimate II is valid. It follows that for any $l = 1,2,3,\ldots$ we obtain $\left\{ u_{k,\varepsilon}^{(l)} \right\} \in W(Q,R)$.
Thus, $\Re : W(Q,R) \to W(Q,R)$.
2. Let us show that $\Re$ is a contraction operator.
Let $\left\{ u_{k,\varepsilon}^{(l)} \right\}, \left\{ u_{k,\varepsilon}^{(l-1)} \right\} \in W(Q,R)$. Consider a new function $\vartheta_{k,\varepsilon}^{(l)} = u_{k,\varepsilon}^{(l)} - u_{k,\varepsilon}^{(l-1)}$, for which the statement of Lemma 2.3 is true, i.e., the IV estimate is valid:

$$\frac{\varepsilon}{\delta_0} \left\langle \frac{\partial \Delta^2 \vartheta_{k,\varepsilon}^{(l)}}{\partial t} \right\rangle_0^2 + \left\langle \vartheta_{k,\varepsilon}^{(l)} \right\rangle_4^2 \le A_2 q^{(l)}.$$

Thus, $\Re$ is a contracting operator. Now, according to the known principle of contraction mappings, problem (2.4)-(2.6) has a unique solution, belonging to the space $W(Q,R)$ at $\varepsilon > 0$. In this case, we have $u_{k,\varepsilon}^{(l)} \to u_{k,\varepsilon}$ as $l \to \infty$ [11], [12], [13], [26].

3. A family of loaded differential equations of mixed type of the second kind, fourth order.

Now we will prove the unique solvability of problem (1.6)-(1.8). In this case, we will use the family of loaded differential equations of the fifth order (2.1) with conditions (2.2), (2.3) as an "ε-regularizing" equation for equation (1.8) with conditions (1.7), (1.8) [3], [4], [5], [6], [7], [11], [9], [10].
Let $\{u_{k,\varepsilon}\} \in W(Q,R)$ for fixed $\varepsilon > 0$ be a unique solution of problem (2.1)-(2.3). Then for $\varepsilon > 0$, inequality IV holds. By the weak compactness theorem [16], [26], from a bounded sequence $\{u_{k,\varepsilon}\}$, one can extract a weakly convergent subsequence of the function $\{u_{k,\varepsilon_j}\}$ such that $u_{k,\varepsilon_j} \to u_k$ weakly in $W(Q,R)$. We will show that the limit function $u_k(x,t)$ satisfies equation (1.6) almost everywhere in $W(Q,\mathbb{R})$.

Indeed, since the subsequence $\{u_{k,\varepsilon_j}\}$ weakly converges in $W(Q,\mathbb{R})$ and $\sqrt{\varepsilon_j}\frac{\partial \Delta u_{k,\varepsilon_j}}{\partial t} \in L_2(Q)$, and the operator $\Im$ is linear, we have:

$$\begin{aligned} \Im u_k - F(u_k) &= \Im u_k - F(u_{k,\varepsilon_j}) - [F(u_k) - F(u_{k,\varepsilon_j})] \\ &= \varepsilon_j \frac{\partial \Delta^2 u_{k,\varepsilon_j}}{\partial t} + L_0(u_k - u_{k,\varepsilon_j}) \\ &\quad + \mu_k^4(u_k - u_{k,\varepsilon_j}) - [F(u_k) - F(u_{k,\varepsilon_j})]. \end{aligned} \tag{3.1}$$

Passing to the limit in (3.1) as $\varepsilon_j \to 0$, we obtain $\Im u_k = F(u_k)$. Therefore, for fixed k, the function $u_k(x,t)$ will be the only solution to problem (1.6)-(1.8) from $W(Q,R)$.
Thus, Theorem 2.4 is proved. Now we will prove Theorem 1.3.
Since all the conditions of Theorem 1.3 are fulfilled, using the Parseval-Steklov equalities [11], [12], [13], [26] to solve problem (1.6)-(1.8), we obtain the only solution to problem (1.1)-(1.5) from the specified class U.

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Optimal quadrature formulas with derivative exact for trigonometric functions

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Abstract. In this article, we study how to find the coefficients of a quadrature formula with derivative using the φ - function method. The φ - function method allows for construction of optimal quadrature formulas in approximate. Also, the functional error was analyzed using the quadrature formula constructed using the φ-function, and the results were supported by precise mathematical expressions. With arbitrary fixed nodes the optimality conditions for the quadrature formula are analyzed and the method for determining the components of that formula are considered. The explicite forms for the coefficients of the optimal quadrature formula are obtained. In particular, for the equally spaced nodes the Euler-Maclaurin type formula is obtained.

Keywords: method φ function, quadrature formula, definite integral, nodal points, optimality.

MSC (2020): 65D30, 65D25, 65D05

1. Introduction

The definite integral is one of the fundamental concepts of mathematical analysis and, in particular, it can be used to get the area or volume of a solid under the graph of a function. However, in many cases, finding an antiderivative function of a integrand is difficult or even impossible. Therefore, several approximate calculation methods for numerical integration have been developed. It is known from the course of mathematical analysis that if the antiderivative function of the function under the integral is known, the integral can be calculated using the Newton - Leibnitz formula. For cases where it is difficult to find the antiderivative of the functions under the integral, the problem of approximating the value of the definite integral arises. To solve this problem, various formulas have been found in mathematics, which are generally called quadrature and cubature formulas[1, 3, 12, 13, 14, 15, 16]. In this work, we construct an optimal quadrature formula in the space $K_2^{(2,0)}$ using the φ - function method. The space $K_2^{(2,0)}$ that we are looking at is a Hilbert space. This space is a special case of the spaces considered in the works [6, 8, 12]. If the nodal points of the quadrature formula are arbitrarily fixed, then the Spline method, the φ function method, and the Sobolev method are available to construct quadrature formulas obtained by minimizing the norm of the error functional depending on the nodes. Using the φ function method in several spaces, A.Chizzetti and A.Ossicini [17], F.Lanzara [13], T.Catina and G.Coman [2] constructed optimal quadrature formulas. In thus paper, we study the problem of constructing an optimal quadrature formula in the sense of Sard. The coefficients of the optimal quadrature formula are calculated using the resulting φ - function. The optimal quadrature formula in this work is exact on the functions $e^{\sigma x}$ and $e^{-\sigma x}$, where σ is a nonzero real parameter [6].

In this work, we consider the following type of quadrature formula

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + R_n(f), \tag{1.1}$$

where A_{0k}, A_{1k}, and x_k are the coefficients and nodes of the quadrature formula (1.1), respectively. Let the nodal points be partitioned on the interval $[a,\, b]$ as follows,

$$a = x_0 < x_1 < \ldots < x_n = b, \tag{1.2}$$

where $R_n\,(f)$ is the error of quadrature formula (1.1).

Let us assume that the function f under the integral we are looking at is taken from the space $K_2^{(2,0)}$, where
$K_2^{(2,0)} := \{f : [a,\, b] \to \mathbb{R} | f' \ -\text{ absolute continuous and } f'' \in L_2\,(a,\, b)\}$. In this space, the inner product for arbitrary functions $f\,(x)$ and $g\,(x)$ is defined as

$$\langle f, g\rangle_{K_2^{(2,0)}} = \int\limits_a^b \left(f''\,(x) + f\,(x)\right)\left(g''\,(x) + g\,(x)\right) dx.$$

The corresponding norm in this space is determined using this formula

$$\|f\|_{K_2^{(2,0)}} = \left(\int\limits_a^b \left(f''\,(x) + f\,(x)\right)^2 dx\right)^{1/2}.$$

We consider the construction of a quadrature formula of the form with the smallest error for functions in the space $K_2^{(2,0)}$ considered in a certain form in the space of quadrature formulas. For convenience, we introduce the following designations

$$A_0 = (A_{00},\, A_{01},\, \ldots,\, A_{0n}),\ \ A_1 = (A_{10},\, A_{11},\, \ldots,\, A_{1n})$$

and

$$X = (x_0,\, x_1,\, \ldots,\, x_n). \tag{1.3}$$

Below we give the definitions of optimality, of the quadrature formulas [3, 6, 17].

Definition 1.1 The quadrature formula of the form (1.1) is called optimal in the Nikolysky sense in the space $K_2^{(2,0)}$ if the quantity

$$F_n\left(K_2^{(2,0)},\, A_0,\, A_1,\, X\right) = \sup_{f \in K_2^{(2,0)}} |R_n\,(f)|$$

reaches its smallest value with respect to A_0, A_1 and X, and A_0, A_1 and X are determined by the equality (1.3).

Definition 1.2. The quadrature formula (1.1) is called optimal in the sense Sard in the space $K_2^{(2,0)}$ if the quantity

$$F_n\left(K_2^{(2,0)},\, A_0,\, A_1\right) = \sup_{f \in K_2^{(2,0)}} |R_n\,(f)|$$

reaches its smallest value relative to A_0, A_1 for fixed X, where A_0, A_1 and X are defined in (1.3).

In this paper we construct optimal quadrature formula of the form (1.1) in the sense of Sard in the Hilbert space $K_2^{(2,0)}$ which is exact for trigonometric functions $\sin\,(x)$ and $\cos\,(x)$. The rest of the work is organized as follows.

2. Method of φ - function in the space $K_2^{(2,0)}$

We discuss the φ - function method for construction of quadrature formulas of the form (1.1) in the space $K_2^{(2,0)}$ In Section 3, we consider the optimization of the quadrature formulas in the form (1.1). We get the explicite expressions of coefficients for the optimal formula. In particular, we get the Euler - Maclaurin type quadrature formula in the space $K_2^{(2,0)}$.

Let the function $f(x)$ be taken from the space $K_2^{(2,0)}$ and the nodes for the given natural numbers n be distributed as in the form (1.2). Then, for each interval $[x_{k-1},\, x_k]$ $(k = 1,\, 2,\, \dots, n)$, we consider the function φ_k, $k = 1,\, 2,\, \dots,\, n$ with the following property

$$\varphi_k''(x) + \varphi_k(x) = 1, \qquad k = 1,\, 2,\, \dots,\, n. \tag{2.1}$$

Then the function φ is defined as follows,

$$\varphi|_{[x_{k-1},\, x_k]} = \varphi_k, \qquad k = 1,\, 2,\, \dots,\, n.$$

Therefore, the reduction of the function φ on the interval $[x_{k-1},\, x_k]$ is equal to φ_k.

We introduce the following notations:

$$I\,(f) := \int\limits_a^b f(x)dx,$$

$$Q_n\,(f) = \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k).$$

Now, using the property of addition of definite integrals and taking into account equality (2.1), we get

$$I\,(f) = \int\limits_a^b f(x)\,dx = \sum_{k=1}^{n} \int\limits_{x_{k-1}}^{x_k} f(x)\,dx = \sum_{k=1}^{n} \int\limits_{x_{k-1}}^{x_k} (\varphi_k''\,(x) + \varphi\,(x)) f(x)\,dx.$$

Then integrating by parts we have

$$I\,(f) = \varphi_n'(x_n) f(x_n) + \sum_{k=1}^{n} \varphi_k'(x_k) f(x_k) - \varphi_1'(x_0) f(x_0) - \sum_{k=1}^{n-1} \varphi_{k+1}'(x_k) f(x_k)$$

$$-\varphi_n(x_n) f'(x_n) - \sum_{k=1}^{n-1} \varphi_k(x_k) f'(x_k) + \varphi_1(x_0) f'(x_0)$$

$$+\sum_{k=1}^{n-1} \varphi_{k+1}\,(x_k)\, f'\,(x_k) + \sum_{k=1}^{n} \int\limits_{x_{k-1}}^{x_k} (f''\,(x) + f\,(x))\, \varphi_k\,(x)\,dx.$$

From this we get the following:

$$I(f) = -\varphi_1'(x_0) + \sum_{k=1}^{n-1} (\varphi_k'(x_k) - \varphi_{k+1}'(x_k)) f(x_k) + \varphi_n'(x_n) f(x_n)$$

$$+\,\varphi_1(x_0) f'(x_0) - \sum_{k=1}^{n-1} (\varphi_k(x_k) - \varphi_{k+1}(x_k)) f'(x_k) - \varphi_n(x_n) f'(x_n) + R_n(f) \tag{2.2}$$

$$= \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + R_n(f)$$

From equality (2.2) we get:

$$\begin{aligned}
&A_{00} = -\varphi_1'(x_0),\\
&A_{0k} = \varphi_k'(x_k) - \varphi_{k+1}'(x_k), \quad k = 1, 2, \ldots, n-1,\\
&A_{0n} = \varphi_n'(x_n),\\
&A_{10} = \varphi_1(x_0),\\
&A_{1k} = \varphi_{k+1}(x_k) - \varphi_k(x_k), \quad k = 1, 2, \ldots, n-1,\\
&A_{1n} = -\varphi_n(x_n),
\end{aligned} \tag{2.3}$$

and the error formula is as follows,

$$R_n\left(f\right) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \left(f''(x) + f\left(x\right)\right) \varphi_k\left(x\right) dx = \int_a^b \left(f''(x) + f(x)\right)\varphi(x) dx. \tag{2.4}$$

Now we solve this nonhomogeneous second-order differential equation:

$$y'' + y = 1 \tag{2.5}$$

and obtain a general solution in the following form

$$y = C_1 \cos\left(x\right) + C_2 \sin\left(x\right) + 1. \tag{2.6}$$

Remark 2.1. Knowing the function φ, one can find the coefficients A_{0k} and A_{1k} , $k = 0, 1, \ldots, n$ from equality (2.3). This method is called the φ function method of constructing a quadrature formula.

Remark 2.2. As can be seen from the expression (2.4) above, the quadrature formula (1.1) is exact to the functions that are solutions of the equation

$$f''(x) + f(x) = 0. \tag{2.7}$$

In the following sections, we deal with finding the coefficients of the optimal quadrature formula (1.1) in the space $K_2^{(2,0)}$.

3. Construction of an optimal quadrature formula

In this section, we consider the optimality of the quadrature formula (1.1) in the space $K_2^{(2,0)}$. Then, using the Cauchy-Schwarz inequality, the absolute value of the error of the expression (2.4) of the quadrature formula (1.1) is determined as follows

$$|R_n\left(f\right)| \le \|f'' + f\|_{L_2(a,b)} \cdot \left(\int_a^b \varphi^2(x) dx\right)^{1/2} = \|f(x)\|_{K_2^{(2,0)}} \cdot \|\varphi(x)\|_{L_2(a,\,b)}.$$

Now we consider the function $\varphi_k(x)$, $x \in [x_{k-1}, x_k]$ is $k = 1, 2, \ldots, n$ as a solution of this equation (2.5). Then based on (9) for φ_k, we have

$$\varphi_k(x) = C_1^{(k)} \cdot \cos x + C_2^{(k)} \cdot \sin x + 1,$$

where $C_1^{(k)}$, $C_2^{(k)}$, $k = 1, 2, \ldots, n$, are arbitrary constants.

In conclusion, to find the functions $\varphi_k(x)$, it is necessary to find the unknowns $C_1^{(k)}$ and $C_2^{(k)}$, $k = 1, 2, \ldots, n$. We find $C_1^{(k)}$ and $C_2^{(k)}$ such that when the integral of the square of the function $\varphi_k(x)$ takes the smallest value. Let's look at this function that is related to these,

$$\mathcal{F}_k(C_1^{(k)}, C_2^{(k)}) = \int\limits_{x_{k-1}}^{x_k} (\varphi_k(x))^2 dx, \qquad k = 1, 2, \ldots, n.$$

Then, taking into account (2.7), we have the following.

$$\mathcal{F}_k(C_1^{(k)}, C_2^{(k)}) = \int\limits_{x_{k-1}}^{x_k} \Big(C_1^{(k)} \cos(x) + C_2^{(k)} \sin(x) + 1\Big)^2 dx$$

$$= \int\limits_{x_{k-1}}^{x_k} \Big(\Big(C_1^{(k)}\Big)^2 \cos^2(x) + \Big(C_2^{(k)}\Big)^2 \sin^2(x) + 1 + 2 \cdot C_1^{(k)} C_2^{(k)} \cos(x)\sin(x)$$

$$+2 \cdot C_1^{(k)} \cos(x) + 2 \cdot C_2^{(k)} \sin(x)\Big)\, dx, k = 1, 2, \ldots, n.$$

We calculate the first particular derivatives of this function with respect to $C_1^{(k)}$ and $C_2^{(k)}$ and equating them to zero we have the following system of equations

$$\begin{cases} a_{11} \cdot C_1^{(k)} + a_{12} \cdot C_2^{(k)} = b_1, \\ a_{21} \cdot C_1^{(k)} + a_{22} \cdot C_2^{(k)} = b_2, \end{cases}$$

where

$$a_{11} = \int\limits_{x_{k-1}}^{x_k} \cos^2(x)\, dx,$$

$$a_{12} = a_{21} = \int\limits_{x_{k-1}}^{x_k} \sin(x)\cos(x)\, dx,$$

$$a_{22} = \int\limits_{x_{k-1}}^{x_k} \sin^2(x)\, dx,$$

$$b_1 = -\int\limits_{x_{k-1}}^{x_k} \cos(x)\, dx,$$

$$b_2 = -\int\limits_{x_{k-1}}^{x_k} \sin(x)\, dx.$$

Taking into account that

$$a_{11}=\tfrac{1}{2}\left(x_k-x_{k-1}+\tfrac{1}{2}\left(\sin\left(2x_k\right)-\sin\left(2x_{k-1}\right)\right)\right),$$

$$a_{12}=a_{21}=-\tfrac{1}{4}\left(\cos\left(2x_k\right)-\cos\left(2x_{k-1}\right)\right),$$

$$a_{22}=\tfrac{1}{2}\left(x_k-x_{k-1}-\tfrac{1}{2}\left(\sin\left(2x_k\right)-\sin\left(2x_{k-1}\right)\right)\right),$$

$$b_1=-\left(\sin\left(x_k\right)-\sin\left(x_{k-1}\right)\right),$$

$$b_2=\cos\left(x_k\right)-\cos\left(x_{k-1}\right).$$

We get

$$\begin{aligned} C_1^{(k)}&=-\tfrac{4\cdot\sin\left(\frac{x_k-x_{k-1}}{2}\right)\cdot\cos\left(\frac{x_k+x_{k-1}}{2}\right)}{x_k-x_{k-1}+\sin(x_k-x_{k-1})},\\ C_2^{(k)}&=-\tfrac{4\cdot\sin\left(\frac{x_k-x_{k-1}}{2}\right)\cdot\sin\left(\frac{x_k+x_{k-1}}{2}\right)}{x_k-x_{k-1}+\sin(x_k-x_{k-1})}. \end{aligned} \tag{3.1}$$

Then using (3.1) from (2.7) we come

$$\begin{aligned} \varphi_k(x)=&-\frac{4\cdot\sin\left(\frac{x_k-x_{k-1}}{2}\right)\cdot\cos\left(\frac{x_k+x_{k-1}}{2}\right)}{x_k-x_{k-1}+\sin\left(x_k-x_{k-1}\right)}\cdot\cos(x)\\ &-\frac{4\cdot\sin\left(\frac{x_k-x_{k-1}}{2}\right)\cdot\sin\left(\frac{x_k+x_{k-1}}{2}\right)}{x_k-x_{k-1}+\sin\left(x_k-x_{k-1}\right)}\cdot\sin(x),\quad k=1,\,2,\,\ldots,\,n. \end{aligned} \tag{3.2}$$

For the first derivative of

$$\begin{aligned} \varphi_k'(x)=&\frac{4\cdot\sin\left(\frac{x_k-x_{k-1}}{2}\right)\cdot\cos\left(\frac{x_k+x_{k-1}}{2}\right)}{x_k-x_{k-1}+\sin\left(x_k-x_{k-1}\right)}\cdot\sin(x)\\ &-\frac{4\cdot\sin\left(\frac{x_k-x_{k-1}}{2}\right)\cdot\sin\left(\frac{x_k+x_{k-1}}{2}\right)}{x_k-x_{k-1}+\sin\left(x_k-x_{k-1}\right)}\cdot\cos(x). \end{aligned} \tag{3.3}$$

Now we can calculate the coefficients $A_{00},\ A_{1k}, k=1,\,2,\,\ldots,\,n$ based on (2.3) for the optimal quadrature formula of the form (1) in the space $K_2^{(2,0)}$. From (6), taking into account (3.2) and (3.3) we get

$$\begin{aligned} A_{00}&=\tfrac{2\cdot(1-\cos(x_1-x_0))}{x_1-x_0+\sin(x_1-x_0)},\\ A_{0k}&=\tfrac{2\cdot(1-\cos(x_k-x_{k-1}))}{x_k-x_{k-1}+\sin(x_k-x_{k-1})}+\tfrac{2\cdot(1-\cos(x_{k+1}-x_k))}{x_{k+1}-x_k+\sin(x_{k+1}-x_k)},\ k=1,\,2,\,\ldots,\,n-1,\\ A_{0n}&=\tfrac{2\cdot(1-\cos(x_n-x_{n-1}))}{x_n-x_{n-1}+\sin(x_n-x_{n-1})},\\ A_{10}&=\tfrac{x_1-x_0-\sin(x_1-x_0)}{x_1-x_0+\sin(x_1-x_0)},\\ A_{1k}&=\tfrac{2\cdot\sin(x_k-x_{k-1})\cdot(x_{k+1}-x_k)-2\cdot\sin(x_{k+1}-x_k)\cdot(x_k-x_{k-1})}{(x_{k+1}-x_k+\sin(x_{k+1}-x_k))\cdot(x_k-x_{k-1}+\sin(x_k-x_{k-1}))},\quad k=1,\,2,\,\ldots,\,n-1,\\ A_{1n}&=\tfrac{\sin(x_n-x_{n-1})-(x_n-x_{n-1})}{x_n-x_{n-1}+\sin(x_n-x_{n-1})}. \end{aligned} \tag{3.4}$$

Thus we have obtained the following main result of thus work.

Theorem 3.1 For fixed nodes $a = x_0 < x_1 < \ldots < x_n = b$ there exists a unique optimal quadrature formula of the form

$$\int_a^b f(x)\,dx \cong \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k), \tag{3.5}$$

in the sense of Sard in the space $K_2^{(2,0)}$ with coefficients (3.4). The formula is exact for trigonometric functions $\sin(x)$ and $\cos(x)$.

From Theorem 3.1 when the nodes are equally spaced we get the optimal quadrature formula of the Euler - Maclaurin type in the space $K_2^{(2,0)}$. That is the following holds.

Corollary 3.1 For equally spaced nodes $x_k = a + kh, \quad k = 0, 1, \ldots, n,\ h = \frac{b-a}{n}$, there exist a unique optimal quadrature formula

$$\int_a^b f(x)\,dx \cong \sum_{k=1}^{n} A_{0k} f(a + kh) + A_{10} f'(a) + A_{1n} f'(b)$$

of the Euler - Maclaurin type with coefficients

$$A_{00} = \frac{2\cdot(1-\cos(h))}{h+\sin(h)},$$

$$A_{0k} = \frac{4\cdot(1-\cos(h))}{h+\sin(h)}, \ k = 1, 2, \ldots, n-1,$$

$$A_{0n} = \frac{2\cdot(1-\cos(h))}{h+\sin(h)},$$

$$A_{10} = \frac{h-\sin(h)}{h+\sin(h)},$$

$$A_{1n} = \frac{\sin(h)-h}{h+\sin(h)}.$$

4. Conclusion

In this work, we constructed an optimal quadrature formula in the space $K_2^{(2,0)}$, where $K_2^{(2,0)}$ is the Hilbert space of absolutely continuous functions whose first-order derivatives are square-integrable on the interval $[a, b]$. The φ-function method and its role in optimizing quadrature formulas presented in this article constitute the main results of the research. The defined quadrature formula can be widely used in theoretical and practical areas of mathematical analysis. In the future, there are opportunities to develop and apply this approach to more complex quadrature formulas. The determined quadrature formula can be widely used in theoretical research and practical mathematical problems.

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Dynamics of a cubic stochastic operator with one discontinuity point

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Abstract. In this paper we consider a population consisting of two species, dynamics of which is defined by a cubic stochastic operator with variable coefficients, making it discontinuous operator at one points. This operator depends on two parameters. It is shown that under suitable conditions on the parameters this operator may have fixed points, convergence of trajectories.

Keywords: Dynamical systems, simplex, fixed point, cubic stochastic operator, trajectory.
MSC (2020): 37N25

1. Introduction

Unlike Markov processes, where the probability distribution of the system evolves in a linear fashion under the action of a stochastic operator, population dynamics are nonlinear because the recombination of genes occurs due to their pairing. The mathematical model of population dynamics was given in seminal works of Kesten [1], where he studied dynamical systems generated by quadratic stochastic operator (QSO) of asexual (multitype) populations and sex-linked systems. He gave conditions on coefficients of QSO under which it has a unique fixed point. Moreover, several results are given for the different mating rules and a stochastic theory for the Mendelian genetics model is given. Kesten's papers form a valuable contribution to the mathematics of population processes.

Note that many dynamical systems that happen naturally in the description of physical and biological processes are piecewise-smooth. That's why the dynamical system containing terms that are non-smooth functions is studied as an important problem. Problems of like this appear, for example, in electrical circuits that have switches, mechanical devices in which components impact with each other or have free play, problems with friction, sliding or squealing, many control systems and models in the social and financial sciences where continuous change can trigger discrete actions [2]. See also motivating examples of a piecewise-smooth systems: generated by the floor function ([3], [4]).

In [1] the random process behind the dynamical system of the type distribution (i.e. the random sizes of each type) is investigated; but, as he admitted himself, the results are somewhat disappointing as the model only yields a dichotomy between extinction and exponential growth, thus not demonstrating stability or adaptation (selection).

In the papers [5], [6] authors gave models of population where the dichotomy between extinction and exponential growth is not the case. These models are given by a function with discontinuity point. They showed that the dynamical systems has a complexity despite the discontinuity point is unique. In [7] and [8] a new notion of a cubic stochastic operator are introduced and investigated.

In this paper we consider a population consisting of two species, dynamics of which is defined by a cubic stochastic operator with variable coefficients, making it discontinuous operator at one points. This operator depends on two parameters. It is shown that under suitable conditions on the parameters this operator may have fixed points, convergence of trajectories.

Let us give some necessary definitions (see [9]). In order to define a discrete-time dynamical system consider a function $f : X \to X$. For $x \in X$ denote by $f^n(x)$ the n-fold composition of

f with itself:

$$f^n(x) = \underbrace{f(f(...f(x)...))}_{n}$$

Definition 1.1. For arbitrary given $x^{(0)} \in X$ and $f : X \to X$ the discrete-time dynamical system (also called forward orbit or trajectory of $x^{(0)}$) is the sequence of points

$$x^{(0)}, x^{(1)} = f\left(x^{(0)}\right), x^{(2)} = f^2\left(x^{(0)}\right), x^{(3)} = f^3\left(x^{(0)}\right), ...$$

Definition 1.2. A point $x \in X$ is called a fixed point for $f : X \to X$ if $f(x) = x$. The point x is called a periodic point of period p if $f^p(x) = x$. The least positive p for which $f^p(x) = x$ is called the prime period of x.

It is clear that the set of all iterates of a periodic point form a periodic sequence (orbit).

There are three kinds of periodic points: attracting, repelling and indifferent. Let x^* be a p -periodic point. If $|(f^p(x^*))'| < 1$, x^* is attracting; $|(f^p(x^*))'| > 1$, $x^*$is repelling; $|(f^p(x^*))'| = 1$, $x^*$is indifferent.

Let $E = \{1, 2, ..., m\}$. By the $(m-1)$-dimensional simplex we mean the set

$$S^{m-1} = \left\{x = (x_1, ..., x_m) \in R^m : x_i \geq 0, \sum_{i=1}^{m} x_i\right\}.$$

Each element $x \in S^{m-1}$ is a probability measure on E, and so it may be looked upon as the state of a biological (or physical) system of m elements.

A *cubic stochastic operator* is a mapping $V : S^{m-1} \to S^{m-1}$ of the form

$$V : x'_l = \sum_{i,j,k=1}^{m} p_{ijk,l} x_i x_j x_k = 1, \ l = 1, ..., m, \tag{1.1}$$

where $p_{ijk,l}$ are coeffcients of heredity such that

$$p_{ijk,l} \geq 0, \sum_{l=1}^{m} p_{ijk,l} = 1, \quad i, j, k, l = 1, ..., m, \tag{1.2}$$

and the coeffcients $p_{ijk,l}$ do not change for any permutation of i, j and k.

For a given $x^{(0)} \in S^{m-1}$ the trajectory $x^{(n)}$, of an initial point $x^{(0)}$ under the action of the CSO (1.1) is defined by $x^{(n+1)} = V\left(x^{(n)}\right)$, where $n = 0, 1, 2, ...$

One of the main problems in mathematical biology consists of the study of the asymptotical behavior of the trajectories. Note that the main problem is open even in two-dimensional case.

The main problem for a given dynamical system is to describe the limit points of $\left\{x^{(n)}\right\}_{n=0}^{\infty}$ for arbitrary given $x^{(0)}$.

2. A CSO WITH ONE DISCONTINUITY POINT

Consider a population of two species, i.e.$m = 2$. For a variable coefficient $p(x)$ define an evolution operator $V_{a,b,c} : z = (x, y) \in S^1 \to z' = (x', y') \in S^1$ as the following:

$$V \equiv V_{a,b} = \begin{cases} x' = x^3 + 3p(x)xy, \\ y' = 3(1 - p(x))xy + y^3, \end{cases} \tag{2.1}$$

where

$$p(x) = \begin{cases} a, & x \le 1/2, \\ b, & x > 1/2, \end{cases}$$

here $a, b \in [0, 1]$.

Given that $x+y=1$, it can be shown that operator (2.1) preserves the simplex S^1 as follows:

$$x' + y' = x^3 + 3p(x)xy + 3(1-p(x))xy + y^3 = x^3 + 3p(x)xy + 3xy(x+y) - 3p(x)xy + y^3 =$$

$$= x^3 + 3x^2y + 3xy^2 + y^3 = (x+y)^3 = 1$$

We are interested to study the dynamical system generated by the evolution operator $V_{a,b}$. Using $x+y=1$ the operator (2.1) can be reduced to the mapping $f_{a,b} : [0,1] \to [0,1]$ defined by

$$f_{a,b}(x) = \begin{cases} x^3 - 3ax^2 + 3ax, & x \le 1/2, \\ x^3 - 3bx^2 + 3bx, & x > 1/2, \end{cases} \tag{2.2}$$

where $a, b \in [0, 1]$.

In case $a = b$ we have $f(x) := f_{a,a}(x) = x^3 - 3ax^2 + 3ax$, $x \in [0, 1]$. The set of the fixed points of this function is

$$Fix(f) = \begin{cases} \{0, 1\} & if \;\; a \in [0, 1/3] \cup [1/3, 1], \\ \{0, \; s_a, 1\} & if \;\; a \in (1/3, 2/3), \end{cases}$$

where $s_a = 3a - 1$.

It is easy to see that

(1) The point 0 is attracting for f, if $a \in [0, 1/3)$; non-hyperbolic if $a = 1/3$ and repelling if $a \in (1/3, 1]$.

(2) The point 1 is repelling for f, if $a \in [0, 2/3)$; non-hyperbolic if $a = 2/3$ and attracting if $a \in (2/3, 1]$.

(3) The point s_a is attracting for f, if $a \in (1/3, 2/3)$.

Moreover, using monotonicity of f one can show that for any initial point $x^{(0)} \in (0, 1)$ the following limits hold

$$\lim_{n\to\infty} f^n\left(x^{(0)}\right) = \begin{cases} 0, & if\, a \in [0, 1/3], \\ s_a, & if\, a \in (1/3, 2/3), \\ 1, & if\, a \in [2/3, 1]. \end{cases}$$

The case $a \ne b$. The following theorem is valid for this case.

Proposition 2.1. *The dynamical system generated by function (2.2) has the following assertions hold*

(1) *If* $a \in [0, 1/3]$, $b \in [0, 1/2]$ *then*

  (a) *The set of fixed points is* $Fix(f) = \{0, 1\}$;

  (b) $\lim_{n\to\infty} x^{(n)} = \lim_{n\to\infty} f^n\left(x^{(0)}\right) = 0$ *for any* $x^{(0)} \in [0, 1)$;

(2) *If* $a \in [0, 1/3]$, $b \in (1/2, 2/3)$ *then*

(a) *The set of fixed points is* $Fix\,(f)=\{0,s_b,1\}$;

(b) $\lim\limits_{n\to\infty}x^{(n)}=\begin{cases}0, & if\ x^{(0)}\in[0,1/2]\,,\\ s_b, & if\ x^{(0)}\in(1/2,1)\,;\end{cases}$

(3) *If* $a\in[0,1/3]\,,\ b\in[2/3,1]$ *then*

(a) *The set of fixed points is* $Fix\,(f)=\{0,1\}$;

(b) $\lim\limits_{n\to\infty}x^{(n)}=\begin{cases}0, & if\ x^{(0)}\in[0,1/2]\,,\\ 1, & if\ x^{(0)}\in(1/2,1]\,;\end{cases}$

(4) *If* $a\in(1/3,1/2]\,,\ b\in[0,1/2]$ *then*

(a) *The set of fixed points is* $Fix\,(f)=\{0,s_a,1\}$;

(b) $\lim\limits_{n\to\infty}x^{(n)}=s_a$ *for any* $x^{(0)}\in(0,1)$;

(5) *If* $a\in(1/3,1/2]\,,\ b\in(1/2,2/3)$ *then*

(a) *The set of fixed points is* $Fix\,(f)=\{0,s_a,s_b,1\}$;

(b) $\lim\limits_{n\to\infty}x^{(n)}=\begin{cases}s_a, & if\ x^{(0)}\in(0,1/2]\,,\\ s_b, & if\ x^{(0)}\in(1/2,1)\,;\end{cases}$

(6) *If* $a\in(1/3,1/2]\,,\ b\in[2/3,1]$ *then*

(a) *The set of fixed points is* $Fix\,(f)=\{0,s_a,1\}$;

(b) $\lim\limits_{n\to\infty}x^{(n)}=\begin{cases}s_a, & if\ x^{(0)}\in(0,1/2]\,,\\ 1, & if\ x^{(0)}\in(1/2,1]\,;\end{cases}$

(7) *If* $a\in(1/2,1]\,,\ b\in[0,1/2)$ *then*

(a) *The set of fixed points is* $Fix\,(f)=\{0,1\}$;

(b) *For sets* $A_1=\left(0,\frac{1+6b}{8}\right)\cup\left(\frac{1+6a}{8},1\right)$, $A_2=\left[\frac{1+6b}{8},\frac{1+6a}{8}\right]$ *the following hold:*

(i) $\forall x^0\in A_1$ *there exists* $n_0\,(x^0)\in\mathbb{N}$, *such that* $f^n\,(x^0)\in A_2$ *for any* $n>n_0$;

(ii) $f\,(A_2)\subset A_2$;

(8) *If* $a\in(1/2,1]\,,\ b\in[1/2,2/3)$ *then*

(a) *The set of fixed points is* $Fix\,(f)=\{0,s_b,1\}$;

(b) $\lim\limits_{n\to\infty}x^{(n)}=s_b$ *for any* $x^{(0)}\in(0,1)$;

(9) *If* $a\in(1/2,1]\,,\ b\in[2/3,1]$ *then*

(a) *The set of fixed points is* $Fix\,(f)=\{0,1\}$;

(b) $\lim\limits_{n\to\infty}x^{(n)}=1$ *for any* $x^{(0)}\in(0,1]$;

Proof. (1).(a) follows from a simple analysis of the equation $f(x) = x$.

(b) Let $x \in [0, 1/2]$. Then the function $f_{a,b}(x)$ is monotone increasing and

$$f_{a,b}(x) - x = x(1-x)(3a-1-x) < 0 \Rightarrow f_{a,b}(x) < x,$$

$0 \le f_{a,b}(x) \le \frac{1+6a}{8}$ for any point $x \in [0, 1/2]$. Therefore the orbit $f^n_{a,b}(x)$ is decreasing and bounded from below by the fixed point $x = 0$. So, the limit of this sequence is 0. Assume that $x \in (1/2, 1]$, then in this case also the function $f_{a,b}(x)$ is monotone increasing and $f_{a,b}(x) < x$, $\frac{1+6b}{8} < f_{a,b}(x) \le 1$ for any point $x \in (1/2, 1]$. Therefore that there exists $k \in N$ such that $f^k_{a,b}(x) \in \left(\frac{1+6b}{8}, \frac{1}{2}\right]$. Thus the limit is 0 in this case as well.

(2). (a) is similar to the proof of case (1).(a).

(b) Let $x \in [0, 1/2]$, then similarly to the proof of case (1).(b).

Let $x \in (1/2, 1]$, then we have $s_b \in (1/2, 1]$. First, we assume that $x \in (1/2, s_b)$. The function $f_{a,b}(x)$ is monotone increasing and

$$f_{a,b}(x) - x = x(1-x)(3b-1-x) > 0 \Rightarrow f_{a,b}(x) > x,$$

$\frac{1+6b}{8} < f_{a,b}(x) < 3b-1$ for any point $x \in (1/2, s_b)$. Therefore the orbit $f^n_{a,b}(x)$ is increasing and bounded from above by the fixed point $x = s_b$. So, the limit of this sequence is s_b. If $x \in (s_b, 1]$, that the function $f_{a,b}(x)$ is monotone increasing and $f_{a,b}(x) < x$, $3b - 1 < f_{a,b}(x) \le 1$ for any point $x \in (s_b, 1]$. Therefore the orbit $f^n_{a,b}(x)$ is decreasing and bounded from below by the fixed point $x = s_b$. So, the limit of this sequence is s_b.

Parts (3)-(6) and (8)-(9) are similar to the proof of the above cases.

(7).(a) is similar to the proof of case (1).(a).

(b).(i). For $x \in A_1$ we have $x \in \left(0, \frac{1+6b}{8}\right)$ or $x \in \left(\frac{1+6a}{8}, 1\right)$. Let's suppose $x \in \left(0, \frac{1+6b}{8}\right)$. In this case the function is $f_{a,b}(x) = x^3 - 3ax^2 + 3ax$ and monotone increasing. Inequality

$$f_{a,b}(x) > x \Rightarrow ((1-3a)x + 1)x(x-1) > 0$$

its solution is any $x \in [0, 1/2]$ for $a \in (1/2, 1]$.

Let's suppose the trajectory of $x_0 \in \left(0, \frac{1+6b}{8}\right)$, i.e., $x_n = f^n(x_0)$ does not go inside of A_2, then this trajectory has its own limit because it is an increasing and bounded by $\frac{1+6b}{8}$ (i.e. the left endpoint of A_2). But $\frac{1+6b}{8}$ isn't fixed point and there is no any fixed point in the left side of A_2. Thus, this trajectory goes inside of A_2.

It can be shown that for $x \in \left(\frac{1+6a}{8}, 1\right)$ the trajectory lies in A_2, as above. ii. We prove that $f(x) \in A_2$ for all $x \in A_2$. For $x \in A_2$ we have

$$x \in \left[\frac{1+6b}{8}, \frac{1}{2}\right] \cup \left(\frac{1}{2}, \frac{1+6a}{8}\right]$$

Let's suppose, $x \in \left[\frac{1+6b}{8}, \frac{1}{2}\right]$. It is easy to check that, $f\left(\frac{1+6b}{8}\right)$ and $f\left(\frac{1}{2}\right)$ are elements of A_2. Then $f(x) \in A_2$ because $f(x)$ is monotone increasing. For the second case, i.e. $x \in \left(\frac{1}{2}, \frac{1+6a}{8}\right]$ we may check similarly. Therefore, $f(A_2) \subset A_2$. Proposition is proved. □

Thus, using **Proposition 2.1** and $y = 1 - x$, we conclude the following theorem for the dynamics of the operator (2.1)

Theorem 2.2. *The dynamical system generated by operator (2.2) has the following assertions hold*

(1) *If* $a \in [0, 1/3]$, $b \in [0, 1/2]$ *then*

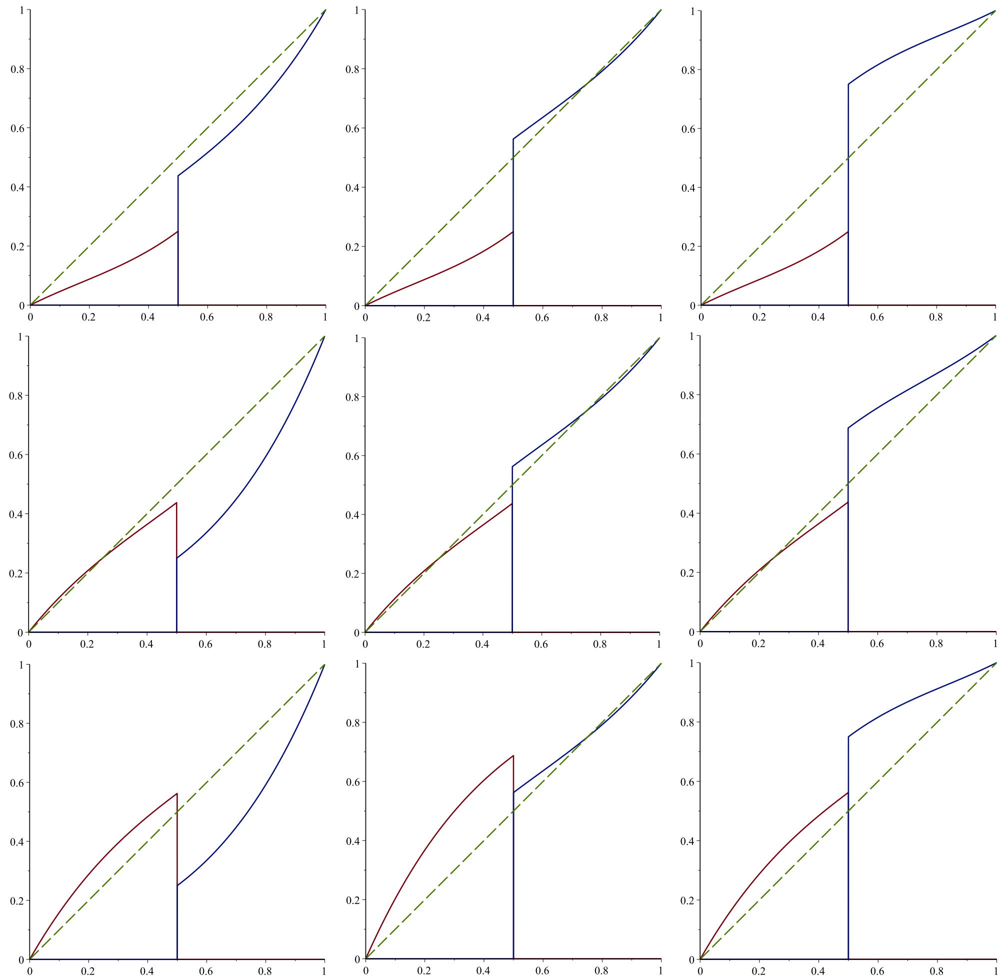

FIGURE 1. The graphs of function (2.2) for the values of parameters according to the corresponding cases of Proposition 2.1.

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, e_2\}$;

(b) $\lim\limits_{n\to\infty} V^n\left(x^{(0)}, y^{(0)}\right) = e_2$ *for any* $x^{(0)} \in [0,1)$;

(2) *If* $a \in [0, 1/3]$, $b \in (1/2, 2/3)$ *then*

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, r_b, e_2\}$;

(b) $\lim\limits_{n\to\infty} V^n\left(x^{(0)}, y^{(0)}\right) = \begin{cases} e_2, & if\ x^{(0)} \in [0, 1/2], \\ r_b, & if\ x^{(0)} \in (1/2, 1); \end{cases}$

(3) *If* $a \in [0, 1/3]$, $b \in [2/3, 1]$ *then*

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, e_2\}$;

(b) $\lim\limits_{n\to\infty} V^n\left(x^{(0)}, y^{(0)}\right) = \begin{cases} e_2, & if\ x^{(0)} \in [0, 1/2], \\ e_1, & if\ x^{(0)} \in (1/2, 1]; \end{cases}$

(4) *If* $a \in (1/3, 1/2]$, $b \in [0, 1/2]$ *then*

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, r_a, e_2\}$;

(b) $\lim\limits_{n\to\infty} V^n\left(x^{(0)}, y^{(0)}\right) = r_a$ *for any* $x^{(0)} \in (0,1)$;

(5) *If* $a \in (1/3, 1/2]\,,\ b \in (1/2, 2/3)$ *then*

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, r_a, r_b, e_2\}$;

(b) $\lim\limits_{n\to\infty} V^n\left(x^{(0)}, y^{(0)}\right) = \begin{cases} r_a, & if\ \ x^{(0)} \in (0, 1/2]\,, \\ r_b, & if\ \ x^{(0)} \in (1/2, 1)\,; \end{cases}$

(6) *If* $a \in (1/3, 1/2]\,,\ b \in [2/3, 1]$ *then*

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, r_a, e_2\}$;

(b) $\lim\limits_{n\to\infty} V^n\left(x^{(0)}, y^{(0)}\right) = \begin{cases} r_a, & if\ \ x^{(0)} \in (0, 1/2]\,, \\ e_1, & if\ \ x^{(0)} \in (1/2, 1]\,; \end{cases}$

(7) *If* $a \in (1/2, 1]\,,\ b \in [0, 1/2)$ *then*

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, e_2\}$;

(b) *For sets* $\Gamma_1 = \{(x,y) \in S^1 : x \in A_1\}$, $\Gamma_2 = \{(x,y) \in S^1 : x \in A_2\}$ *the following hold:*

(i) $\forall\left(x^{(0)}, y^{(0)}\right) \in \Gamma_1$ *there exists* $n_0\left(x^{(0)}, y^{(0)}\right) \in \mathbb{N}$, *such that* $V^n\left(x^{(0)}, y^{(0)}\right) \in \Gamma_2$ *for any* $n > n_0$;

(ii) $V\left(\Gamma_2\right) \subset \Gamma_2$;

(8) *If* $a \in (1/2, 1]\,,\ b \in [1/2, 2/3)$ *then*

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, r_b, e_2\}$;

(b) $\lim\limits_{n\to\infty} V^n\left(x^{(0)}, y^{(0)}\right) = r_b$ *for any* $x^{(0)} \in (0,1)$;

(9) *If* $a \in (1/2, 1]\,,\ b \in [2/3, 1]$ *then*

(a) *The set of fixed points is* $Fix\,(V) = \{e_1, e_2\}$;

(b) $\lim\limits_{n\to\infty} V^n\left(x^{(0)}, y^{(0)}\right) = e_1$ *for any* $x^{(0)} \in (0,1]$;

where $e_1 = (1,0)$, $e_2 = (0,1)$, $r_a = (3a-1, 2-3a)$, $r_b = (3b-1, 2-3b)$.

Let us study 2-periodic points of the function (2.2). It is clear that a 2-periodic orbit x_1, x_2 exists if they satisfy the following system of equations:

$$\begin{cases} x_1^3 - 3ax_1^2 + 3ax_1 = x_2, \\ x_2^3 - 3bx_2^2 + 3bx_2 = x_1. \end{cases} \tag{2.3}$$

Substituting x_2 from the first equation of (2.3) into the second equation leads to a polynomial equation of degree 9 in x_1. Which may have up to nine solutions, two of them are $x_1 = 0$ and $x_1 = 1$ independently on the values of parameters. These solutions are fixed points. To find 2-periodic (except fixed) points we divide the polynomial by x_1 and $x_1 - 1$ and get an equation of degree 7 of which has the following form

$$x_1^7 + (1-9a)\,x_1^6 + \left(1+27a^2\right)x_1^5 + \left(1-27a^2-27a^3-3b\right)x_1^4 + \left(1+54a^3-3b+18ab\right)x_1^3 +$$
$$+\left(1-27a^3-3b-27a^2b\right)x_1^2 + \left(1+27a^2b\right)x_1 - 9ab + 1 = 0$$

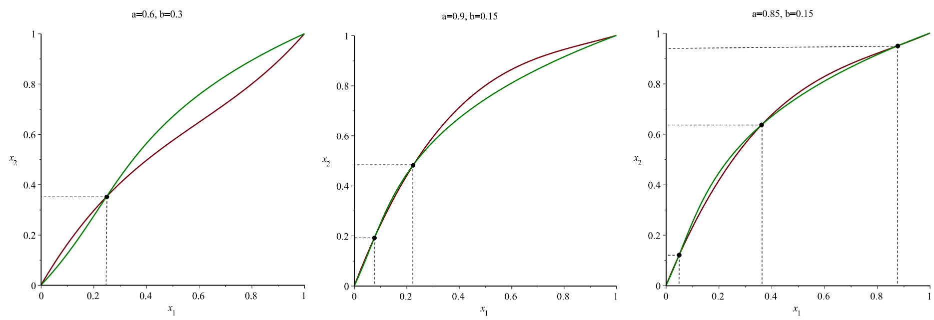

FIGURE 2. The red line is the graph of $x_1^3 - 3ax_1^2 + 3ax_1 = x_2$ and the green line is the graph of $x_2^3 - 3ax_2^2 + 3ax_2 = x_1$.

Due to the difficulty in obtaining a general solution for this equation to find 2-periodic points, we consider results for specific parameter values. A numerical analysis reveals the following:

If $a = 0.6$ and $b = 0.3$, there is a unique pair of 2-periodic orbit $x_1 = 0.24875$, $x_2 = 0.35176$.

If $a = 0.9$ and $b = 0.15$, then there are two pairs of 2-periodic orbits $x_1 = 0.07723$, $x_2 = 0.19287$ and $x_1 = 0.22803$, $x_2 = 0.48715$.

If $a = 0.85$ and $b = 0.15$, then there are three pairs of 2-periodic orbits $x_1 = 0.04754$, $x_2 = 0.11558$, $x_1 = 0.36278$, $x_2 = 0.63722$ and $x_1 = 0.88442$, $x_2 = 0.95246$.

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A Hybrid Inertial S - iteration Algorithm for Quasi -ϕ- Asymptotically Nonexpansive mappings, Maximal Monotone Operators and Generalized Mixed Equilibrium Problems

Lawal Umar, Harbau M. H., Yusuf Ibrahim, and Abdulazeez Idris

Abstract. In this paper, we propose a hybrid inertial S-iteration algorithm for two quasi -ϕ- asymptotically nonexpansive mappings, maximal monotone operator and generalized mixed equilibrium problems in a real Banach space. We also, established a strong convergence theorem of the propose iterative scheme. The results present in this paper extend and improve some recent results in the literature.

Keywords: Hybrid S-Inertial Algorithm, Quasi-ϕ- asymptotically nonexpansive mappings, Maximal Monotone Operators, Generalized Mixed Equilibrium Problems.
MSC (2020): 47H09, 47J25.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E. We consider E^* and $\| . \|$ as the dual space of E and induced norm on E respectively. The generalised mixed equilibrium problem (GMEP) [3] is to find $z \in C$ such that

$$f(z,y) + \Theta(y) - \Theta(z) + \langle Bz, y-z\rangle \geq 0, \forall y \in C,$$

The set of solutions of generalized mixed equilibrium problem is denoted by

$$GMEP(f,B,\Theta) = \{z \in C : f(z,y) + \Theta(y) - \Theta(z) + \langle Bz, y-z\rangle \geq 0, \forall y \in C\}.$$

where $f : C \times C \longrightarrow \mathbb{R}$ is a bifunction, $\Theta : C \longrightarrow \mathbb{R}$ is a real valued function and $B : C \longrightarrow E^*$ is a nonlinear mapping. Let $S : C \longrightarrow C$ be a nonlinear mapping. A point $x \in C$ is a fixed point of S provided that $Sx = x$. $F(S) = \{x \in C : Sx = x\}$ denote the fixed point set of S. A point $x \in C$ is called an asymptotic fixed point of S [20] if C contains a sequence $\{x_n\}$ which converges weakly to x such that $\lim_{n\to\infty} \| x_n - Sx_n \| = 0$. $\hat{F}(S)$ denote the asymptotic fixed point set of S. A set- valued mapping $N : E \longrightarrow 2^{E^*}$ with domain $D(N) = \{x \in E : N(x)\} \neq \emptyset$ and range $R(N) = \{x^* \in E^* : x^* \in N(x),\ x \in D(N)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$, for all $x^* \in N(x),\ y^* \in N(y)$. We denote the set $\{x \in E : 0 \in Nx\}$ by $N^{-1}0$. A monotone N is said to be maximal if its graph $G(N) = \{(x,y) : y \in Nx\}$ is not properly contained in the graph of any other monotone operator. If N is maximal monotone, then the solution set $N^{-1}0$ is closed and convex. Let E be a reflexive, strictly convex and smooth Banach space, it is well known that N is a maximal monotone if and only if $R(J + rN) = E^*$, for all $r > 0$. Defined the resolvent of N by $J_r = (J + rN)^{-1}J$, for all $r > 0$. J_r denote a single valued mapping from E to $D(N)$. Notice that $N^{-1}(0) = F(J_r)$, for all $r > 0$, where $F(J_r)$ is the set of all fixed point of J_r. Defined the Yosida approximation of N by $N_r = (J - JJ_r)/r$, for all $r > 0$. It is well known that $N_r x \in N(J_r x)$, for all $r > 0$ and $x \in E$.

Definition 1.1. i) Let $\{S_i\}_{i=1}^{\infty} : C \longrightarrow C$ be a sequence of mapping. $\{S_i\}_{i=1}^{\infty}$ is said to be a family of uniformly quasi-ϕ-asymptotically nonexpansive mapping [5], if $\Upsilon := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \longrightarrow 1$ (as $n \to \infty$) such that for each $i \geq 1$

$$\phi(p, S_i^n x) \leq k_n \phi(p,x), \quad \forall p \in \Upsilon,\ x \in C,\ \forall n \geq 1.$$

ii) A mapping $S : C \longrightarrow C$ is said to be uniformly $L - Lipschitz$ continuous [5], if there exists a constant $L > 0$ such that for all $x, y \in C$,

$$\| S^n x - S^n y \| \leq L \| x - y \|, \; n \geq 1.$$

iii) A mapping $S : C \longrightarrow C$ is said to be closed [21] if, for any sequence $\{x_n\} \subset C$ with $x_n \longrightarrow x$ and $Sx_n \longrightarrow y$ then $y = Sx$.

The equilibrium problems and fixed point problems are closely related problems that investigating various solutions and applications of a large number of problems arising in optimization,economics, physics, game theory, theory of differential equations and many other related areas in sciences (for more details see Alansari et al. [3], Blum and Oettli [4], Change et al.[5], Chidume et al. [6], Combettes Hirstoage [7]). Several techniques have been proposed for approximating solutions of equilibrium problems and fixed point problems with generalization in different spaces (for more details see [3, 5, 6, 13, 15, 21, 25] and the references therein).

In 2007 Agarwal et al [1] introduced and proved the strong convergence theorem for approximating fixed point of contraction mapping in Hilbert space using the following iterative process called S-iteration method.

$$\begin{cases} x_0 \in C; \\ y_n = (1 - \alpha_n^1) x_n + \alpha_n^1 S x_n; \\ x_{n+1} = (1 - \alpha_n^0) S x_n + \alpha_n^0 y_n, \; n \geq 1 \end{cases}$$

where $\{\alpha_n^0\}$, $\{\alpha_n^1\}$ are sequences in $[0, 1]$
Sahu [22] Proposed and studied S-iteration method and established strong convergence theorem for contraction mappings. Suparatulatorn et al [23] proposed S-iteration process by considering two G-nonexpansive maps S_1 and S_2 in a Hilbert space. However, to increase the speed rate of convergence, an inertial type - algorithm was first introduced and studied by Polyak [19] for approximating a smooth convex minimization problem. Due to it important many researchers have developed interests and constructed several fast iterative algorithms using inertial extrapolation method (for more details see [3, 6, 8, 9, 16] and the references therein).
In 2008, Mainge [22] proposed the following inertial Mann algorithm by combining Mann algorithm and inertial extrapolation for acceleration process

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}); \\ x_{n+1} = (1 - \beta_n) x_n + \beta_n T w_n, \quad n \geq 1. \end{cases}$$

The author established strong convergence theorem.
Phonon et al [18], proposed and studied the following modified inertial S-iteration process by combining modified S-iteration and inertial extrapolation for accelerating the convergence process of the modified S-iteration algorithm for two arbitrary nonexpansive maps S_1 and S_2 in a Hilbert space.

$$\begin{cases} w_n = x_n + \alpha_n (x_n - x_{n-1}); \\ y_n = (1 - \beta_n) w_n + \beta_n S_1 w_n, \\ x_{n+1} = (1 - \gamma_n) S_1 w_n + \gamma_n S_2 y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1), \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. They proved strong convergence result.
Recently, Chidume et al [6] proposed the following inertial algorithm for approximating a

common fixed point of a countable family of relatively nonexpansive mappings and the set of zeros of a maximal monotone operator in a real Banach space

$$\begin{cases} x_0, x_1 \in B, \quad C_0 = B, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = J^{-1}((1-\beta)Jw_n + \beta JSJ_{r_n}w_n) \\ u_n = J^{-1}((1-\gamma)Jw_n + \gamma JTz_n) \\ c_{n+1} = \{z \in C : \psi(z, u_n) \leq \psi(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}, \{\alpha_n\} \subset [0,1), \beta, \gamma \in (0,1)$ and $\{r_n\} \subset [a, \infty)$, for some $a > 0$. It has been proved that the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.
Very recently, Harbau and Abdulwahab [10] introduced and studied the following inertial hybrid S-iteration process for two asymptotically nonexpansive mappings and a system of equilibrium problems in a real Hilbert space

$$\begin{cases} x_0, x_1 \in C, \quad C_0 = C, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = (1-\beta_n)w_n + \beta_n T_1^n w_n, \\ z_n = (1-\gamma_n)T_1^n + \gamma_n T_2^n y_n, \\ u_n = T_{r_n} z_n, \\ c_{n+1} = \{z \in C : \| u_n - z \| \leq \| w_n - z \| + \theta_n\}, \\ x_{n+1} = P_{C+1}x_0, \ \forall n \geq 0, \end{cases} \tag{1.1}$$

where $\{\alpha_n\} \subset (0,1)$, $\beta_n, \gamma_n \in [\zeta, 1-\zeta]$, for some $\zeta \in (0,1)$ and $T_i : C \longrightarrow C, \ i = 1,2$ are asymptotically nonexpansive mappings with sequence $\{k_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} k_{n,i} = 0$. They proved that $\{x_n\}$ converges strongly to $P_\Gamma x_0$, where $\Gamma := \cap_{i=1}^2 F(T) \cap EP(f, C) \neq \emptyset$.
Also, Chidume et al. [6] introduced inertial techniques to improve the speed of convergence, inspired by their application in convex minimization problems. By operating in Banach spaces, their work applies to a wider range of mathematical and practical problems where Hilbert space assumptions may not hold. Their approach unifies fixed-point theory and maximal monotone operator techniques, providing a robust framework for solving equilibrium and optimization problems. In the work Harbau and Abdulwahab [10], asymptotically nonexpansive mappings pose specific difficulties in convergence, which their $S-$ iteration process successfully overcomes. By combining inertial and hybrid iteration strategies, they advanced iterative methods for solving equilibrium and fixed-point problems. Their work is motivated by applications in optimization and variational inequalities, where equilibrium problems frequently arise.

Motivated and inspired by the work of Chidume et al [6], and Harbau and Abdulwahab [10], the purpose of this paper is to generalize the process (1.1) from Hilbert space to Banach space by introducing a hybrid inertial $S-$-iteration algorithm for approximating fixed point problem of two quasi-ϕ-asymptotically nonexpansive mappings, maximal monotone operator and system of generalized mixed equilibrium problems in Banach space. our results improve and extend the result of Chidume et al [6], and Harbau and Abdulwahab [10] and some recent results announced in the literature, in the following sense:
1. Our main Theorem incorporates quasi$-\phi-$ asymptotically nonexpansive mappings, which generalize the relatively nonexpansive mappings and asymptotically nonexpansive mappings studied in [6] and [10], respectively. This extension allows the inclusion of a wider class of operators, making the results more comprehensive. 2. Our proposed results achieves strong convergence results in settings that involve both equilibrium problems and maximal monotone operators, which are not directly addressed in [6] and [10], respectively. This adds a layer of complexity and novelty to the results.

2. Preliminaries

Let E be a real Banach space and E^* be the dual space of E. Observe that for any $\{x_n\} \subset E$ and a point $x \in E$, we denotes $x_n \longrightarrow x$ and $x_n \rightharpoonup x$ as strong and weak convergences respectively. A mapping $J : E \longrightarrow 2^{E^*}$ is said to be normalized duality if:

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|\}, \forall x \in E.$$

Let $G := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then E is said to be smooth if the $\lim\limits_{t\to 0} \frac{\| x + ty \| - \| x \|}{t}$ exists for all $x, y \in G$, it is also said to be uniformly smooth if the limit exists uniformly in $x, y \in G$. A Banach space E said to be strictly convex if $\frac{\| x + y \|}{2} < 1$ for all $x, y \in G$ with $\|x\| = \|y\| = 1$ and $x \neq y$ and E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\| x + y \|}{2} \leq 1 - \delta$ for all $x, y \in G$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. For any $\{x_n\} \subset E$, E is said to satisfies Kadec - Klee property if $x_n \rightharpoonup x$ and $\| x_n \| \longrightarrow \| x \|$, then $x_n \longrightarrow x, \quad \forall x \in E$. The function $\delta : [0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\varepsilon) = inf\{1 - \|\frac{x + y}{2}\| : x, y \in G, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}.$$

is called modulus of convexity of E. The function $\rho_E : [0, \infty) \longrightarrow [0, \infty)$ defined by

$$\rho_E(t) = sup\{\frac{\| x + y \| + \| x - y \|}{2} - 1; \|x\| = 1 : x \in G, \|y\| \leq t\}.$$

is called the modulus of smoothness of E. Let E be a smooth Banach space, then a map $\phi : E \times E \longrightarrow \mathbb{R}$ is said to be Lyapunov functional if :

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \forall x, y \in E. \tag{2.1}$$

By considering the framework of Hilbert space H, we observe that (2.1) reduces to $\phi(x, y) = \|x - y\|^2, \forall x, y \in H$. Also for all $x, y, \in E$, from (2.1) we have the following properties :

$$(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2, \tag{2.2}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \tag{2.3}$$

and

$$\phi(x, y) \leq \|x\|\|Jx - Jy\| + \|y\|\|x - y\|. \tag{2.4}$$

Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive real Banach space E. Following the Alber [2], the map $\Pi_C : E \longrightarrow C$ defined by $\hat{x}; = \Pi_C(x)$ such that $\phi(\hat{x}, x) = \inf\limits_{y \in C} \phi(y, x)$ is called generalized projection of x onto C. Furthermore, the generalized projection Π_C and the metric projection P_C are equivalent in a real Hilbert space.

Lemma 2.1. [11] *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim\limits_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim\limits_{n\to\infty} \| x_n - y_n \| = 0$.*

Remark 2.2. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded, then by considering (2.4) it is observe that the converse of Lemma 3.1 is also true.

Lemma 2.3. [2] *Let E be a reflexive, smooth, and strictly convex Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.4. [5] *Let E be a real uniformly smooth and strictly convex Banach space, and C be a nonempty closed convex subset of E. Let $S : C \longrightarrow C$ be a closed and quasi - ϕ- asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty), k_n \longrightarrow 1$. Than, $F(S)$ is a closed convex subset of C.*

Lemma 2.5. [12] *Let E be a smooth, strictly convex and reflexive Banach space and let $N : E \longrightarrow 2^{E^*}$ be a monotone operator. Then N is maximal if and only if $R(J + rN) = E^*$ for all $r > 0$.*

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E with $N : E \longrightarrow 2^{E^*}$ as a monotone operator satisfying

$$D(N) \subset C \subset J^{-1}\big(\bigcap_{r>0} R(J + rN)\big).$$

Then, the resolvent $J_r : C \longrightarrow D(N)$ of N can be define by

$$J_r x = \{z \in D(N) : Jx \in Jz + rNz\}, \forall x \in C.$$

Recall that $J_r x$ consists of one point. Now, for $r > 0$, the Yosida approximation $N_r : C \longrightarrow E^*$ is defined by

$$N_r x = (Jx - JJ_r x)/r, \quad \forall x \in C.$$

Lemma 2.6. [17] *Let E be a smooth Banach space. Then*

$$\phi(v, J^{-1}[\gamma Jt + (1 - \gamma)Jy]) \leq \gamma\phi(v, t) + (1 - \gamma)\phi(v, y), \quad \forall \gamma \in [0, 1], v, t, y \in E.$$

Lemma 2.7. [12, 14] *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E and let $N : E \longrightarrow 2^{E^*}$ be a monotone operator satisfying*

$$D(N) \subset C \subset J^{-1}\big(\bigcap_{r>0} R(J + rN)\big).$$

Observe that for $r > 0$, let J_r and N_r be the resolvent and the Yosida approximation of N, respectively. Then, the following hold:
(i) $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$, *for all* $x \in C$, $u \in N^{-1}0$;
(ii) $\phi(J_r x, N_r x) \in N$, *for all* $x \in C$; *where* $(x, x^*) \in N$ *denotes the value of* x^* *at* $x(x^* \in Nx)$.
(iii) $F(J_r) = N^{-1}0$.

Assumption Z: Consider $f : C \times C \longrightarrow \mathbb{R}$ as a bifunction for solving equilibrium problem which satisfies the following assumptions [4]:
(z_1) $f(x, x) = 0, \forall x \in C$;
(z_2) f is monotone, 1.e, $f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C$;
(z_3) for each $x, y, z \in C, \limsup_{\delta \to 0} f(\delta z + (1 - \delta)x, y) \leq f(x, y)$;
(z_4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.8. [24] *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\| \delta s + (1 - \delta)y \|^2 \leq \delta \| s \|^2 + (1 - \delta) \| y \|^2 - \delta(1 - \delta)g(\| s - y \|),$$

for all $s, y \in B_r(0)$ and $\delta \in [0, 1]$

Lemma 2.9. [4, 21] *Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $f : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying assumptions $(z_1) - (z_4)$. For any given number $r > 0$ and any given point $x \in E$, then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C.$$

Substituting x with $J^{-1}(Jx - rBx)$, then there exists $z \in C$ such that

$$f(z, y) + \langle y - z, Bz\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0. \forall y \in C.$$

where B is a monotone mapping from C into E^*.

Lemma 2.10. [21, 25] *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E. Let $\Theta : C \longrightarrow \mathbb{R}$ be a proper, convex and lower semi-continuous function, $B : C \longrightarrow E^*$ be a continuous and monotone mapping and $f : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the assumptions $(z_1) - (z_4)$. For any given number $r > 0$ and any given point $x \in E$, a mapping $S_r : E \longrightarrow C$ is define by*

$$S_r(x) = \{z \in C : f(z, y) + \Theta(y) - \Theta(z) + \langle Bz, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}, \forall x \in E,$$

for all $x \in C$. The mapping S_r has the following properties:
(q_1) S_r is single-valued;
(q_2) S_r is a firmly nonexpansive - type mapping, for all $x \in E, y \in C$

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \leq \langle S_r x - S_r y, Jx - Jy\rangle$$

(q_3) $F(S_r) = GMEP(f, B, \Theta)$;
(q_4) $GMEP(f, B, \Theta)$ is a closed convex set of C.
(q_5) $\phi(v^*, S_r x) + \phi(S_r x, x) \leq \phi(v^*, x), \quad \forall v^* \in F(S_r), \quad x \in E$.

3. Main result

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $N : E \longrightarrow 2^{E^*}$ be a maximal monotone operator satisfying $D(N) \subset C$ and $J_r = (J + rN)^{-1}J$, for all $r > 0$. Let $f : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying assumptions $(z_1) - (z_4)$, let a nonlinear mapping $B : C \longrightarrow E^*$ be continuous and monotone, and $\Theta : C \longrightarrow \mathbb{R}$ be a convex and lower semi-continuous function. Let $S_i : C \longrightarrow C$, for $i = 1, 2$, be a finite family of uniformly quasi-ϕ-asymptotically nonexpansive mappings with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $k_{n,i} \longrightarrow 1$ as $n \to \infty$ and S_i is L_i-Lipschitz continuous. Assume that $\Gamma := \cap_{i=1}^{2} F(S_i) \cap N^{-1}0 \cap GMEP(f, B, \Theta) \neq \emptyset$. Define the sequence $\{x_n\}$ generated by the algorithm:*

$$\begin{cases} \text{(i)} & x_0, x_1 \in C, \quad C_0 = C, \\ \text{(ii)} & w_n = x_n + \Omega_n(x_n - x_{n-1}), \\ \text{(iii)} & y_n = J^{-1}\big((1 - \mu_n)Jw_n + \mu_n JS_1^n J_{r_n} w_n\big), \\ \text{(iv)} & z_n = J^{-1}\big((1 - \eta_n)JS_1^n J_{r_n} w_n + \eta_n JS_2^n y_n\big), \\ \text{(v)} & u_n = S_{r_n} z_n, \\ \text{(vi)} & C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq k_{n,1}k_{n,2}\phi(u, w_n)\}, \\ \text{(vii)} & x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where:

- $\Omega_n \in (0,1)$,
- $\{\mu_n\}$ *and* $\{\eta_n\}$ *are sequences in* $[0,1]$ *such that* $\liminf_{n\to\infty} \mu_n(1-\mu_n) > 0$ *and* $\liminf_{n\to\infty} \eta_n(1-\eta_n) > 0$,
- $\{r_n\} \subset [d,\infty)$ *for some* $d > 0$,
- $0 < d \leq \eta_n < 1$.

Then, the sequence $\{x_n\}$ *converges strongly to* $\Pi_\Gamma x_0$.

Proof. We divide the proof into six steps as follows:

Step 1 : We show that (vi) is closed and convex, and $\Gamma \subset C_n, \forall n \geq 0$. We observe from (i) that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. We notice that for any $u \in C_n$, we have

$$\begin{aligned} \phi(u,u_n) \leq k_{n,1}k_{n,2}\phi(u,w_n) \iff & (1-k_{n,1}k_{n,2}) \| u \|^2 - 2(1-k_{n,1}k_{n,2})\langle u, Ju_n\rangle \\ + \quad & 2k_{n,1}k_{n,2}\langle u, Jw_n - Ju_n\rangle \leq k_{n,1}k_{n,2} \| w_n \|^2 - \| u_n \|^2 \end{aligned} \tag{3.2}$$

this implies that C_{n+1} is closed and convex. Next, we show that $\Gamma \subset C_n, \quad \forall n \geq 0$. Let $u_n = S_{r_n}z_n$, putting $v_n = J_{r_n}w_n$ and $p \in \Gamma$. Now by Lemma 2.6, Lemma2.7(i) and from the fact that $S_i,\ i = 1,2$, are quasi-ϕ-asymptotically nonexpansive, we obtain the following estimate:

$$\begin{aligned} \phi(p,u_n) &= \phi(p, S_{r_n}z_n) \\ &\leq \phi(p,z_n) \\ &= \phi(p, J^{-1}((1-\eta_n)JS_1^n v_n + \eta_n JS_2^n y_n)) \\ &= \| p \|^2 - 2\langle p, (1-\eta_n)JS_1^n v_n + \eta_n JS_2^n y_n\rangle + \|(1-\eta_n)JS_1^n v_n + \eta_n JS_2^n y_n\|^2 \\ &\leq \| p \|^2 - 2\big(\langle p, (1-\eta_n)JS_1^n v_n\rangle + \langle p, \eta_n JS_2^n y_n\rangle\big) + (1-\eta_n) \| JS_1^n v_n \|^2 + \eta_n \| JS_2^n y_n \|^2 \\ &= (1-\eta_n)\big[\, \| p \|^2 - 2\langle p, JS_1^n v_n\rangle + \|S_1^n v_n\|^2\big] + \eta_n\big[\, \| p \|^2 - 2\langle p, JS_2^n y_n\rangle + \|S_2^n y_n\|^2\big] \\ &= (1-\eta_n)\phi(p, S_1^n v_n) + \eta_n \phi(p, S_2^n y_n) \\ &\leq (1-\eta_n)k_{n,1}\phi(p,v_n) + \eta_n k_{n,2}\phi(p,y_n) \\ &= (1-\eta_n)k_{n,1}\phi(p, J_{r_n}w_n) + \eta_n k_{n,2}\phi(p,y_n) \\ &\leq (1-\eta_n)k_{n,1}\phi(p,w_n) + \eta_n k_{n,2}\phi(p,y_n) \end{aligned} \tag{3.3}$$

Similarly, since $S_i,\ i = 1,2$, are quasi-ϕ-asymptotically nonexpansive mappings, by Lemma 2.6 and Lemma2.7(i), we have

$$\begin{aligned} \phi(p,y_n) &= \phi\big(p, J^{-1}\big((1-\mu_n)Jw_n + \mu_n JS_1^n v_n\big)\big) \\ &= \| p \|^2 - 2\langle p, (1-\mu_n)Jw_n + \mu_n JS_1^n v_n\rangle + \|(1-\mu_n)Jw_n + \mu_n JS_1^n v_n\|^2 \\ &\leq \| p \|^2 - 2\big(\langle p, (1-\mu_n)Jw_n\rangle + \langle p, \mu_n JS_1^n v_n\rangle\big) + (1-\mu_n) \| Jw_n \|^2 \\ &+ \mu_n \| JS_1^n v_n \|^2 \\ &= (1-\mu_n)\big[\, \| p \|^2 - 2\langle p, Jw_n\rangle + \|w_n\|^2\big] + \mu_n\big[\, \| p \|^2 - 2\langle p, JS_1^n v_n\rangle \\ &+ \|S_1^n v_n\|^2\big] \\ &= (1-\mu_n)\phi(p,w_n) + \mu_n \phi(p, S_1^n v_n) \\ &\leq (1-\mu_n)\phi(p,w_n) + \mu_n k_{n,1}\phi(p,v_n) \qquad (3.4) \\ &= (1-\mu_n)\phi(p,w_n) + \mu_n k_{n,1}\phi(p, J_{r_n}w_n) \\ &\leq (1-\mu_n)k_{n,1}\phi(p,w_n) + \mu_n k_{n,1}\phi(p,w_n) \\ &= k_{n,1}\phi(p,w_n). \qquad (3.5) \end{aligned}$$

Therefore, from (3.1) and (3.3), we obtain

$$\begin{aligned} \phi(p,u_n) &\leq (1-\eta_n)k_{n,1}\phi(p,w_n)+\eta_n k_{n,2}\big[k_{n,1}\phi(p,w_n)\big] \\ &= (1-\eta_n)k_{n,1}\phi(p,w_n)+\eta_n k_{n,1}k_{n,2}\phi(p,w_n) \\ &\leq k_{n,2}(1-\eta_n)k_{n,1}\phi(p,w_n)+\eta_n k_{n,1}k_{n,2}\phi(p,w_n) \\ &= k_{n,1}k_{n,2}\big((1-\eta_n)+\eta_n\big)\phi(p,w_n) \\ &= k_{n,1}k_{n,2}\phi(p,w_n), \end{aligned}$$

implies that $p \in C_{n+1}$. This shows that $\Gamma \subset C_{n+1}$ and $\Gamma \subset C_n$ for all $n \geq 0$. Thus, the sequence $\{x_n\}$ is well defined.

Step 2 : We show that (ii)-(v) are bounded and (2.1) is bounded and hence Cauchy. It follows from (vii), and $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$ that

$$\phi(x_n,x_0) \leq \phi(x_{n+1},x_0), \quad \forall n \geq 0.$$

Hence $\{\phi(x_n,x_0)\}$ is nondecreasing. Furthermore we obtain

$$\phi(x_n,x_0) = \phi(\Pi_{C_n}x_0,x_0) \leq \phi(p,x_0)-\phi(p,x_n) \leq \phi(p,x_0).$$

Which give that $\{\phi(x_n,x_0)\}$ is bounded. This implies that $\lim_{n\to\infty}\phi(x_n,x_0)$ exists and from (2.2), we conclude that $\{x_n\}$ is bounded. Also, since $\{\phi(x_n,x_0)\}$ is nondecreasing, $\{\phi(x_n,x_0)\}$ is convergent. Now since $\{x_n\}$ is bounded, this implies that $\{w_n\}$ is bounded. Furthermore by considering (2.2), inequalities (3.1) and (3.3) imply that $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. Next, we show that $\{x_n\}$ is cauchy sequence in C. Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$ for $m > n$, by Lemma 5.1, we have

$$\phi(x_m,x_n) = \phi(x_m,\Pi_{C_n}x_0) \leq \phi(x_m,x_0)-\phi(x_n,x_0) \longrightarrow 0, \quad as\ n,m \longrightarrow \infty.$$

Therefore it follows from Lemma 3.1 that $\lim_{n\to\infty} \parallel x_m - x_n \parallel = 0$. This implies that the $\{x_n\}$ is a cauchy sequence and from the completeness of E, there exists a point $\varpi \in C$ such that

$$\lim_{n\to\infty} x_n = \varpi \tag{3.6}$$

Step 3 : We show that $\varpi \in \cap_{i=1}^{2}F(S_i)$. Now from the definition of $x_n = \Pi_{C_n}x_0$, we have

$$\begin{aligned} \phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{C_n}x_0) \\ &\leq \phi(x_{n+1},x_0)-\phi(\Pi_{C_n}x_0,x_0) \\ &= \phi(x_{n+1},x_0)-\phi(x_n,x_0) \end{aligned}$$

Since $\lim_{n\to\infty}\phi(x_n,x_0)$ exists, then we also have

$$\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0.$$

It follows from Lemma 3.1 that

$$\lim_{n\to\infty} \parallel x_{n+1}-x_n \parallel = 0. \tag{3.7}$$

We observe from the definition of w_n that

$$\parallel w_n - x_n \parallel = \Omega_n \parallel x_n - x_{n-1} \parallel \leq \parallel x_n - x_{n-1} \parallel$$

Using (3.5), we get

$$\lim_{n\to\infty} \| w_n - x_n \| = 0. \tag{3.8}$$

From (3.4) and (3.6), we obtain

$$\lim_{n\to\infty} w_n = \varpi \tag{3.9}$$

By (3.5) and (3.6), we have

$$\lim_{n\to\infty} \| x_{n+1} - w_n \| = 0. \tag{3.10}$$

Since $\{w_n\}$ is bounded and from the Remark 2.2 and (4.1), we conclude that

$$\lim_{n\to\infty} \phi(x_{n+1}, w_n) = 0. \tag{3.11}$$

From the definition of $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_{n+1}, u_n) \leq k_{n,1} k_{n,2} \phi(x_{n+1}, w_n)$$

Using (4.2) and $k_{n,i} \longrightarrow 1$ as $n \longrightarrow \infty$, we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

By Lemma 3.1, we get

$$\lim_{n\to\infty} \| x_{n+1} - u_n \| = 0. \tag{3.12}$$

Taking into account that

$$\| x_n - u_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - u_n \|$$

Using (3.5) and (5.1), lead to

$$\lim_{n\to\infty} \| x_n - u_n \| = 0. \tag{3.13}$$

Also, by triangular inequality, we have

$$\| w_n - u_n \| \leq \| w_n - x_n \| + \| x_n - u_n \|$$

From (3.6) and (3.13), we conclude that

$$\lim_{n\to\infty} \| w_n - u_n \| = 0. \tag{3.14}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \| Jw_n - Ju_n \| = 0. \tag{3.15}$$

Using (3.7) and (3.14), we get that

$$\lim_{n\to\infty} u_n = \varpi. \tag{3.16}$$

On the other hand

$$\phi(x_{n+1}, z_n) \leq k_{n,1}k_{n,2}\phi(x_{n+1}, w_n)$$

By (4.2), we get

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0.$$

From Lemma 3.1, we conclude that

$$\lim_{n\to\infty} \| x_{n+1} - z_n \| = 0. \tag{3.17}$$

Consider the following inequality

$$\| x_n - z_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \|$$

Using (3.5) and (3.17), we obtain

$$\lim_{n\to\infty} \| x_n - z_n \| = 0. \tag{3.18}$$

Also from (3.4) and (3.18)

$$\lim_{n\to\infty} z_n = \varpi. \tag{3.19}$$

It follows from (3.13) and (3.18) that

$$\lim_{n\to\infty} \| u_n - z_n \| = 0.$$

Since J is uniformly norm-to-norm continuous, we have

$$\lim_{n\to\infty} \| Ju_n - Jz_n \| = 0. \tag{3.20}$$

Similarly, since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, it follows that

$$\phi(x_{n+1}, y_n) \leq k_{n,1}k_{n,2}\phi(x_{n+1}, w_n).$$

From (4.2), we conclude that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

By Lemma 3.1, we get

$$\lim_{n\to\infty} \| x_{n+1} - y_n \| = 0. \tag{3.21}$$

Taking into account that

$$\| x_n - y_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - y_n \| .$$

Using (3.5) and (3.21), we obtain

$$\lim_{n\to\infty} \| x_n - y_n \| = 0. \tag{3.22}$$

From (3.4) and (3.22), we get that

$$\lim_{n\to\infty} y_n = \varpi. \tag{3.23}$$

Now, since $p \in \Gamma$ and using Lemma 2.8, we obtain the following estimate

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}((1-\mu_n)Jw_n + \mu_n JS_1^n v_n)) \\
&= \| p \|^2 - 2\langle p, (1-\mu_n)Jw_n + \mu_n JS_1^n v_n\rangle + \|(1-\mu_n)Jw_n + \mu_n JS_1^n v_n \|^2 \\
&\leq \| p \|^2 - 2\langle p, (1-\mu_n)Jw_n\rangle - 2\langle p, \mu_n JS_1^n v_n\rangle + (1-\mu_n) \| Jw_n \|^2 \\
&+ \mu_n \| JS_1^n v_n \|^2 - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|^2) \\
&= (1-\mu_n)\big[\| p \|^2 - 2\langle p, Jw_n\rangle + \|w_n\|^2\big] + \mu_n\big[\| p \|^2 - 2\langle p, JS_1^n v_n\rangle \\
&+ \|S_1^n v_n\|^2\big] - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&= (1-\mu_n)\phi(p, w_n) + \mu_n\phi(p, S_1^n v_n) - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&\leq (1-\mu_n)\phi(p, w_n) + \mu_n k_{n,1}\phi(p, v_n) - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&= (1-\mu_n)\phi(p, w_n) + \mu_n k_{n,1}\phi(p, J_{r_n} w_n) - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&\leq (1-\mu_n)\phi(p, w_n) + \mu_n k_{n,1}\phi(p, w_n) - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&\leq k_{n,1}(1-\mu_n)\phi(p, w_n) + \mu_n k_{n,1}\phi(p, w_n) - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&= k_{n,1}\phi(p, w_n) - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|).
\end{aligned} \tag{3.24}$$

Furthermore, putting (3.24) in (3.1), we have

$$\begin{aligned}
\phi(p, u_n) &\leq (1-\eta_n)k_{n,1}\phi(p, w_n) + \eta_n k_{n,2}\big[k_{n,1}\phi(p, w_n) - \mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|)\big] \\
&= (1-\eta_n)k_{n,1}\phi(p, w_n) + \eta_n k_{n,1}k_{n,2}\phi(p, w_n) - \eta_n k_{n,2}\mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&\leq k_{n,2}(1-\eta_n)k_{n,1}\phi(p, w_n) + \eta_n k_{n,1}k_{n,2}\phi(p, w_n) - \eta_n k_{n,2}\mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&= k_{n,1}k_{n,2}(1-\eta_n)\phi(p, w_n) + \eta_n k_{n,1}k_{n,2}\phi(p, w_n) - \eta_n k_{n,2}\mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \\
&= k_{n,1}k_{n,2}\phi(p, w_n) - \eta_n k_{n,2}\mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|).
\end{aligned}$$

Implies that

$$\eta_n k_{n,2}\mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \leq k_{n,1}k_{n,2}\phi(p, w_n) - \phi(p, u_n).$$

Since $0 < a \leq \eta_n < 1$, then we have

$$a k_{n,2}\mu_n(1-\mu_n)g(\| Jw_n - JS_1^n v_n \|) \leq k_{n,1}k_{n,2}\phi(p, w_n) - \phi(p, u_n). \tag{3.25}$$

On the other hand

$$\begin{aligned}
k_{n,1}k_{n,2}\phi(p, w_n) - \phi(p, u_n) &= (1 + (k_{n,1}k_{n,2} - 1))\phi(p, w_n) - \phi(p, u_n) \\
&= \phi(p, w_n) - \phi(p, u_n) + (k_{n,1}k_{n,2} - 1)\phi(p, w_n) \\
&= \| p \|^2 - 2\langle p, Jw_n\rangle + \|w_n\|^2 - \big[\|p\|^2 - 2\langle p, Ju_n\rangle + \|u_n\|^2\big] \\
&+ (k_{n,1}k_{n,2} - 1)\big[\|p\|^2 - 2\langle p, Jw_n\rangle + \|w_n\|^2\big] \\
&= \|w_n\|^2 - \|u_n\|^2 - 2\langle p, Jw_n - Ju_n\rangle \\
&+ (k_{n,1}k_{n,2} - 1)\big[\|p\|^2 - 2\langle p, Jw_n\rangle + \|w_n\|^2\big] \\
&\leq \| w_n - u_n \| (\| w_n \| + \| u_n \|) + 2 \| p \| \| Jw_n - Ju_n \| \\
&+ (k_{n,1}k_{n,2} - 1)\big[\|p\|^2 - 2\langle p, Jw_n\rangle + \|w_n\|^2\big].
\end{aligned} \tag{3.26}$$

Since $k_{n,i} \longrightarrow 1$ as $n \longrightarrow \infty$, using (3.14) and (3.15) in (3.26), we conclude that

$$\lim_{n\to\infty}\big(k_{n,1}k_{n,2}\phi(p,w_n) - \phi(p,u_n)\big) = 0. \tag{3.27}$$

Also, since $\liminf_{n\to\infty}\mu_n(1-\mu_n) > 0$, using (3.27) it follows from (3.25) that

$$\lim_{n\to\infty} g(\| Jw_n - JS_1^n v_n \|) = 0.$$

By the property of g, we obtain

$$\lim_{n\to\infty} \| Jw_n - JS_1^n v_n \| = 0. \tag{3.28}$$

From uniform continuity of J^{-1} on bounded sets, we get that

$$\lim_{n\to\infty} \| w_n - S_1^n v_n \| = 0. \tag{3.29}$$

We notice that

$$\begin{aligned} \| Jy_n - Jw_n \| &= \| (1-\mu_n)Jw_n + \mu_n JS_1^n v_n - Jw_n \| \\ &= \| \mu_n (JS_1^n v_n - Jw_n) \| \\ &\leq \mu_n \| JS_1^n v_n - Jw_n \| . \end{aligned}$$

It follows from (3.28) that

$$\lim_{n\to\infty} \| Jy_n - Jw_n \| = 0. \tag{3.30}$$

By the uniform continuity of J^{-1} on bounded sets, we obtain

$$\lim_{n\to\infty} \| y_n - w_n \| = 0. \tag{3.31}$$

From (3.2), we have

$$\frac{1}{k_{n,i}\mu_n}\big(\phi(p,y_n) - (1-\mu_n)\phi(p,w_n)\big) \leq \phi(p,v_n). \tag{3.32}$$

Putting $v_n = J_{r_n}w_n$, using Lemma 2.7(i), we obtain

$$\phi(v_n,w_n) = \phi(J_{r_n}w_n,w_n) \leq \phi(p,w_n) - \phi(p,J_{r_n}w_n) = \phi(p,w_n) - \phi(p,v_n).$$

Therefore, by using (3.32), we have

$$\begin{aligned} \phi(v_n,w_n) &\leq \phi(p,w_n) - \phi(p,v_n) \\ &\leq \phi(p,w_n) - \frac{1}{k_{n,1}\mu_n}\big[\phi(p,y_n) - (1-\mu_n)\phi(p,w_n)\big] \\ &= (1 - \frac{1}{k_{n,1}})\phi(p,w_n) + \frac{1}{k_{n,1}\mu_n}\big[\phi(p,w_n) - \phi(p,y_n)\big] \\ &= (1 - \frac{1}{k_{n,1}})\phi(p,w_n) + \frac{1}{k_{n,1}\mu_n}\big[\, \| w_n \|^2 - \| y_n \|^2 - 2\langle p, Jw_n - Jy_n\rangle\big] \\ &\leq (1 - \frac{1}{k_{n,1}})\phi(p,w_n) + \frac{1}{k_{n,1}\mu_n}\big[|\|w_n\| - \|y_n\||(\|w_n\| + \|y_n\|) + 2\|p\|\|Jw_n - Jy_n\|\big] \\ &\leq (1 - \frac{1}{k_{n,1}})\phi(p,w_n) + \frac{1}{k_{n,1}\mu_n}\big[\|w_n - y_n\|(\|w_n\| + \|y_n\|) + 2\|p\|\|Jw_n - Jy_n\|\big]. \end{aligned}$$

Since $k_{n,i} \longrightarrow 1$ as $n \longrightarrow \infty$, it follows from (3.30) and (3.31) that

$$\lim_{n\to\infty} \phi(v_n, w_n) = 0.$$

From Lemma 3.1, we get

$$\lim_{n\to\infty} \| v_n - w_n \| = 0. \tag{3.33}$$

By triangular inequality, we have

$$\| v_n - x_n \| \leq \| v_n - w_n \| + \| w_n - x_n \|$$

Taking the advantage of (3.6) and (3.33), we obtain

$$\lim_{n\to\infty} \| v_n - x_n \| = 0. \tag{3.34}$$

Taking into account that

$$\| v_n - S_1^n v_n \| \leq \| v_n - w_n \| + \| w_n - S_1^n v_n \|$$

Using (3.29) and (3.33), we conclude that

$$\lim_{n\to\infty} \| v_n - S_1^n v_n \| = 0. \tag{3.35}$$

Notice that

$$\| x_n - S_1^n v_n \| \leq \| v_n - S_1^n v_n \| + \| v_n - x_n \| .$$

By (3.34) and (3.35), we get

$$\lim_{n\to\infty} \| x_n - S_1^n v_n \| = 0. \tag{3.36}$$

Also

$$\begin{aligned} \| x_n - S_1^n x_n \| &= \| x_n - S_1^n v_n + S_1^n v_n - S_1^n x_n \| \\ &\leq \| x_n - S_1^n v_n \| + \| S_1^n v_n - S_1^n x_n \| \\ &\leq \| x_n - S_1^n v_n \| + L_1 \| v_n - x_n \| . \end{aligned}$$

Using (3.34) and (3.36), we have

$$\lim_{n\to\infty} \| x_n - S_1^n x_n \| = 0. \tag{3.37}$$

Furthermore

$$\begin{aligned} \| x_{n+1} - S_1^n x_{n+1} \| &\leq \| x_{n+1} - x_n \| + \| x_n - S_1^n x_n \| + \| S_1^n x_n - S_1^n x_{n+1} \| \\ &\leq \| x_{n+1} - x_n \| + \| x_n - S_1^n x_n \| + L_1 \| x_n - x_{n+1} \| \\ &= (1 + L_1) \| x_n - x_{n+1} \| + \| x_n - S_1^n x_n \| . \end{aligned}$$

Using (3.5) and (3.37), we obtain

$$\lim_{n\to\infty} \| x_{n+1} - S_1^n x_{n+1} \| = 0. \tag{3.38}$$

Now

$$\begin{aligned}\| x_{n+1} - S_1 x_{n+1} \| &= \| x_{n+1} + S_1^{n+1} x_{n+1} - S_1^{n+1} x_{n+1} - S_1 x_{n+1} \| \\ &\leq \| x_{n+1} - S_1^{n+1} x_{n+1} \| + \| S_1 x_{n+1} - S_1^{n+1} x_{n+1} \| \\ &= \| x_{n+1} - S_1^{n+1} x_{n+1} \| + \| S_1 x_{n+1} - S_1(S_1^n x_{n+1}) \| \\ &\leq \| x_{n+1} - S_1^{n+1} x_{n+1} \| + L_1 \| x_{n+1} - S_1^n x_{n+1} \| .\end{aligned}$$

Taking the advantage of (3.37) and (3.38), we conclude that

$$\lim_{n\to\infty} \| x_n - S_1 x_n \| = 0. \tag{3.39}$$

Similarly, since $p \in \Gamma$, $u_n = S_{r_n} z_n$, by using (3.3) and Lemma 2.8. we obtain the following estimate

$$\begin{aligned}\phi(p, u_n) &= \phi(p, S_{r_n} z_n) \\ &\leq \phi\big(p, J^{-1}((1-\eta_n) J S_1^n v_n + \eta_n J S_2^n y_n)\big) \\ &= \| p \|^2 - 2\langle p, (1-\eta_n) J S_1^n v_n + \eta_n J S_2^n y_n \rangle + \|(1-\eta_n) J S_1^n v_n + \eta_n J S_2^n y_n \|^2 \\ &\leq \| p \|^2 - 2\langle p, (1-\eta_n) J S_1^n v_n \rangle - 2\langle p, \eta_n J S_2^n y_n \rangle + (1-\eta_n) \| J S_1^n v_n \|^2 \\ &+ \eta_n \| J S_2^n y_n \|^2 - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &= (1-\eta_n)\big[\| p \|^2 - 2\langle p, J S_1^n v_n \rangle + \|S_1^n v_n\|^2\big] + \eta_n \big[\| p \|^2 - 2\langle p, J S_2^n y_n \rangle + \|S_2^n y_n\|^2\big] \\ &- \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &= (1-\eta_n)\phi(p, S_1^n v_n) + \eta_n \phi(p, S_2^n y_n) - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &\leq (1-\eta_n) k_{n,1} \phi(p, v_n) + \eta_n k_{n,2} \phi(p, y_n) - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &= (1-\eta_n) k_{n,1} \phi(p, J_{r_n} w_n) + \eta_n k_{n,2} \phi(p, y_n) - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &\leq (1-\eta_n) k_{n,1} \phi(p, w_n) + \eta_n k_{n,2} \phi(p, y_n) - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &\leq (1-\eta_n) k_{n,1} \phi(p, w_n) + \eta_n k_{n,2} [k_{n,1} \phi(p, w_n)] - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &= (1-\eta_n) k_{n,1} \phi(p, w_n) + \eta_n k_{n,1} k_{n,2} \phi(p, w_n) - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &\leq k_{n,2}(1-\eta_n) k_{n,1} \phi(p, w_n) + \eta_n k_{n,1} k_{n,2} \phi(p, w_n) - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &= (1-\eta_n) k_{n,1} k_{n,2} \phi(p, w_n) + \eta_n k_{n,1} k_{n,2} \phi(p, w_n) - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \\ &= k_{n,1} k_{n,2} \phi(p, w_n) - \eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|).\end{aligned}$$

Implies that

$$\eta_n (1-\eta_n) g(\| J S_1^n v_n - J S_2^n y_n \|) \leq k_{n,1} k_{n,2} \phi(p, w_n) - \phi(p, u_n.) \tag{3.40}$$

Using (3.27) and $\liminf_{n\to\infty} \eta_n (1-\eta_n) > 0$, It follows from (3.40) that

$$\lim_{n\to\infty} g(\| J S_1^n v_n - J S_2^n y_n \|) = 0.$$

From the properties of the function g, we obtain

$$\lim_{n\to\infty} \| J S_1^n v_n - J S_2^n y_n \| = 0.$$

Taking the advantage of J^{-1} as uniformly continuity on bounded sets, we get

$$\lim_{n\to\infty} \| S_1^n v_n - S_2^n y_n \| = 0. \tag{3.41}$$

Observe that

$$\begin{aligned}\| x_n - S_2^n x_n \| &\le \| x_n - S_1^n v_n \| + \| S_1^n v_n - S_2^n y_n \| + \| S_2^n y_n - S_2^n x_n \| \\ &\le \| x_n - S_1^n v_n \| + \| S_1^n v_n - S_2^n y_n \| + L_2 \| y_n - x_n \| .\end{aligned}$$

Using (3.22), (3.36) and (3.41), we obtain

$$\lim_{n\to\infty} \| x_n - S_2^n x_n \| = 0. \tag{3.42}$$

Also

$$\begin{aligned}\| x_{n+1} - S_2^n x_{n+1} \| &\le \| x_{n+1} - x_n \| + \| x_n - S_2^n x_n \| + \| S_2^n x_n - S_2^n x_{n+1} \| \\ &\le \| x_{n+1} - x_n \| + \| x_n - S_2^n x_n \| + L_2 \| x_n - x_{n+1} \| \\ &= (1 + L_2) \| x_{n+1} - x_n \| + \| x_n - S_2^n x_n \| .\end{aligned}$$

By (3.5) and (3.42), we have

$$\lim_{n\to\infty} \| x_{n+1} - S_2^n x_{n+1} \| = 0. \tag{3.43}$$

Similarly

$$\begin{aligned}\| x_{n+1} - S_2 x_{n+1} \| &= \| x_{n+1} + S_2^{n+1} x_{n+1} - S_2^{n+1} x_{n+1} - S_2 x_{n+1} \| \\ &\le \| x_{n+1} - S_2^{n+1} x_{n+1} \| + \| S_2 x_{n+1} - S_2^{n+1} x_{n+1} \| \\ &= \| x_{n+1} - S_2^{n+1} x_{n+1} \| + \| S_2 x_{n+1} - S_2 (S_2^n x_{n+1}) \| \\ &= \| x_{n+1} - S_2^{n+1} x_{n+1} \| + L_2 \| x_{n+1} - S_2^n x_{n+1} \| .\end{aligned}$$

From the idea of (3.42) and (3.43), we conclude that

$$\lim_{n\to\infty} \| x_n - S_2 x_n \| = 0. \tag{3.44}$$

Now, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \varpi$ as $j \to \infty$. Furthermore, it follows from (3.6), (3.13), (3.22) and (3.34) that $w_{n_j} \rightharpoonup \varpi$, $u_{n_j} \rightharpoonup \varpi$, $y_{n_j} \rightharpoonup \varpi$ and $v_{n_j} \rightharpoonup \varpi$ as ($j \longrightarrow \infty$) respectively.
Then, it follows from (3.39) and (3.44) that

$$\lim_{n\to\infty} \| x_n - S_i x_n \| = 0, \; for i = 1, 2.$$

Implies that $S_i \varpi = \varpi$, for $i = 1, 2$. Hence $\varpi \in \cap_{i=1}^2 F(S_i)$.

Step 4 : We show that $\varpi \in N^{-1}0$. It follows from the uniform continuity of J on bounded sets and (3.33) that

$$\lim_{n\to\infty} \| J v_n - J w_n \| = 0.$$

Since $r_n \ge d$, so we have

$$\lim_{n\to\infty} \frac{1}{r_n} \| J w_n - J v_n \| = 0.$$

Therefore

$$\lim_{n\to\infty} \| N_{r_n} w_n \| = \lim_{n\to\infty} \frac{1}{r_n} \| J w_n - J v_n \| = 0.$$

It follows from the fact that N is monotone and by Lemma 2.7(ii), we have

$$\langle z - v_n, z^* - N_{r_n} w_n \rangle \geq 0, \quad \forall n \geq 0, \ (z, z^*) \in N.$$

This implies that

$$\langle z - v_{n_j}, z^* - N_{r_{n_j}} w_{n_j} \rangle = \langle z - \varpi, z^* \rangle \geq 0.$$

Since N is maximal monotone, we have $\varpi \in N^{-1}0$.

Step 5 : We show that $\varpi \in GMEP(f, B, \Theta)$. By (3.20) and from the assumption that $r_n \geq d$ and $d > 0$, we obtain

$$\lim_{n\to\infty} \frac{\| Ju_n - Jz_n \|}{r_n} = 0. \tag{3.45}$$

Since $u_n = S_{r_n} z_n$, we have

$$\tau(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n \rangle \geq 0, \ \forall y \in C.$$

Where

$$\tau(u_n, y) = f(u_n, y) + \Theta(y) - \Theta(u_n) + \langle Bu_n, y - u_n \rangle$$

By applying assumption (z_2), we get

$$\frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n \rangle \geq -\tau(u_n, y) \geq \tau(y, u_n), \ \forall y \in C.$$

It follows from (3.45), (z_4) and $u_n \longrightarrow \varpi$ as $n \to \infty$ that

$$\tau(y, \varpi) \leq 0, \ \forall y \in C.$$

Consider $y_\delta = \delta y + (1-\delta)\varpi$, for $\delta \in (0, 1]$ and $y \in C$. This implies that $y_\delta \in C$ and $\tau(y_\delta, \varpi) \leq 0$. Hence, it follows from (z_1) and (z_4) that

$$\begin{aligned} 0 &= \tau(y_\delta, y_\delta) \\ &\leq \delta\tau(y_\delta, y) + (1-\delta)\tau(y_\delta, \varpi) \\ &\leq \tau(y_\delta, y) \end{aligned}$$

So

$$\tau(y_\delta, y) \geq 0, \ \forall y \in C.$$

Taking the advantage of (z_3) and limit as $\delta \longrightarrow 0$, we conclude that

$$\tau(\varpi, y) \geq 0, \ \forall y \in C.$$

This implies that $\varpi \in GMEP(f, B, \Theta)$. Hence $\varpi \in \cap_{i=1}^{2} F(S_i) \cap N^{-1}0 \cap GMEP(f, B, \Theta)$.

Step 6 : We show that $x_n \longrightarrow \varpi = \Pi_\Gamma x_0$. Now Setting $q^* = \Pi_\Gamma x_0$. From the fact that

$x_n = \Pi_{C_n} x_0$ and $\Gamma \subset C_n$, $\forall n \geq 0$, we have $\phi(x_n, x_0) \leq \phi(q^*, x_0)$. Since the norm is weakly lower semi-continuous, we have the following estimate

$$\begin{aligned} \phi(\varpi, x_0) &= \parallel \varpi \parallel^2 - 2\langle \varpi, Jx_0 \rangle + \|x_0\|^2 \\ &\leq \liminf_{j\to\infty} \big(\parallel x_{n_j} \parallel^2 - 2\langle x_{n_j}, Jx_0 \rangle + \|x_0\|^2 \big) \\ &\leq \liminf_{j\to\infty} \phi(x_{n_j}, x_0) \\ &\leq \limsup_{j\to\infty} \phi(x_{n_j}, x_0) \\ &\leq \phi(q^*, x_0). \end{aligned} \tag{3.46}$$

So

$$\phi(q^*, x_0) \leq \phi(u, x_0), \quad \forall u \in \Gamma. \tag{3.47}$$

Therefore $\phi(\varpi, x_0) = \phi(q^*, x_0)$. From the uniqueness of $\Pi_\Gamma x_0$, we have $\varpi = q^*$. Finally we show that $x_{n_j} \longrightarrow \varpi$, as $j \to \infty$. From (3.46) and (3.47), we conclude that $\phi(x_{n_j}, x_0) \longrightarrow \phi(\varpi, x_0)$, as $j \to \infty$. Therefore, $\parallel x_{n_j} \parallel \longrightarrow \parallel \varpi \parallel$, as $j \to \infty$. Taking the advantage of Kadec-Klee property of E, we obtain that $x_{n_j} \longrightarrow \varpi$, as $j \to \infty$. Hence $x_n \longrightarrow \varpi = \Pi_\Gamma x_0$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $N : E \longrightarrow 2^{E^*}$ be a maximal monotone operators satisfying $D(N) \subset C$ and $J_r = (J + rN)^{-1}J$, for all $r > 0$. Let $f : C \times C \longrightarrow \mathbb{R}$ be a bifunctions satisfying assumptions $(z_1) - (z_4)$, let a nonlinear mapping $B : C \longrightarrow E^*$ be a continuous and monotone, and $\Theta : C \longrightarrow \mathbb{R}$ be a convex and lower semi-continuous function. Let $S_i : C \longrightarrow C, i = 1, 2,$ be a finite family of uniformly quasi-ϕ-nonexpansive mappings. Assume that $\Gamma := \cap_{i=1}^2 F(S_i) \cap N^{-1}0 \cap GMEP(f, B, \Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0, x_1 \in C, \quad C_0 = C, \\ w_n = x_n + \Omega_n(x_n - x_{n-1}), \\ y_n = J^{-1}((1-\mu_n)Jw_n + \mu_n JS_1 J_{r_n} w_n), \\ z_n = J^{-1}((1-\eta_n)JS_1^n J_{r_n} w_n + \eta_n JS_2 y_n), \\ u_n = S_{r_n} z_n, \\ C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq \phi(u, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\Omega_n \subset (0,1)$, $\{\mu_n\}$ and $\{\eta_n\}$ are sequence in $[0,1]$ such that $\liminf_{n\to\infty} \mu_n(1-\mu_n) > 0$, $\liminf_{n\to\infty} \eta_n(1-\eta_n) > 0$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Suppose that $0 < a \leq \eta_n < 1$, then $\{x_n\}$ converges strongly to $\Pi_\Gamma x_0$.

Corollary 3.3. *Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $f : C \times C \longrightarrow \mathbb{R}$ be a bifunctions satisfying assumptions $(z_1) - (z_4)$, let a nonlinear mapping $B : C \longrightarrow E^*$ be a continuous and monotone, and $\Theta : C \longrightarrow \mathbb{R}$ be a convex and lower semi-continuous function. Let $S_i : C \longrightarrow C, i = 1, 2,$ be a finite family of uniformly quasi-ϕ-asymptotically nonexpansive mappings mappings with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $k_{n,i} \longrightarrow 1$ as $n \to \infty$ and S_i is L_i- Lipschitz continuous. Assume that $\Gamma := \cap_{i=1}^2 F(S_i) \cap GMEP(f, B, \Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following* *algorithm:*

$$\begin{cases} x_0, x_1 \in C, \quad C_0 = C, \\ w_n = x_n + \Omega_n(x_n - x_{n-1}), \\ y_n = J^{-1}((1-\mu_n)Jw_n + \mu_n JS_1^n w_n), \\ z_n = J^{-1}((1-\eta_n)JS_1^n w_n + \eta_n JS_2^n y_n), \\ u_n = S_{r_n} z_n, \\ C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq k_{n,1}k_{n,2}\phi(u, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\Omega_n \subset (0,1)$, $\{\mu_n\}$ *and* $\{\eta_n\}$ *are sequence in* $[0,1]$ *such that* $\liminf_{n\to\infty}\mu_n(1-\mu_n) > 0$, $\liminf_{n\to\infty}\eta_n(1-\eta_n) > 0$ *and* $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$. *Suppose that* $0 < a \leq \eta_n < 1$, *then* $\{x_n\}$ *converges strongly to* $\Pi_\Gamma x_0$.

Corollary 3.4. *Let* C *be a nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space* E. *Let* $f : C \times C \longrightarrow \mathbb{R}$ *be a bifunctions satisfying assumptions* $(z_1)-(z_4)$, *let a nonlinear mapping* $B : C \longrightarrow E^*$ *be a continuous and monotone, and* $\Theta : C \longrightarrow \mathbb{R}$ *be a convex and lower semi-continuous function. Let* $S_i : C \longrightarrow C, i = 1, 2,$ *be a finite family of uniformly quasi-*ϕ*-nonexpansive mappings. Assume that* $\Gamma := \cap_{i=1}^2 F(S_i) \cap GMEP(f, B, \Theta) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} x_0, x_1 \in C, \quad C_0 = C, \\ w_n = x_n + \Omega_n(x_n - x_{n-1}), \\ y_n = J^{-1}((1-\mu_n)Jw_n + \mu_n JS_1 w_n), \\ z_n = J^{-1}((1-\eta_n)JS_1 w_n + \eta_n JS_2 y_n), \\ u_n = S_{r_n} z_n, \\ C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq \phi(u, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\Omega_n \subset (0,1)$, $\{\mu_n\}$ *and* $\{\eta_n\}$ *are sequence in* $[0,1]$ *such that* $\liminf_{n\to\infty}\mu_n(1-\mu_n) > 0$, $\liminf_{n\to\infty}\eta_n(1-\eta_n) > 0$ *and* $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$. *Suppose that* $0 < a \leq \eta_n < 1$, *then* $\{x_n\}$ *converges strongly to* $\Pi_\Gamma x_0$.

4. A Numerical Example

Maximal monotone operator $N : E \to 2^{E^*}$ satisfying the required properties can be constructed as follows:
Consider $E = \mathbb{R}$, and let $C = [0, 1]$, a closed and convex subset of $\mathbb{R}$.
Define the set-valued operator $N : \mathbb{R} \to 2^{\mathbb{R}}$ by:

$$N(x) = \begin{cases} [0, \infty), & x = 1, \\ [-\infty, 0], & x = 0, \\ \{0\}, & x \in (0, 1). \end{cases}$$

Verification:
Maximal Monotonicity:
- N is monotone because for any $x, y \in \mathbb{R}$ and $x^* \in N(x)$, $y^* \in N(y)$, we have:

$$\langle x - y, x^* - y^* \rangle = (x - y)(x^* - y^*) \geq 0,$$

which is satisfied due to the structure of N.
- N is maximal because there is no monotone operator M with a graph strictly containing the

graph of N.
Domain Inclusion: The domain of N is $D(N) = [0, 1]$, which is included in C.
Resolvent Operator:
The resolvent operator $J_r = (J + rN)^{-1}J$, for $r > 0$, maps $x \in C$ to the unique point $z \in D(N)$ such that:

$$Jx \in Jz + rNz.$$

For this N, J_r corresponds to a projection-like mapping ensuring single-valuedness on C.
Yosida Approximation: The Yosida approximation of N is given by:

$$N_r(x) = \frac{Jx - JJ_r(x)}{r},$$

which ensures $N_r(x) \in N(J_r(x))$.

This operator N meets the maximal monotone condition, domain restriction, and resolvent requirements as outlined in the main result.

A bifunction $f : C \times C \to \mathbb{R}$ that satisfies assumptions $(z_1) - (z_4)$, based on the framework in the main result, is as follows:
Let $C = [0, 1] \subset \mathbb{R}$. Define:

$$f(x, y) = x^2 - xy.$$

Verification of Assumptions:
Assumption (z_1): $f(x, x) = 0$ for all $x \in C$.

$$f(x, x) = x^2 - x \cdot x = 0.$$

Assumption (z_2): f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$.

$$f(x, y) + f(y, x) = (x^2 - xy) + (y^2 - yx) = x^2 + y^2 - 2xy = (x - y)^2 \geq 0.$$

Assumption (z_3): For each $x, y, z \in C$,

$$\limsup_{\delta \to 0} f(\delta z + (1 - \delta)x, y) \leq f(x, y).$$

Let $x, y, z \in C$. Compute:

$$f(\delta z + (1 - \delta)x, y) = (\delta z + (1 - \delta)x)^2 - (\delta z + (1 - \delta)x)y.$$

Expanding and taking the limit as $\delta \to 0$:

$$\limsup_{\delta \to 0} f(\delta z + (1 - \delta)x, y) = x^2 - xy = f(x, y).$$

Assumption (z_4): For each $x \in C$, the mapping $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Fix $x \in C$. The function $y \mapsto f(x, y) = x^2 - xy$ is affine in y, hence it is convex. It is also continuous, and therefore lower semicontinuous.
Thus, $f(x, y) = x^2 - xy$ satisfies assumptions $(z_1) - (z_4)$.

A nonlinear mapping $B : C \to E^*$ that is continuous and monotone, given the context of the main result, is as follows:
Let $C = [0, 1] \subset \mathbb{R}$ (a closed, convex subset of $\mathbb{R}$) and define $B : C \to \mathbb{R}$ by:

$$B(x) = x^3.$$

Verification of Properties:
Monotonicity: B is monotone if for all $x, y \in C$,

$$\langle B(x) - B(y), x - y \rangle \geq 0.$$

Compute:

$$\langle B(x) - B(y), x - y \rangle = (x^3 - y^3)(x - y).$$

Factorize:

$$(x^3 - y^3)(x - y) = (x - y)^2(x^2 + xy + y^2).$$

Since $(x - y)^2 \geq 0$ and $x^2 + xy + y^2 \geq 0$, it follows that $\langle B(x) - B(y), x - y \rangle \geq 0$, so B is monotone.
Continuity: $B(x) = x^3$ is a polynomial function and therefore continuous on $\mathbb{R}$, particularly on C.
This $B(x)$ satisfies the requirements for continuity and monotonicity within the specified framework.

An example of $\Theta : C \to \mathbb{R}$, which is convex and lower semi-continuous, based on the main result, is:

Let $C = [0, 1] \subset \mathbb{R}$. Define:

$$\Theta(x) = x^2.$$

Verification:
Convexity: To check convexity, consider any $x, y \in C$ and $\lambda \in [0, 1]$. Compute:

$$\Theta(\lambda x + (1 - \lambda)y) = (\lambda x + (1 - \lambda)y)^2.$$

Expanding:

$$(\lambda x + (1 - \lambda)y)^2 = \lambda^2 x^2 + 2\lambda(1 - \lambda)xy + (1 - \lambda)^2 y^2.$$

Since $\Theta(x) = x^2$ is quadratic, and the quadratic function is convex, we have:

$$\Theta(\lambda x + (1 - \lambda)y) \leq \lambda\Theta(x) + (1 - \lambda)\Theta(y).$$

Lower Semi-Continuity: The function $\Theta(x) = x^2$ is continuous on $\mathbb{R}$, and continuity implies lower semi-continuity. Thus, $\Theta(x) = x^2$ satisfies the requirements of being convex and lower semi-continuous on $C = [0, 1]$.

Let $C = [0, 1] \subset \mathbb{R}$ (a closed and convex subset of $\mathbb{R}$).
Define the mapping $S_{1,2} : C \to C$ by:

$$S_1(x) = \frac{x}{2} + \frac{1}{4} \textit{ and } S_2(x) = \frac{3y_n}{4} + \frac{1}{8}.$$

Now, consider the Lyapunov functional:

$$\phi(p, x) = (x - p)^2.$$

Verification of Uniformly Quasi-ϕ-Asymptotically Nonexpansiveness:
The fixed point of S_1 is found by solving $S_1(x) = x$:

$$x = \frac{x}{2} + \frac{1}{4} \implies x = \frac{1}{2}.$$

Thus, $F(S_1) = \{\frac{1}{2}\}$.
Quasi-ϕ-Asymptotic Nonexpansiveness:
For all $p \in F(S_1)$ and $x \in C$:

$$\phi(p, S_1(x)) = (S_1(x) - p)^2 = \left(\frac{x}{2} + \frac{1}{4} - \frac{1}{2}\right)^2 = \left(\frac{x}{2} - \frac{1}{4}\right)^2.$$

Similarly:

$$\phi(p, x) = (x - p)^2 = \left(x - \frac{1}{2}\right)^2.$$

Clearly, there exists a sequence $k_n = 1 + \frac{1}{n}$ such that:

$$\phi(p, S_1(x)) \leq k_n \phi(p, x), \quad \text{for all } n \geq 1 \text{ and } x \in C.$$

Uniform Quasi-ϕ-Asymptotic Behavior: Since $k_n \to 1$ as $n \to \infty$, S_1 (and similarly S_2) is uniformly quasi-ϕ-asymptotically nonexpansive.

To numerically verify the main theorem in this paper, let us proceed step by step with explicit examples and computations:

Algorithm Parameters
- Relaxation Parameters:

$$\Omega_n = \frac{1}{n+2}, \quad \mu_n = \frac{1}{n+3}, \quad \eta_n = \frac{1}{n+4}.$$

- Step Sizes: $r_n = 1 + \frac{1}{n}$.
- Lyapunov Functional: $\phi(p, x) = (x - p)^2$.
- Projection Operator: For $x_0 \in C$, $\Pi_{C_{n+1}}(x_0)$ is the Euclidean projection onto C_{n+1}, computed as:

$$\Pi_{C_{n+1}}(x_0) = \arg \min_{u \in C_{n+1}} \|u - x_0\|^2.$$

Set $x_0 = 0.5$, $x_1 = 0.6$ with initial subset $C_0 = [0, 1]$. We now substitute into the iterative algorithm:

$$w_n = x_n + \frac{1}{n+2}(x_n - x_{n-1})$$

$$y_n = (1 - \frac{1}{n+3})w_n + \frac{1}{n+3}(\frac{w_n}{2} + \frac{1}{4})$$

$$z_n = (1 - \frac{1}{n+4})(\frac{w_n}{2} + \frac{1}{4}) + \frac{1}{n+4}(\frac{3y_n}{4} + \frac{1}{8})$$

$$u_n = \frac{z_n}{2} + \frac{1}{4},$$

$C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq k_{n,1} k_{n,2} \phi(u, w_n)\}$, with $k_{n,1} = k_{n,2} = 1 + \frac{1}{n}$

$$x_{n+1} = \min\left(\max(u_n, C_0(1)), C_0(2)\right).$$

Using Python, we simulate the above steps for $n = 0, 1, 2, \ldots, 10$ with the specified parameters and compute the iterates $\{x_n\}$. The sequence $\{x_n\}$ converges quickly to a fixed point $x_n \approx 0.6157$ in $\Gamma = \bigcap_{i=1}^{2} F(S_i) \cap N^{-1}(0) \cap GMEP(f, B, \Theta)$. See figure 1 below.

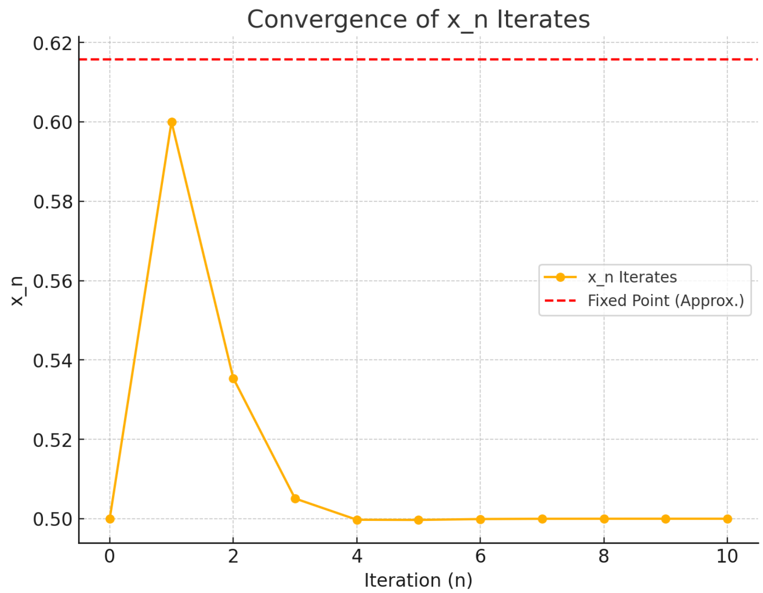

FIGURE 3.

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Geometry of some completely integrable Hamiltonian systems

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Abstract. The paper studies the geometry of a Liouville foliation generated by a completely integrable Hamiltonian system. It is shown that regular leaves are two dimensional submanifolds with zero Gaussian curvature and zero Gaussian torsion. It is studied a geometry of the distribution which generates orthogonal foliation to the Liouville foliation.

Keywords: Poisson bracket, Hamiltonian system, Liouville foliation, Gauss curvature, Gauss torsion.

MSC (2020): 37J35, 53A07, 53D17, 57R30, 57R25.

1. Introduction

The basic concept of a Hamiltonian system of differential equations forms the basis of much of the more advanced work in classical mechanics, including motions of rigid bodies, celestial mechanics, quantization theory and so on. More recently, Hamiltonian methods have become increasingly important in the study of the equations of continuum mechanics, including fluids, plasmas and elastic media [1, 6].

We are interested on the geometry of the Liouville foliation generated by Hamiltonian systems.

In this paper the geometry of the Liouville foliation generated by a completely integrable Hamiltonian system is studied.

1.1. Preliminaries.

Definition 1.1. [1]. A *Poisson bracket* on a smooth manifold M is an operation that assigns a smooth real-valued function {F,H} on M to each pair F, H of smooth, real-valued functions, with the basic properties:

(a) *Bilineality*:

$$\{cF + c' P, H\} = c\{F, H\} + c'\{P, H\},$$

$$\{F, cH + c' P\} = c\{F, H\} + c'\{F, P\}, \quad c, c' \in \mathbb{R};$$

(b) *Skew-Symmetry*:

$$\{F, H\} = -\{H, F\};$$

(c) *Jacobian Identity*:

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0;$$

(d) *Leibnitz' Rule*:

$$\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}.$$

(Here P is an arbitrary smooth real-valued function and $\cdot$ denotes the ordinary multiplication of real-valued functions.)

A manifold M with a Poisson bracket is called as a *Poisson manifold* and the Poisson bracket defines a *Poisson structure* on M.

Example 1.2. Let M be the Euclidean space $\mathbb{R}^m$, $m = 2n + l$ with coordinates
$(p, q, z) = (p^1, ..., p^n, q^1, ..., q^n, z^1, ..., z^l)$. If $F(p, q, z)$ and $H(p, q, z)$ are smooth functions, we define their Poisson bracket to be the function:

$$\{F, H\} = \sum_{i=1}^{n} \left\{ \frac{\partial H}{\partial p^i} \frac{\partial F}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial F}{\partial p^i} \right\} \tag{1.1}$$

We note the particular bracket identities:

$$\{p^i, p^j\} = 0, \{q^i, q^j\} = 0, \{q^i, p^j\} = \delta^i_j,$$

$$\{p^i, z^k\} = \{q^i, z^k\} = \{z^t, z^k\} = 0.$$

in which i and j run from 1 to n, when t and k run from 1 to l. Here δ^i_j is the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise.

Definition 1.3. Linear space V is called *symplectic*, if there is a non-degenerate skew-symmetric bilinear form ω.

If there was chosen a basis $e_1, ..., e_m$ in V, then the form ω is uniquely defined by its matrix $\Omega = (\omega_{ij})$ where $\omega_{ij} = \omega(e_i, e_j)$.

A differential 2-form ω is called a *symplectic structure* on a smooth manifold M if it satisfies two conditions:

1) ω is closed, that is $d\omega = 0$,

2) ω is non-degenerated at each point of the manifold, i.e., in local coordinates, $det\Omega(x) \neq 0$, where $\Omega(x) = (\omega_{ij}(x))$ is the matrix of this form.

The manifold endowed with a symplectic structure is called a *symplectic manifold.*

All symplectic manifolds are manifolds of even-dimension and orientable.

Definition 1.4. [2, 5]. Let M be a Poisson manifold and $H\colon M \to \mathbb{R}$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $sgradH$ on M satisfying

$$sgradH(F) = \{F, H\} \tag{1.2}$$

for every smooth function $F\colon M \to \mathbb{R}$.

The equations governing the flow of $sgradH$ are referred to as *Hamilton's equations* for the *Hamiltonian function* H.

In the case of the Poisson bracket on $\mathbb{R}^m$ $(m = 2n + l)$, the Hamiltonian vector field to any $H(p, q, z)$, as clearly, corresponds

$$sgradH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right) \tag{1.3}$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\begin{cases} \dfrac{dp^i}{dt} = -\dfrac{\partial H}{\partial q^i}, \\[2ex] \dfrac{dq^i}{dt} = \dfrac{\partial H}{\partial p^i}, \\[2ex] \dfrac{dz^j}{dt} = 0, \end{cases} \tag{1.4}$$

where $i = 1, \dots, n$ and $j = 1, \dots, l$.

The system (1.4) is called a *Hamiltonian system* with Hamiltonian $H(p, q, z)$[2, 1, 4].

The following property of the Poisson bracket is known[2].

Proposition 1.5. *Let M be a Poisson manifold and $F, H \colon M \to \mathbb{R}$ are smooth functions with corresponding Hamiltonian vector fields $sgradF$, $sgradH$. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of the two Hamiltonian vector fields:*

$$sgrad\{F, H\} = [sgradH, sgradF].$$

Definition 1.6. The Hamiltonian vector field associated with $H(x)$ has the form

$$sgradH = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \{x^i, x^j\} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i} \right).$$

Let $F(x)$ be a second smooth function. We obtain the basic formula

$$\{F, H\} = \sum_{i=1}^{n} \sum_{j=1}^{n} \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial x^j} \tag{1.5}$$

for the Poisson bracket.

This basic brackets $A^{ij}(x) = \{x^i, x^j\}$ $i, j = 1, ..., m$ are called the *structure functions* of the Poisson manifold M with respect to the given local coordinates.

A skew-symmetric $m \times m$ matrix $A(x)$ called the *structure matrix* of M.

If the Poisson bracket is non-degenerate ($det(A^{ij}) \neq 0$ everywhere on M), then the Poisson manifold is called a *symplectic manifold*. The symplectic structure in this case has the form $\omega = A_{ij} dx^i \wedge dx^j$ where A_{ij} are components of the matrix inverse to (A^{ij}).

Definition 1.7. [3]. Let M^m (where $m = 2n$) be a symplectic manifold and $sgradH$ the Hamiltonian vector field with a smooth Hamiltonian function H.

A Hamiltonian system $sgradH$ is called *completely integrable in the sense of Liouville or completely integrable*, if there exists a set of smooth functions $f_1, \dots, f_n$ such that:

1) $f_1, ..., f_n$ are first integrals of $sgradH$ Hamiltonian vector field,

2) they are functionally independent on M, that is, almost everywhere on M their gradients are linearly independent,

3) $\{f_i, f_j\} = 0$ for any i and j,

4) the vector fields $sgradf_i$ are complete, that is a natural parameter on their integral trajectories is defined on the whole number line.

Definition 1.8. [2]. A partition of the manifold M^m (where $m = 2n$) into connected components of joint level surfaces of the integrals $f_1, ..., f_n$ is called *the Liouville foliation* corresponding to the completely integrated system.

Since the system $f_1, \dots, f_n$ is preserved by $sgradH$, each leaf of the Liouville foliation is an invariant surface. Any Liouville foliation consists of regular leaves (which fill almost all M) and singular leaves (a set with zero measure).

Let M^{2n} be a Poisson manifold with the integrable Hamiltonian vector field $sgradH$ in sense of Liouville and $f_1, \dots, f_n$ be its independent first integrals.

Let us recall some notions on the geometry of two dimensional submanifolds of four dimensional Euclidean space.

Let two dimensional surface F in $\mathbb{R}^4$ is given with vector function $\mathbf{r} = \mathbf{r}(u, v) \in C^2$.

The *Gaussian curvature* of two dimensional surface F is given by the formula [7]

$$K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} + \frac{c_{11}c_{22} - c_{12}^2}{g_{11}g_{22} - g_{12}^2}. \tag{1.6}$$

where g_{ij} are coefficients of the first quadratic form, b_{ij} are coefficients of the second quadratic form in the direction of the first normal and c_{ij} are coefficients of the second quadratic form in the direction of the second normal.

The *Gaussian torsion* of two dimensional surface F in $\mathbb{R}^4$ is a function that assigns a value to each point of the surface, calculated by the following formula [8]:

$$\sigma_{\mathbf{G}} = \frac{\sum\limits_{i,j=1}^{2} (b_{i1}c_{j2} - b_{i2}c_{j1})g_{ij}}{\sqrt{g_{11}g_{22} - g_{12}^2}}. \tag{1.7}$$

2. Gaussian curvature and Gaussian torsion of regular leaves

Let $sgradH$ be a completely integrable Hamiltonian vector field with the Hamiltonian function $H\colon \mathbb{R}^4 \to \mathbb{R}$ on the four dimensional Euclidean space with the Cartesian coordinates (p_1, p_2, q_1, q_2)

$$H = H(p_1, p_2, q_1, q_2). \tag{2.1}$$

The Hamiltonian vector field corresponding to H is

$$sgradH = -\frac{\partial H}{\partial q^1}\cdot\frac{\partial}{\partial p^1} - \frac{\partial H}{\partial q^2}\cdot\frac{\partial}{\partial p^2} + \frac{\partial H}{\partial p^1}\cdot\frac{\partial}{\partial q^1} + \frac{\partial H}{\partial p^2}\cdot\frac{\partial}{\partial q^2}, \tag{2.2}$$

where the Hamiltonian system has the following form

$$\begin{cases} p_1' = -\dfrac{\partial H}{\partial q^1}, \\ p_2' = -\dfrac{\partial H}{\partial q^2}, \\ q_1' = \dfrac{\partial H}{\partial p^1}, \\ q_2' = \dfrac{\partial H}{\partial p^2}. \end{cases} \tag{2.3}$$

We assume that the following functions

$$\begin{aligned} f_1 &= f_1(p_1, q_1), \\ f_2 &= f_2(p_2, q_2) \end{aligned} \tag{2.4}$$

are the first integrals of Hamiltonian system (2.3).

Level surfaces of these first integrals generates a Liouville foliation $\mathcal{F}$.

Theorem 2.1. *Regular leaves of a Liouville foliation $\mathcal{F}$ generated by Hamiltonian system (2.3) are two dimensional submanifolds of four dimensional Euclidean manifold with zero Gauss curvature and zero Gauss torsion.*

Proof. A regular leaf of the Liouville foliation is a two dimensional submanifold with equations:

$$\begin{cases} f_1(p_1, q_1) = c_1, \\ \\ f_2(p_2, q_2) = c_2. \end{cases} \tag{2.5}$$

Now we can check metric characteristics of this two dimensional submanifold.

It can be parameterized as:

$$\begin{cases} p_1 = p_1(u), \\ p_2 = p_2(v), \\ q_1 = q_1(u), \\ q_2 = q_2(v). \end{cases} \tag{2.6}$$

Now we find

$$\frac{\partial r}{\partial u} = r_1 = \{p_1'(u); 0; q_1'(u); 0\}, \quad \frac{\partial r}{\partial v} = r_2 = \{0; p_2'(v); 0; q_2'(v)\}$$

and coefficients of the first quadratic form

$$g_{11} = \langle r_1, r_1 \rangle = p_1'^2(u) + q_1'^2(u),$$

$$g_{12} = \langle r_1, r_2 \rangle = \langle r_2, r_1 \rangle = g_{21} = 0,$$

$$g_{22} = \langle r_2, r_2 \rangle = p_2'^2(v) + q_2'^2(v).$$

To find coefficients of the second quadratic forms we need two normal vectors. We choose them as:

$$n_1 = \{-q_1'(u); 0; p_1'(u); 0\}$$

and

$$n_2 = \{0; -q_2'(v); 0; p_2'(v)\}.$$

Now we can find two second quadratic forms of the regular leaf. Coefficients of the first of them is calculated by the formula

$$b_{ij} = -\frac{1}{|n_1|} \langle \partial_i r, \partial_j n_1 \rangle.$$

By using these equation we find that

$$b_{11} = -\frac{1}{\sqrt{p_1'^2(u) + q_1'^2(u)}} (q_1'(u) p_1''(u) - p_1'(u) q_1''(u)), \qquad b_{12} = b_{21} = b_{22} = 0.$$

Coefficients of the second of them are calculated by the formula

$$c_{ij} = -\frac{1}{|n_2|} \langle \partial_i r, \partial_j n_2 \rangle.$$

From here

$$c_{11} = c_{12} = c_{21} = 0, \qquad c_{22} = -\frac{1}{\sqrt{p_2'^2(v) + q_2'^2(v)}} (q_2'(v) p_2''(v) - p_2'(v) q_2''(v).$$

Now we are ready to calculate the Gaussian curvature

$$K = \frac{(b_{11} \cdot 0 - 0) + (0 \cdot c_{22} - 0)}{(p_1'^2(u) + q_1'^2(u))(p_2'^2(v) + q_2'^2(v))} = 0 \tag{2.7}$$

and the Gaussian torsion

$$\sigma_{\mathbf{G}} = \frac{(b_{11}\cdot 0 - 0)\cdot g_{11} + (b_{11}c_{22} - 0)\cdot 0 + (0\cdot c_{22} - 0)\cdot g_{22}}{\sqrt{(p_1'^2(u) + q_1'^2(u))(p_2'^2(v) + q_2'^2(v))}} = 0 \tag{2.8}$$

of two dimensional sub manifold with equations (2.5).

Theorem 2.1 is proved. □

Example 2.2. Let us consider the Hamiltonian $H : \mathbb{R}^4 \to \mathbb{R}$ on the Euclidean four dimensional space $\mathbb{R}^4$ which is given by the formula

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2 - q_1^2 + q_2^2). \tag{2.9}$$

The Hamiltonian vector field corresponding to H is

$$sgradH = q_1\frac{\partial}{\partial p_1} - q_2\frac{\partial}{\partial p_2} + p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}.$$

It is not difficult to check corresponding Hamiltonian system

$$\begin{cases} p_1' = q_1, \\ p_2' = -q_2, \\ q_1' = p_1, \\ q_2' = p_2 \end{cases} \tag{2.10}$$

is completely integrable.

We have two functionally independent first integrals of the hamiltonian system

$$\begin{aligned} f_1 &= p_1^2 - q_1^2, \\ f_2 &= p_2^2 + q_2^2. \end{aligned} \tag{2.11}$$

A leaf of the Liouville foliation is given by the following system of equations

$$\begin{cases} p_1^2 - q_1^2 = c_1, \\ p_2^2 + q_2^2 = c_2 \end{cases} \tag{2.12}$$

and it is regular when $c_1 \neq 0$ and $c_2 > 0$.

Comparing with Theorem 2.1 we know that regular leaves of the Liouville foliation are two dimensional submanifolds with zero Gauss curvature and zero Gauss torsion.

Singular two dimensional leaves are surfaces obtained by Cartesian multiplication of circles with intersecting lines, where one dimensional ones are conjugate hyperbolas and intersecting lines. (Figure 1.)

3. Orthogonal foliation of completely integrable Hamiltonian system

Let M be a C^∞ manifold of dimension m,

Definition 3.1. [4]. A *distribution* P on M is a map which assigns to every point $x \in M$ vector subspace $P(x)$ of T_xM.

Every set of smooth vector fields D generates a distribution, where for every point $x \in M$ matches subspace $P(x) \subset T_xM$, that generated by set of vectors $D(x) = \{X(x)\colon X \in D\}$.

The distribution P is called *completely integrable*, if for every $x \in M$ there is a submanifold L_x of the manifold M such, that $T_yL_x = P(y)$ for all $y \in L_x$. The submanifold L_x of M is

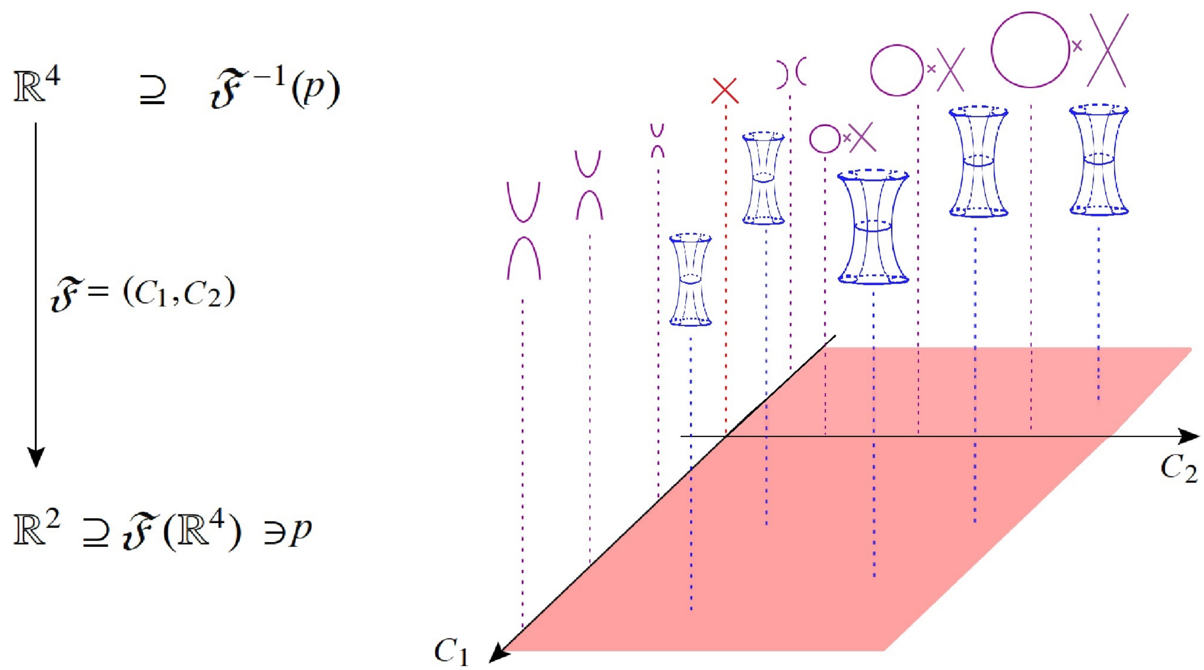

Figure 1. Leaves of the Liouville foliation (2.12)

called an *integral submanifold (or integral manifold)* of the distribution P. A *maximal integral manifold of* P is a connected submanifold L of M such that

(a) L is an integral manifold of P,

(b) every connected integral manifold of P which intersects L is an open submanifold of L.

We say that P is completely integrable if through every point $x \in M$ there passes a maximal integral manifold of P.

Theorem 3.2 (Hermann)**.** [10] *In order a system of smooth vector fields* $D = \{X_1, X_2, ..., X_k\}$ *to generate completely integrable distribution, it is necessary and sufficient that it be involutive.*

Involutiveness of $D = \{X_1, X_2, \ldots, X_k\}$ on M means, that for each pair (X_i, X_j) of vector fields there exist smooth real-valued functions $f_{ij}^l(x)$, such that it takes

$$[X_i, X_j] = \sum_{l=1}^{k} f_{ij}^l(x) X_l,$$

$x \in M$, $i, j, l = 1, \ldots, k$ (Here $[\cdot, \cdot]$ denotes Lie bracket of smooth vector fields.)

$D \subset V(M)$ be a set of vector fields of all smooth (class C^∞) vector field $V(M)$ and $t \to X^t(x)$ be an integral curve of the vector field X with the initial point x for $t = 0$, which is defined in some region $I(x)$ of real line.

Definition 3.3. [9]. *The orbit* $L(x)$ of a system D of vector fields through a point x is the set of points y in M such that there exist $t_1, t_2, \ldots, t_k \in R$ and vector fields $X_1, X_2, \ldots, X_k \in D$ such that

$$y = X_k^{t_k}(X_{k-1}^{t_{k-1}}(\ldots(X_1^{t_1}(x))))$$

where k is an arbitrary positive integer.

The fundamental result in study of orbits is the Sussman theorem.

Theorem 3.4 (Sussman)**.** [9] *Let M be a smooth manifold, and let D be a set of vector fields. Then*

(a) L is an orbit of D, then L admits a unique differentiable structure such that L is a submanifold of M. The dimension of L is equal to its rank.

(b) With the topology and differentiable structure of (a), every orbit of D is a maximal integral submanifold of distribution P.

(c) P has the maximal integral manifolds property.

(d) P is involutive.

Definition 3.5. A partition F of the manifold M by path-connected immersed submanifolds L_α is called *a singular foliation of M* if it verifies condition:

for each leaf L_α and each vector $v \in T_pL_\alpha$ at the point p there is $X \in XF$ such that $X(p) = v$, where T_pL_α is the tangent space of the leaf L_α at the point p, XF is the module of smooth vector fields on M tangent to leaves (XF acts transitively on each leaf).

If the dimension of L is maximal, it is called regular, otherwise L is called singular. It is known that orbits of vector fields generate singular foliation.

Let us denote by P the distribution generated by vector fields

$$\begin{aligned} grad f^1 &= \{p_1'(u); 0; q_1'(u); 0\}, \\ grad f^2 &= \{0, p_2'(v), 0, q_2'(v)\}. \end{aligned} \tag{3.1}$$

Theorem 3.6. *The distribution P generates foliation $\mathcal{F}^\perp$, which is orthogonal to Liouville foliation $\mathcal{F}$ and regular leaves of singular foliation $\mathcal{F}^\perp$ generated by integral submanifolds of P are two dimensional surfaces of zero Gauss curvature and zero Gauss torsion.*

Proof. The system of vector fields $D = \{X_1, X_2\}$, where $X_1 = grad f_1$, $X_2 = grad f_2$ is involutive as Lie bracket of vector fields is

$$[X_1, X_2] = 0.$$

It follows from Sussman theorem the distribution P is completely integrable.

Now we assume a functions

$$\begin{aligned} \mathfrak{F}_1 &= \mathfrak{F}_1(p_1, q_1), \\ \mathfrak{F}_2 &= \mathfrak{F}_2(p_2, q_2) \end{aligned} \tag{3.2}$$

satisfy following conditions

$$\frac{\partial f_1}{\partial p_1}\frac{\partial \mathfrak{F}_1}{\partial p_1} + \frac{\partial f_1}{\partial q_1}\frac{\partial \mathfrak{F}_1}{\partial q_1} = 0, \quad \frac{\partial f_2}{\partial p_2}\frac{\partial \mathfrak{F}_2}{\partial p_2} + \frac{\partial f_2}{\partial q_2}\frac{\partial \mathfrak{F}_2}{\partial q_2} = 0.$$

From this conditions we have got the following equalities

$$grad f_1(\mathfrak{F}_1) = 0, \quad grad f_1(\mathfrak{F}_2) = 0$$

$$grad f_2(\mathfrak{F}_1) = 0, \quad grad f_2(\mathfrak{F}_2) = 0$$

for the functions

$$\mathfrak{F}_1 = \mathfrak{F}_1(p_1, q_1), \quad \mathfrak{F}_2 = \mathfrak{F}_2(p_2, q_2).$$

It follows from those equalities integral submanifolds of the the distribution P are given by equations

$$\begin{cases} \mathfrak{F}_1(p_1, q_1) = s_1, \\ \\ \mathfrak{F}_2(p_2, q_2) = s_2. \end{cases} \tag{3.3}$$

By technics from proof of Theorem 2.1 we have got that regular leaves of $F^{\perp}$ are two dimensional surfaces of zero Gauss curvature and zero Gauss torsion.

Theorem 3.6 is proved. □

Example 3.7. As an example we will take Hamiltonian system (2.9) which is given in the Example 2.2.

It is shown that, functions

$$f_1 = p_1^2 - q_1^2, \quad f_2 = p_2^2 + q_2^2$$

are functionally independent first integrals of the Hamiltonian system (2.9).

In this case, the system of vector fields $D = \{X_1, X_2\}$ consists of vector fields

$$X_1 = gradf_1 = \{p_1, 0, -q_1, 0\},$$

$$X_2 = gradf_2 = \{0, p_2, 0, q_2\}.$$

D is involutive since the Lie bracket of the vector fields is equal to zero.

It follows from the Sussman theorem that the distribution P is completely integrable.

It is easy to check that functions

$$\mathfrak{F}_1(p_1, q_1) = p_1 \cdot q_1, \quad \mathfrak{F}_2(p_2, q_2) = \frac{p_2}{q_2} \tag{3.4}$$

are invariant functions for the system of the vector fields $\{X_1, X_2\}$.

Distribution P generates foliation leaves of which are given by equations

$$\begin{cases} p_1 \cdot q_1 = c_1, \\ \\ \dfrac{p_2}{q_2} = c_2, \end{cases}$$

where c_1, c_2 are real numbers. It follows from Theorem 2.1 that integral submanifolds of P are two-dimensional surfaces of zero Gauss curvature and zero Gauss torsion.

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2-local *-automorphisms on semifinite von Neumann algebras

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Abstract. In the article we consider 2-local *-automorphisms of a complex and real W*-algebras. Unlike the derivation, a *-automorphism of W*-algebras does not necessarily have to be inner. Therefore, we consider approximately inner automorphisms of W*-algebras. It is proved that in a semifinite W*-algebra any 2-locally approximate inner *-automorphism is a Jordan *-automorphism. As a result, the following was obtained: 1) if M is a semifinite W*-algebra with $\overline{Int}(M) = Aut(M)$, then every 2-local *-automorphism is a Jordan *-automorphism; 2) in a semifinite factor, every 2-local approximate inner *-automorphism is a *-automorphism. Since the condition $\overline{Int}(M) = Aut(M)$ is satisfied for the injective factors of types II_1 and II_∞, then any 2-local *-automorphism of the algebra M is a *-automorphism. Hence, in factor of type II_1 any 2-local *-automorphism is a *-automorphism. At the end of the article, similar results are obtained for real W*-algebras.

Keywords: Jordan *-automorphism, 2-local *-automorphism, 2-local approximate inner *-automorphism, injective factor of type II_1 and type II_∞.
MSC (2020): 46L05, 46L10.

1. Introduction

Let $\mathcal{A}$ be a *-algebra. Linear map $\theta : \mathcal{A} \to \mathcal{A}$ is called a **-automorphism*, if $\theta(x^*) = \theta(x)^*$ and $\theta(xy) = \theta(x)\theta(y)$, for all $x, y \in \mathcal{A}$. Let $Aut(\mathcal{A})$ denote the set of all *-automorphisms of the algebra $\mathcal{A}$. Every invertible (or unitary) element $u \in \mathcal{A}$ realizes a *-automorphism of Adu onto $\mathcal{A}$, defined as $Adu(x) = uxu^{-1}$, $x \in \mathcal{A}$. Such *-automorphisms are called *inner *-automorphisms*. The set of all inner *-automorphisms is denoted as $Int(\mathcal{A})$. A linear map $\Theta : \mathcal{A} \to \mathcal{A}$ is called a **-local *-automorphism* if for each $x \in \mathcal{A}$, there is a *-automorphism θ_x of $\mathcal{A}$ such that $\Theta(x) = \theta_x(x)$. It follows, the map $\Theta : \mathcal{A} \to \mathcal{A}$ (not linear in general) is called *-*2-local *-automorphism* if for any $x, y \in \mathcal{A}$, there is a *-automorphism $\theta_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Theta(x) = \theta_{x,y}(x)$ and $\Theta(y) = \theta_{x,y}(y)$. If we require the linearity of Θ in the definition, then, in many cases, it is easy to show that Θ is a *-automorphism. In addition, also easily follows that for $n \geq 3$ every n-local *-automorphism is a *-automorphism.

In the article [1] R. Kadison introduced the concept of local derivation and, in particular, proved that any continuous local derivation on a W*-algebra is derivation. In the work [2] B. Johnson generalized this result for C*-algebras. In the article [3] D. Larson and A. Surur introduced the concept of local *-automorphism and proved that local *-automorphisms on the algebra $B(X)$ of all bounded linear operators on an infinite-dimensional (complex) Banach space X are *-automorphisms. For real C*- and W*-algebras, local derivations are considered in the work of U.Karimov (see [4]).

In the article [5] P.Semrl studied 2-local *-automorphisms and described them on the algebra $B(H)$ of all linear bounded operators in the infinite-dimensional separable (complex) Hilbert space H. A similar description for the finite-dimensional case appeared later in the work [6]. A real analogue of these results, in particular cases, was obtained in the work [7], in which. Considered the case when H is a real Hilbert space.

The article considers 2-local *-automorphisms of a complex and real W*-algebras. Unlike the derivation, a *-automorphism of W*-algebras does not necessarily have to be inner. Therefore, we consider approximately inner automorphisms of W*-algebras. It is proved that in a semifinite

W*-algebra any 2-local approximate inner *-automorphism is a Jordan *-automorphism. As a result, the following was obtained: 1) if M is a semi-finite W*-algebra with $\overline{Int}(M) = Aut(M)$, then every 2-local *-automorphism is a Jordan *-automorphism; 2) in a semifinite factor, every 2-local approximate inner *-automorphism is a *-automorphism. Since the condition $\overline{Int}(M) = Aut(M)$ is satisfied in the injective factors of types II_1 and II_∞, then any 2-local *-automorphism of the algebra M is a *-automorphism. Hence, in factor of type II_1 any 2-local *-automorphism is a *-automorphism. At the end of the article, similar results are obtained for real W*-algebras.

2. Preliminaries

A real Banach *-algebra R is called a *real* C*-*algebra*, if $\|xx^*\| = \|x\|^2$ and the element $\mathbb{1} + xx^*$ is invertible for any $x \in R$. It is easy to show that R is a real C*-algebra if and only if the norm can be extended to the complexification $\mathcal{A} = R + iR$ of the algebra R, so that the algebra $\mathcal{A}$ was a C*-algebra (see [8, 5.1.1]). A real *-subalgebra $R \subset B(H)$ is called a *real* W*-*algebra* if it is closed in the weak operator topology, $\mathbb{1} \in R$ and $R \cap iR = \{0\}$ [9], [10], [8].

Let us note the following facts:

(1) *every 2-local *-automorphism is homogeneous.*

Indeed, if we apply the 2-locality Θ to the elements x and λx (where λ is an arbitrary scalar), then there is an *-automorphism $\theta_{x,\lambda x}$ such that

$$\Theta(\lambda x) = \theta_{x,\lambda x}(\lambda x) = \lambda\theta_{x,\lambda x}(x) = \lambda\Theta(x) \tag{2.1}$$

(2) *if a 2-local *-automorphism is linear, then it is a Jordan *-automorphism.*

Let us recall that linear map $\theta : \mathcal{A} \to \mathcal{A}$ with the property $\theta(x \circ y) = \theta(x) \circ \theta(y)$ is called a *Jordan *-automorphism*, where $x \circ y = (xy + yx)/2$. Obviously, if $\theta(x^2) = \theta(x)^2$ (for all x), then by the linearity of θ we obtain

$$\theta(x^2) + \theta(xy) + \theta(yx) + \theta(y^2) = \theta(x^2 + xy + yx + y^2) = \theta((x + y)^2) = \theta(x + y)^2$$

$$= \theta(x)^2 + \theta(x)\theta(y) + \theta(y)\theta(x) + \theta(y)^2,$$

i.e $\theta(xy + yx) = \theta(x)\theta(y) + \theta(y)\theta(x)$. Thus the linear map with $\theta(x^2) = \theta(x)^2$ $(\forall x \in \mathcal{A})$ is a Jordan *-automorphism.

It is easy to show that if Θ is a 2-local *-automorphism of the algebra $\mathcal{A}$, then $\Theta(x^2) = \Theta(x)^2$, for any $x \in \mathcal{A}$. Indeed, if we apply the 2-locality Θ to the elements x and x^2, then there exists an *-automorphism θ_{x,x^2} such that

$$\Theta(x^2) = \theta_{x,x^2}(x^2) = \theta_{x,x^2}(x)\theta_{x,x^2}(x) = \theta_{x,x^2}(x)^2 = \Theta(x)^2. \tag{2.2}$$

Thus, if a 2-local *-automorphism is linear, then it is a Jordan *-automorphism.

(3) *Nontrivial 2-local automorphism.*

There are few examples in the literature of nontrivial local mappings on operator algebras. Below we briefly present such examples from [11].

Let $\mathcal{A} = \langle I, E_{12}, E_{13}\rangle \subset M_3(\mathbb{C})$. An automorphism of this algebra is a natural linear extension of the mapping θ, acting on the basis elements as:

$$\theta(I) = I, \quad \theta(E_{12}) = aE_{12} + bE_{13}, \quad \theta(E_{13}) = cE_{12} + dE_{13}, \qquad ad - bc \neq 0.$$

Put

$$\Theta\big(a_{11}I + a_{12}E_{12} + a_{13}E_{13}\big) := a_{11}I + a_{12}^3E_{12} + a_{13}^3E_{13}.$$

This is nonlinear, so Θ is not an automorphism, but a 2-local automorphism. Really, let $A = a_{11}I+a_{12}E_{12}+a_{13}E_{13}$ and $B = b_{11}I+b_{12}E_{12}+b_{13}E_{13}$. If they are linearly dependent, then it is easy to obtain the needed automorphism. If they are linearly independent, then $\Theta(A)$ and $\Theta(B)$ are also linearly independent. In particular, $\left\{\left(\begin{smallmatrix} a_{12} \\ a_{13}\end{smallmatrix}\right), \left(\begin{smallmatrix} b_{12} \\ b_{13}\end{smallmatrix}\right)\right\}$ and $\left\{\left(\begin{smallmatrix} a_{12}^3 \\ a_{13}^3\end{smallmatrix}\right), \left(\begin{smallmatrix} b_{12}^3 \\ b_{13}^3\end{smallmatrix}\right)\right\}$ are bases for $\mathbb{C}^2$. , $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ such that

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} a_{12} \\ a_{13}\end{pmatrix} = \begin{pmatrix} a_{12}^3 \\ a_{13}^3\end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} b_{12} \\ b_{13}\end{pmatrix} = \begin{pmatrix} b_{12}^3 \\ b_{13}^3\end{pmatrix}.$$

Hence, there is automorphism $\beta_{A,B}$ of the algebra $\mathcal{A}$, defined through a, b, c, d, such that $\beta_{A,B}(A) = \Theta(A)$ and $\beta_{A,B}(B) = \Theta(B)$.

3. 2-local automorphism on semi-finite real von Neumann algebras

Let M be a semi-finite von Neumann algebra, and τ be an exact normal semi-finite trace on M. *-Automorphism θ on M is called *approximately inner* if there exists sequence $\{\theta_n\}$ of inner automorphisms of M such that $\theta(x) = \lim\lim \theta_n(x)$, for all $x \in M$. In this case, there is a sequence $\{u_n : n \in \mathbb{N}\}$ of unitary elements in M such that $\theta(x) = \lim u_n x u_n^*$. It is clear that the set of all approximately inner *-automorphisms of M coincides with $\overline{Int}(M)$ of the closure of the set $Int(M)$.

We consider 2-local *-automorphisms with the following property: the map $\Theta : M \to M$ is called a *2-local approximate inner *-automorphism* (in short, a 2-*l.a.i.**-automorphism), if for any $x, y \in M$ there exists approximate inner *-automorphism $\theta_{x,y} \in \overline{Int}(M)$ such that $\Theta(x) = \theta_{x,y}(x)$ and $\Theta(y) = \theta_{x,y}(y)$.

Theorem 3.1. *Let M be a semifinite W^*-algebra, and let $\Theta : M \to M$ be a 2-l.a.i.*-automorphism. Then Θ is a Jordan *-automorphism.*

Proof. Let $x, y \in M$. Then there exists a sequence of $\{u_n : n \in \mathbb{N}\}$ unitary elements in M such that $\Theta(x) = \lim u_n x u_n^*$ and $\Theta(y) = \lim u_n y u_n^*$. Since $\Theta(x)\Theta(y)^* = \lim u_n xy^* u_n^*$, than $\tau\big(\Theta(x)\Theta(y)^*\big) = \tau(xy^*)$, $\forall x, y \in M$. Hence, due to the linearity of the trace, for all $z \in M$, we have

$$\tau\big(\big(\Theta(x+y) - \Theta(x) - \Theta(y)\big)\Theta(z)^*\big) = \tau\big(\big(x+y-x-y\big)z^*\big) = 0,$$

in particular we get

$$\tau\big(\big(\Theta(x+y) - \Theta(x) - \Theta(y)\big)\big(\Theta(x+y) - \Theta(x) - \Theta(y)\big)^*\big) = 0.$$

Consequently, $\Theta(x+y) - \Theta(x) - \Theta(y) = 0$, i.e. Θ is additive. By (2.1) Θ is homogeneous, and hence Θ is linear. Since the set of eigenvalues of $\Theta(x)$ according to multiplicity is the same as that of $x \in M$ and

$$\Theta(x^*) = \theta_{x,x^*}(x^*) = \theta_{x,x^*}(x)^* = \Theta(x)^*.$$

Since approximetely inner *-automorphisms are a special case of *-automorphisms, then 2-local approximetely inner *-automorphisms are a special case of 2-local *-automorphisms. Therefore formula (2.2) also holds for 2-local approximetely inner *-automorphisms. Then according to (2.2) the map Θ a Jordan *-automorphism. $\square$

Corollary 3.2. *If M is a semi-finite W^*-algebra with $\overline{Int}(M) = Aut(M)$, then every 2-local *-automorphism is a Jordan *-automorphism.*

Let us present an auxiliary result from [12, Theorem H]:

Theorem 3.3. *If Θ is a Jordan homomorphism of a ring R onto a prime ring R' of characteristic different from 2 and 3, then Θ is either a homomorphism or an anti-homomorphism.*

Recall that an algebra (or ring) A is called *semiprime* if $aAa = \{0\}$ implies $a = 0$; is called *prime* if $aAb = \{0\}$ implies $a = 0$ or $b = 0$. It is easy to show that any W*-algebra (real or complex) is semiprime, and it is prime if and only if it is a factor.

By Theorems 2.2 and 3.3, any 2-*l.a.i.**-automorphism of a semi-finite factor is either an *-automorphism or an *-antiautomorphism. Further, following similarly to the end of the proof of [7, Theorem 2], we obtain the following result

Corollary 3.4. *In a semifinite factor, every 2-l.a.i.*-automorphism is a *-automorphism.*

Recall that a W*-algebra M in $B(H)$ is called *injective* if there exists a projection P from $B(H)$ onto M with $\|P\| = 1$, $P(\mathbb{1}) = \mathbb{1}$. This is equivalent to the fact that M is approximately finite-dimensional, i.e. there exists an increasing sequence $\{M_n\}$ of finite-dimensional W*-subalgebras in M with $\mathbb{1} \in M_n$ such that $\cup_n M_n$ is weakly dense in M. It is known that there exists a unique (up to isomorphism) injective (complex and real) factor of each type II_1, II_∞, and they satisfy the condition $\overline{Int}(M) = Aut(M)$. Therefore, by Corollaries 3.2 and 3.4 we obtain the following result

Corollary 3.5. *Let M injective factor of type II_1 or II_∞. Then any 2-local *-automorphism of the algebra M is a *-automorphism.*

On the other side, it is also known, that if M factor of type II_1, then $\tau\theta = \tau$, for all $\theta \in Aut(M)$, where τ is the canonical finite trace on M. Hence, in this case, the proof of Theorem 2.2 proceeds without the condition of approximatively inner local *-automorphism.

Theorem 3.6. *Let M be a factor of type II_1. Then every 2-local *-automorphism is a *-automorphism.*

Since all the facts used are also valid for real W*-algebras, the proof of the obtained results can be easily transferred to the real case. Let us formulate the real analogue of the results.

Theorem 3.7. *1) In a semifinite real W*-algebra, every 2-l.a.i.*-automorphism is a Jordan *-automorphism;*
*2) in a semifinite real factor, every 2-l.a.i. *-automorphism is a *-automorphism;*
*3) in a real factor of type II_1, every 2-local *-automorphism is a *-automorphism;*
*4) in an injective real factor of type II_∞, every 2-local *-automorphism is a *-automorphism.*

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On a mixed problem with integral conditions for third-order equation

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Abstract. The solvability of mixed problems with integral conditions for third-order partial differential equations is an important area of research in the theory of differential equations and their applications. Such problems arise in various fields of physics, continuum mechanics, oscillation theory, and other processes.

In this paper, we consider a non-local problem with an integral condition for a third-order partial differential equation with the thermal conductivity operator in the main part.

The theorems on the existence and uniqueness of the solution of the studied non-local problem are proved here. Methods of the theory of differential equations, the Green function and the theory of integral equations are used to the prove the solvability of the problem. The problem under study is reduced to an equivalent second kind Volterra integral equation which is certainly solvable.

Keywords: boundary value problem; regular solution; non–local condition; integral condition; non–local problem; equation of thermal conductivity; Green's function; integral equation; the Volterra equation; the Abel equation.

MSC (2020): 65D30, 65D32

1. Introduction

The study of the solvability of nonlocal problems with integral conditions for parabolic equations began with the research presented in references [1] and [2]. Mixed problems with integral conditions for parabolic equations have been explored in studies [3] – [10], primarily focusing on second-order equations in one-dimensional [3] – [8] and multidimensional [9] – [10] domains. Various nonlocal problems with integral conditions for different types of third-order partial differential equations have been examined in [11] – [14]. Investigating the solvability of nonlocal problems for third-order differential equations is significant, both for advancing the theory of initial-boundary value problems related to partial differential equations and for the practical applications of mathematical modeling in various processes.

In this paper, we study nonlocal boundary value problems with integral conditions for a third-order equation that includes a heat conductivity operator in its principal part.

2. Statement of the problem

In the domain $D = \{(x,t):\ 0 < x < \ell,\ 0 < t < T\}$, we consider a third-order partial differential equation of the following form:

$$Lu \equiv \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) = f(x,t), \tag{2.1}$$

where $f(x,t)$ is given function.

Note that equation (2.1) belongs to the first canonical form with respect to the highest derivatives specified in [15], i.e. the characteristic equation has one common integral, and a triple one. This factor significantly affects both the correctness of the problem statement and its solvability.

In this paper, the following problems are investigated for equation (2.1):

Nonlocal problem. *Find solution $u(x,t)$ to equation (2.1) in domain D that satisfies the initial condition*

$$u(x,0)=\varphi(x), \qquad 0\le x\le \ell, \tag{2.2}$$

boundary conditions

$$u(0,t)=\mu_1(t), \quad u_x(0,t)=\mu_2(t), \qquad 0\le t\le T, \tag{2.3}$$

and integral condition

$$\int\limits_0^\ell k(x)u(x,t)dx=\mu_3(t), \qquad 0\le t\le T, \tag{2.4}$$

where $\varphi(x)$, $k(x)$, $\mu_i(t)$, $(i=\overline{1,3})$ are the given functions, continuous at $[0,\ell]$ and $[0,T]$, satisfying the matching conditions:

$$\varphi(0)=\mu_1(0), \quad \varphi^{'}(\ell)=\mu_2(0), \quad \int\limits_0^\ell k(x)\varphi(x)dx=\mu_3(0).$$

In condition (2.4), $k(x)$ is the given function, continuous at $[0,\ell]$, having the smoothness required for the upcoming transformations.

In the nonlocal problem (2.1)–(2.4), the boundary conditions contain non-locality in time, first considered in [16]. Note that in the studies conducted by A.I. Kozhanov and his students, the solvability of boundary value problems combining problems with A.A. Samarskii nonlocal conditions and problems with integral conditions was investigated.

3. Solvability of a non-local problem (2.1)–(2.4)

We denote by $C^{k,l}(D)$, the class of functions $u(x,t)$ continuous with their partial derivatives of order $\partial^{m+n}u(x,y)/\partial x^m\partial y^n$ for all $m=\overline{0,k}$, $n=\overline{0,l}$; $C^{0,0}(D)$; and we denote by $C(D)$.

By class $C^{(k,\nu)}(D)$, we mean functions defined in domain D, for which all partial derivatives of order k exist and satisfy the Holder condition with exponent ν, $0<\nu<1$.

Definition 3.1. *A regular solution to equation (2.1) in domain D is function $u(x,t)$ from class $C^{3,1}(D)\cup C^{2,0}(\overline{D})$ satisfying it in the usual sense.*

We study problem (2.1)–(2.4) in the space $C^{3,1}(D)\cap C^{2,0}(\overline{D})$, in this case, the following theorem on the solvability of the non-local problem (2.1)–(2.4) is valid:

Theorem 3.1. *Let condition $f(x,t)\in C(\overline{D})$ be satisfied and the given functions $\varphi(x)$, $\mu_i(t),(i=1,2,3)$ satisfy the following conditions:*

$$\varphi(x),k(x)\in C^2[0,\ell], \quad \mu_1(t),\mu_3(t)\in C^1[0,T],\mu_2(t)\in C[0,T].$$

Then there exists a unique continuous solution to the non-local problem (2.1)-(2.4).

Let us construct an explicit solution to the problem (2.1)-(2.4) using the Green function for the heat conduction equation. Introducing the following notation

$$\frac{\partial u(x,t)}{\partial x}=v(x,t), \tag{3.1}$$

from equation (2.1), we obtain

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}=f(x,t). \tag{3.2}$$

Let us denote $v(\ell,t)$ by $\mu(t)$ and for equation (3.2), we will solve the following problem: *Find in D solution $v(x,t)$ to the parabolic equation (3.2) satisfying the following conditions:*

$$\begin{cases} v(x,0)=\varphi^{'}(x), & 0\le x\le \ell, \\ v(0,t)=\mu_2(t), & 0\le t\le T, \\ v(\ell,t)=\mu(t), & 0\le t\le T. \end{cases} \tag{3.3}$$

Then we will define function $\mu(t)$ from condition (2.4).

It is required that function $\mu(t)$ satisfies the following assumptions: function $\mu(t)$ is continuously differentiable and integrable in $[0,T], \mu(0)=0$.

We will represent the solution to problem (3.2)–(3.3) as $v(x,t)=\mathrm{w}(x,t)+v_1(x,t)$, where the following function

$$v_1(x,t)=\left(1-\frac{x}{\ell}\right)\mu_2(t)+\frac{x}{\ell}\,\mu(t)$$

satisfies boundary conditions (3.3), and function $\mathrm{w}(x,t)$ satisfies zero boundary conditions.

Thus, the solution to problem (3.2)–(3.3) is reduced to the solution of the following boundary value problem. *Find in D solution $\mathrm{w}(x,t)$ to the inhomogeneous heat conduction equation*

$$\frac{\partial \mathrm{w}}{\partial t}-\frac{\partial^2 \mathrm{w}}{\partial x^2}=F(x,t), \tag{3.4}$$

satisfying the following conditions:

$$\mathrm{w}(x,0)=\varphi_1(x), \quad \mathrm{w}(0,t)=\mathrm{w}(\ell,t)=0, \tag{3.5}$$

where

$$F(x,t)=f(x,t)-\left(1-\frac{x}{\ell}\right)\mu_2^{'}(t)-\frac{x}{\ell}\,\mu^{'}(t), \quad \varphi_1(x)=\varphi^{'}(x)-v_1(x,0).$$

Now we transform condition (2.4) in new notations; for this integrating equality (2.5) and using condition (2.3), we obtain:

$$u(x,t)=\mu_1(t)+\int\limits_0^x v(z,t)dz.$$

With expressions for function $v(x,t)$, from the last equality, we obtain:

$$u(x,t)=\mu_1(t)+\left(x-\frac{x^2}{2\ell}\right)\mu_2(t)+\frac{x^2}{2\ell}\,\mu(t)+\int\limits_0^x \mathrm{w}(z,t)dz. \tag{3.6}$$

We multiply both sides of (3.6) by function $k(x)$ and integrate the resulting expression from 0 to L after some transformations, we have

$$\int\limits_0^\ell k(x)u(x,t)dx=\mu_1(t)\int\limits_0^\ell k(x)dx+\mu_2(t)\int\limits_0^\ell\left(x-\frac{x^2}{2\ell}\right)k(x)dx+$$

$$+\mu(t)\int\limits_0^\ell \frac{x^2}{2\ell}k(x)+\int\limits_0^\ell k(x)dx\int\limits_0^x \mathrm{w}(z,t)dzdx.$$

Thus, condition (2.4) in new notations has the following form:

$$\int_0^\ell k_0(x)\mathrm{w}(x,t)dx = \mu_4(t) - \mu(t)\int_0^\ell \frac{x^2}{2\ell}\,k(x)dx, \tag{3.7}$$

here,

$$k_0(x) = \int_x^\ell k(z)dz; \quad \mu_4(t) = \mu_3(t) - \mu_1(t)\int_0^\ell k(x)dx - \mu_2(t)\int_0^\ell \left(x - \frac{x^2}{2\ell}\right)k(x)dx.$$

From [17], it follows that with Green's function from the first boundary value problem for the heat equation, the solution to problem (3.4)–(3.5) can be written as:

$$\mathrm{w}(x,t) = \int_0^\ell G(x,t;\xi,0)\varphi_1(\xi)d\xi + \int_0^t\int_0^\ell G(x,t;\xi,\tau)F(\xi,\tau)d\xi d\tau; \tag{3.8}$$

where

$$G(x,t;\xi,\tau) = \frac{2}{\ell}\sum_{n=1}^{+\infty}\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\sin\frac{n\pi}{\ell}\,x\sin\frac{n\pi}{\ell}\,\xi. \tag{$*$}$$

The proof of the absolute and uniform convergence of the series (*) and the same convergence of the series obtained by direct differentiation arbitrarily many times with respect to x and t can be found in [18].

Substitute (3.8) in the left part of (3.7), we have

$$\int_0^\ell k_0(x)\mathrm{w}(x,t)dx = \int_0^\ell k_0(x)dx\int_0^\ell G(x,t;\xi,0)\varphi_1(\xi)d\xi +$$

$$+\int_0^\ell k_0(x)dx\int_0^t\int_0^\ell G(x,t;\xi,\tau)F(\xi,\tau)d\xi d\tau = J_1 + J_2, \tag{3.9}$$

here

$$J_1 = \int_0^\ell k_0(x)dx\int_0^\ell G(x,t;\xi,0)\varphi_1(\xi)d\xi;$$

$$J_2 = \int_0^\ell k_0(x)dx\int_0^t\int_0^\ell G(x,t;\xi,\tau)F(\xi,\tau)d\xi d\tau.$$

Considering the explicit form of Green's function $G(x,t;\xi,\tau)$, we rewrite integral J_2 as:

$$J_2 = \int_0^\ell k_0(x)dx\int_0^t\int_0^\ell\sum_{n=1}^{\infty}\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\times$$

$$\times\sin\left(\frac{n\pi}{\ell}\,x\right)\sin\left(\frac{n\pi}{\ell}\,\xi\right)f(\xi,\tau)d\xi d\tau -$$

$$-\frac{2}{\ell}\int\limits_0^\ell k_0(x)\sin\frac{n\pi}{\ell}\,x\,dx\int\limits_0^\ell\left(1-\frac{\xi}{\ell}\right)\sin\frac{n\pi}{\ell}\,\xi\,d\xi\times$$

$$\times\int\limits_0^t \exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\mu_2^{'}(\tau)d\tau+$$

$$+\frac{2}{\ell}\int\limits_0^l k_0(x)\sin\frac{n\pi}{\ell}\,x\,dx\int\limits_0^\ell\frac{\xi}{\ell}\,\sin\frac{n\pi}{\ell}\,\xi\,d\xi\int\limits_0^t\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\mu^{'}(\tau)d\tau.$$

Transforming the integrals, from the last expression, we obtain:

$$J_2=\int\limits_0^\ell k_0(x)dx\int\limits_0^t\int\limits_0^\ell\sum_{n=1}^\infty\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\times$$

$$\times\sin\left(\frac{n\pi}{\ell}\,x\right)\sin\left(\frac{n\pi}{\ell}\,\xi\right)f(\xi,\tau)d\xi d\tau-$$

$$-\frac{\ell}{\pi}\sum_{n=1}^\infty\frac{\cos\pi n}{n}\,k_0^n\left[\mu_2(t)-\left(\frac{\pi n}{\ell}\right)^2\int\limits_0^t\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\mu_2(\tau)d\tau\right]-$$

$$-\frac{\ell}{\pi}\sum_{n=1}^\infty\frac{\cos\pi n}{n}\,k_0^n\left[\mu(t)-\left(\frac{\pi n}{\ell}\right)^2\int\limits_0^t\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\mu(\tau)d\tau\right],\tag{3.10}$$

where

$$k_0^n=\frac{2}{\ell}\int\limits_0^\ell k_0(x)\sin\left(\frac{n\pi}{\ell}\,x\right)dx.\tag{3.11}$$

Substitute (3.10) into condition (3.7), we obtain the Volterra integral equation of the second kind for function:

$$\alpha\mu(t)-\int\limits_0^t K(t,\tau)\mu(\tau)d\tau=g(t),\tag{3.12}$$

with kernel

$$K(t,\tau)=\left(\frac{\ell}{\pi}\right)^2\sum_{n=1}^\infty\frac{\cos\pi n}{n}k_0^n\,\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\},\tag{3.13}$$

and the right part has the form

$$g(t)=\mu_4(t)-\frac{2}{\ell}\int\limits_0^\ell k_0(x)dx\int\limits_0^\ell\sum_{n=1}^\infty\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2t\right\}\times$$

$$\times\sin\left(\frac{n\pi}{\ell}x\right)\,\sin\left(\frac{n\pi}{\ell}\,\xi\right)\varphi_1(\xi)d\xi-$$

$$-\frac{2}{\ell}\int\limits_0^\ell k_0(x)dx\int\limits_0^t\int\limits_0^\ell\sum_{n=1}^\infty\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\times$$

$$\times \sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{n\pi}{\ell}\xi\right)f(\xi,\tau)d\xi d\tau-$$

$$-\frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{\cos\pi n}{n}k_0^n\Bigg[\mu_2(t)-$$

$$-\left(\frac{n\pi}{\ell}\right)^2\int_0^t \exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}\mu_2(\tau)d\tau\Bigg], \tag{3.14}$$

here

$$\alpha=\int_0^\ell \frac{x^2}{2\ell}k(x)dx+\left(\frac{\ell}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{\cos\pi n}{n}k_0^n. \tag{3.15}$$

Now we will prove the existence and uniqueness of the solution to this equation.

Theorem 3.2. *If all the conditions of the theorem are satisfied, and*

$$k_0(0)=k_0(\ell)=0,\quad k_0''(0)=k_0''(\ell)=0, \tag{3.16}$$

then the integral equation (3.12) has a unique solution $\mu(t)$ in the class of functions $C^1[0,T]$.

Proof.

1. Integrating by parts three times the integral from formula (3.11) taking into account the following conditions (3.16), we obtain:

$$k_0^n=\frac{2}{\ell}\left(\frac{\ell}{n\pi}\right)^3\int_0^\ell k_0'''(x)\cos\frac{n\pi}{\ell}xdx=\left(\frac{\ell}{\pi}\right)^3\frac{p_n}{n^3}, \tag{3.17}$$

Since function $k_0'''(x)$ is continuous on segment $[0,\ell]$, then, as it is known from the theory of Fourier series, by virtue of Bessel's inequality, the following series

$$\sum_{n=1}^{\infty}p_n^2\le\frac{2}{\ell}\int_0^\ell [k_0'''(x)]^2dx.$$

converges, and therefore, $p_n\to 0$ as $n\to\infty$. From here we have $|k_0^n|\le\dfrac{\varepsilon_n}{n^3}$, where $\varepsilon_n>0$ and $\varepsilon_n\to 0$ for $n\to\infty$.

Considering (3.17), we rewrite the series (3.13) as

$$|K(t,\tau)|\le\left(\frac{\ell}{\pi}\right)^2\sum_{n=1}^{\infty}\left|\frac{\cos\pi n}{n^4}\right||p_n|\exp\left\{-\left(\frac{n\pi}{\ell}\right)^2(t-\tau)\right\}; \tag{3.18}$$

that is, the series (3.18) converges uniformly at $n\to\infty$. In fact, the number series $\dfrac{p_n}{n^4}$ converges at $n\to\infty$. Then the kernel $K(t,\tau)$ is continuous on the set $0\le\tau\le t\le T$.

Therefore, series (3.17) converges uniformly at $0\le\tau\le t\le T$. Then function $K(t,\tau)$ is continuous on the specified set.

By (3.15), it is easy to show the continuity of the right-hand side of equation (3.12).

Thus, equation (3.12) is a Volterra integral equation of the second kind with a continuous kernel and a continuous right-hand side, and therefore it has a unique solution in the class of continuous functions on $[0,T]$.

2. Now consider the case when in the equation (3.12) $\alpha = 0$, then, we obtain first kind the Volterra integral equation with respect to the unknown function $\mu(t)$

$$\int_0^t K(t,\tau)\mu(\tau)d\tau = g(t), \tag{3.19}$$

where the functions $K(t,\tau)$ and $g(t)$ are defined in (3.13) and (3.14), respectively.

If the equation (3.19) is solvable in the class $C[0,T]$, then $g(t) \in C^1[0,T]$. Due to the convergence of the numerical series (3.17), the kernal (17) converges uniformly and allows for term by term differentiation by with respect to t when $0 \leq \tau \leq t \leq T$.

Then the function $K'_t(t,\tau)$ is continuous on the specified set.

Differentiating both parts of equality (3.19), we have

$$K(t,t)\mu(t) + \int_0^t \frac{\partial K(t,\tau)}{\partial t}\mu(\tau)d\tau = g'(t). \tag{3.20}$$

Let's put in (3.13) $\tau = t$, then we get

$$K(t,t) = \left(\frac{l}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{\cos \pi n}{n} k_0^n, \tag{3.21}$$

4. Conclusion

By virtue of the assessment $|k_0^n| \leq \dfrac{\varepsilon_n}{n^3}$, The series (3.21) converges uniformly, so equation (3.20) is the second kind Volterra integral equation of the second kind with a continuous kernal and a continuous right-hand side, therefore equation (3.19) has a unique solution $\mu(t) \in C[0,T]$).

Thus, the solvability of the non-local problem (2.1)–(2.4) has been proved.

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Numerical implementation of optimal water resource management problems in open channels using Python

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Abstract. 2D flow modeling plays a crucial role in understanding and predicting hydrological processes such as floods and droughts. A mathematical model is formulated based on initial and boundary conditions using 2D Saint-Venant differential equations to simulate flood movements. The model is discretized using the explicit finite difference method and implemented using Python programming. For the experiment, a rectangular-shaped flow channel was selected. The water surface elevation z(m), flow velocity components u(m/s) and v(m/s) were computed over specific time intervals and visualized.

Keywords: Open channel, flood wave, Saint-Venant equations, shallow water equations, numerical methods, Python programming.

1. Introduction

Floods and droughts are among the most frequent natural disasters affecting large populations and agricultural lands. These disasters not only cause loss of human lives and damage to infrastructure but also lead to variations in river flows, either increasing or decreasing. Prolonged rainfall and inadequate river capacity are major causes of flooding, while insufficient precipitation leads to drought conditions. Studying flood flow behavior is crucial for early risk mitigation and saving lives. Mathematical models are valuable in managing various hydrological phenomena, including rainfall, snowmelt, groundwater flow, and hydraulic structure design. Many studies have been conducted on flood modeling. One-dimensional models describing the propagation of flood waves use dynamic equations known as the Saint-Venant (SV) equations [1, 2]. These equations are widely applied for predicting flow velocity, depth, or discharge.

For two-dimensional surface flow modeling, the Saint-Venant equations are derived from the Navier-Stokes equations and are sometimes referred to as the Shallow Water (SW) equations. Due to the nonlinear nature of the Saint-Venant equations, various numerical methods are employed for surface flow modeling [3, 4]. The LaxWendroff method is used for modeling two-dimensional flow over micro-relief.

The uneven slope of the channel bed affects water depth over time, making certain regions deeper or shallower. The Saint-Venant equations are applied to simulate flood waves in water bodies[5, 6]. Several numerical methods have been introduced for solving one-dimensional Saint-Venant equations to analyze flood waves, depth variations, and water surface profiles over time [7, 8, 9, 10].

The applied numerical scheme is primarily designed for analyzing and predicting shallow water flows and may introduce errors in deep or turbulent flow predictions. Nevertheless, these schemes are simple and computationally efficient. The influence of various parameters, including friction, initial conditions, and computational accuracy, plays a significant role in ensuring the models alignment with real-world physical conditions [11, 12, 13, 14]. This study examines these parameters in open channel flow modeling and presents results through visualization.

Thus, this research aims to develop a simple flood modeling system and implement a Python-based user-defined code to analyze and control hydrodynamic processes.

2. Methodology

The models for solving 2D Saint-Venant equations include numerical methods (FDM, FVM, FEM, Godunov), physics-informed neural networks (PINNs), and commercial software (HEC-RAS, MIKE21, TUFLOW, TELEMAC, FLOW-3D). Numerical methods provide fast, accurate, reliable, and mathematically well-founded solutions. Physics-Informed Neural Networks (PINNs) require more computational resources and have slower training processes compared to numerical methods. Commercial software like InfoWorks RS or HEC-RAS, which are commonly used to solve the 2D Saint-Venant equation system, rely on predefined models where the user has limited direct control over the mathematical formulations. In contrast, solving the Saint-Venant equations using Python provides full control over each parameter, allowing the model to be customized and simplified as needed. Two-dimensional hydrodynamic flow equations, known as the 2D Saint-Venant equations, are derived from the Navier-Stokes equations by averaging over depth, applying kinematic boundary conditions, and making certain assumptions. These equations are used for water flow management, relying on the principles of mass and momentum conservation.

The LaxWendroff method used to solve the Saint-Venant differential equations is considered highly effective for advection, wave propagation, and shallow water equations (SWEs). The explicit finite difference scheme proposed in this study is clearly linked to the analytical expressions, incorporating second-order differential terms $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 g}{\partial y^2}$, which enhances the accuracy of convergence.

In this physical model, the water surface elevation $z(x,y,t)$ and the velocities $u(x,y,t)$, $v(x,y,t)$ are time-dependent, and the solution domain is defined as follows:

$$\Omega_T = \left\{(x,y,t) \in \mathbb{R}^3 \,\middle|\, 0 \le x \le L,\ 0 \le y \le W,\ 0 \le t \le T\right\}$$

In this domain, each function is defined as follows: $z(x,y,t)$ water surface elevation, $u(x,y,t)$ velocity in the x-direction, $v(x,y,t)$ velocity in the y-direction

Discretized Solution Domain. To solve the space-time domain numerically, it is necessary to divide it into a finite number of grid points. Therefore, the domain Ω_T is discretized as follows:

$$\Omega_T^h = \left\{(x_i, y_j, t^k) \,\middle|\, x_i = i\cdot\Delta x,\ y_j = j\cdot\Delta y,\ t^k = k\cdot\Delta t\right\}$$

Here:

- $i = 0,1,2,\ldots,N_x$ number of subdivisions along the x-direction,
- $j = 0,1,2,\ldots,N_y$ number of subdivisions along the y-direction,
- $k = 0,1,2,\ldots,N_t$ number of time steps.

Grid Spacing.

- $\Delta x = \frac{L}{N_x}$ grid spacing in the x-direction,
- $\Delta y = \frac{W}{N_y}$ grid spacing in the y-direction,
- $\Delta t = \frac{T}{N_t}$ time step size.

This grid-based structure discretizes the computational domain such that the Saint-Venant equations are solved approximately at each grid point using finite difference formulations. The numerical algorithm updates the solution at every time step across the grid, thereby simulating the evolution of wave propagation, flow dynamics, surface elevation, and other hydraulic phenomena over time.

The governing equations for surface flow are expressed as follows:

$$\frac{\partial}{\partial t}h + \frac{\partial}{\partial x}hu + \frac{\partial}{\partial y}hv = 0 \tag{2.1}$$

$$\frac{\partial}{\partial t}hu + \frac{\partial}{\partial x}\left(hu^2 + \frac{gh^2}{2}\right) + \frac{\partial}{\partial y}huv = gh(S_{0x} - S_{fx}) \tag{2.2}$$

$$\frac{\partial}{\partial t}hv + \frac{\partial}{\partial x}huv + \frac{\partial}{\partial y}\left(hv^2 + \frac{gh^2}{2}\right) = gh(S_{0y} - S_{fy}) \tag{2.3}$$

The first equation represents the continuity equation based on the principle of mass conservation, while the second and third equations describe the momentum equations in the x and y directions, respectively.

Where:

z (m) is the water surface elevation (depth), u (m/s) and v (m/s) are the depth-averaged velocity components in the x and y directions, respectively, g (m/s) is the gravitational acceleration, S_{0x} and S_{fx} are the water surface gradient and frictional resistance in the x direction, S_{0y} and S_{fy} in the y direction, and t is the time step. By applying the product rule of differentiation, the equations (2.1)–(2.3) can be transformed into the following form:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} + v\frac{\partial h}{\partial y} + h\frac{\partial v}{\partial y} = 0 \tag{2.4}$$

$$h\frac{\partial u}{\partial t} + u\frac{\partial h}{\partial t} + u^2\frac{\partial h}{\partial x} + 2hu\frac{\partial u}{\partial x} + gu\frac{\partial u}{\partial x} + uv\frac{\partial h}{\partial y} + hv\frac{\partial u}{\partial y} + hu\frac{\partial v}{\partial y} = gh(S_{0x} - S_{fx}) \tag{2.5}$$

$$h\frac{\partial v}{\partial t} + v\frac{\partial h}{\partial t} + uv\frac{\partial h}{\partial x} + hv\frac{\partial u}{\partial x} + hu\frac{\partial v}{\partial x} + v^2\frac{\partial h}{\partial y} + 2hv\frac{\partial v}{\partial y} + gh\frac{\partial h}{\partial y} = gh(S_{0y} - S_{fy}) \tag{2.6}$$

To obtain the finite difference solution, equations (2.4)(2.6) are discretized using explicit finite difference schemes, where the temporal and spatial derivatives are approximated by the following expressions:

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\Delta t} + \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\Delta t, \quad \frac{\partial u}{\partial x} \approx \frac{u_{i+1,j}^{k} - u_{i-1,j}^{k}}{2\Delta x} - \frac{\Delta t}{2}\frac{\partial^2 f}{\partial x^2}, \quad \textit{and} \tag{2.7}$$

$$\frac{\partial u}{\partial y} \approx \frac{u_{i,j+1}^{k} - u_{i,j-1}^{k}}{2\Delta y} - \frac{\Delta t}{2}\frac{\partial^2 g}{\partial y^2}$$

The discretized form of the water surface elevation equation is given as follows:

$$z_{i,j,n+1} = z_{i,j,n} - \frac{\Delta t}{2}\left[\frac{z_{i+1,j,n} - z_{i-1,j,n}}{2\Delta x}u_{i,j,n} + \frac{z_{i,j+1,n} - z_{i,j-1,n}}{2\Delta y}v_{i,j,n}\right] \tag{2.8}$$

$$+\frac{\Delta t^2}{2}\left[u_{i,j,n}^2\frac{z_{i+1,j,n} - 2z_{i,j,n} + z_{i-1,j,n}}{\Delta x^2} + v_{i,j,n}^2\frac{z_{i,j+1,n} - 2z_{i,j,n} + z_{i,j-1,n}}{\Delta y^2}\right]$$

$$+2u_{i,j,n}\frac{v_{i,j+1,n}-v_{i,j-1,n}}{2\Delta y}+2v_{i,j,n}\frac{u_{i+1,j,n}-u_{i-1,j,n}}{2\Delta x}.$$

The velocity in the x direction is expressed as follows:

$$u_{i,j,n+1}=u_{i,j,n}-\frac{\Delta t}{z_{i,j,n+1}}\left[\frac{z_{i,j,n+1}-z_{i,j,n}}{\Delta t}+u_{i,j,n}^2\frac{z_{i+1,j,n}-z_{i-1,j,n}}{2\Delta x}+2u_{i,j,n}\frac{u_{i+1,j,n}-u_{i-1,j,n}}{2\Delta x}\right. \tag{2.9}$$

$$\left.+gz_{i,j,n}\frac{z_{i+1,j,n}-z_{i-1,j,n}}{2\Delta x}-gz_{i,j,n}(S_{0x}-S_{fx})\right].$$

The velocity in the y direction is expressed as follows:

$$v_{i,j,n+1}=v_{i,j,n}-\frac{\Delta t}{z_{i,j,n+1}}\left[\frac{z_{i,j,n+1}-z_{i,j,n}}{\Delta t}+u_{i,j,n}v_{i,j,n}\frac{z_{i+1,j,n}-z_{i-1,j,n}}{2\Delta x}+v_{i,j,n}^2\frac{z_{i,j+1,n}-z_{i,j-1,n}}{2\Delta y}\right. \tag{2.10}$$

$$\left.+2v_{i,j,n}\frac{v_{i,j+1,n}-v_{i,j-1,n}}{2\Delta y}+gz_{i,j,n}\frac{z_{i,j+1,n}-z_{i,j-1,n}}{2\Delta y}-gz_{i,j,n}(S_{0y}-S_{fy})\right].$$

The parameters u, v, and h in equations (2.8)–(2.10) are assigned initial values for all nodes based on an exponential function, and then the solution is updated at each time step $t_k \to t_{k+1}$ using the values from the previous time step. The convergence of finite differences is performed at each branch point (x, y, t). The velocities of the water flow and the height of the water level are calculated for each time stage and along the distance of the flow channel.

The flowchart of the calculation algorithm is presented below.

For testing and implementation, a rectangular flow channel is considered. The finite-difference discretization of the computational domain is shown in Fig. 2.

The implementation of the numerical solution was coded using the Python program, and the following data were used for the model:

Flow Channel Dimensions

Length: $L = 300$ m, Width: $W = 50$ m, Height: $H = 5$ m

Initial Conditions

Water surface height: $z_0(x, y) = 0$ m, Initial velocities: $u_0 = 0$ m/s, $v_0 = 0$ m/s

Slopes

$S_{0x} = 0$, $S_{fx} = 0.1$, $S_{0y} = 0$, $S_{fy} = 0.1$

The flood wave at the initial point is generated as follows:

$$z = h \cdot e^{-\frac{(a\cdot dx-a_0)^2+(b\cdot dy-b_0)^2}{2b^2}}$$

Where:

- a and b are the node indices in the x and y directions, respectively.
- a_0 and b_0 are the initial positions of the water surface in the x and y directions, respectively.

The following values are accepted for the test state: $a_0 = 20$, $b_0 = 20$, h=5

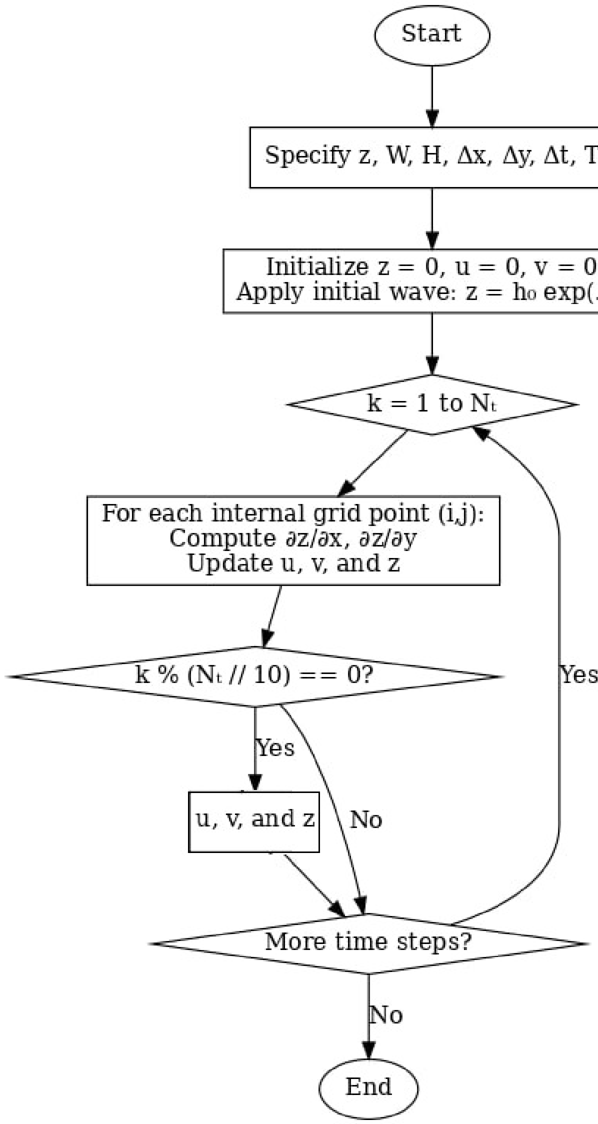

Figure 1.Simulation algorithm.

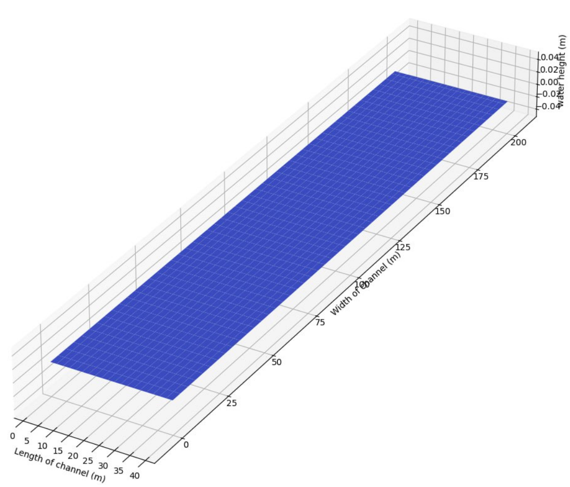

Figure 2.Schematic of an open channel with free surface.

3. Results and Discussion

In modeling water movement, the results of numerical methods were used to calculate the water level h (m) and velocity components $u(x, y)$ and $v(x, y)$ at different time stages, and visualization was carried out based on the numerical solutions. The graphs generated by the Python program show the simulation states based on the water level and velocity parameters at various time stages. These simulation states are useful in understanding the changes in water levels and flow velocities in different parts of the flow channel.

From the point where the wave is generated, the propagation of the wave along the flow can be studied and analyzed based on its height and velocity.

The results can be observed in the range from 2 m to 198 m along the length of the flow channel and from 2 m to 38 m along its width.

As seen in Fig. 3, based on the color scales, the maximum water height in the flow channel reaches 5 meters at time ($t = 10$ s).

Fig. 4 shows the simulation results for the velocity components u and v, respectively. The influence of velocity components in the flow channel areas during different time intervals can be easily compared.

The distribution of water velocity at different time stages is useful in predicting water movement. If initial data are obtained, this can be useful for managing flood warning systems. According to the analysis of the obtained results, in cases where no wave is generated, the simple flow channel state is limited, and only the effect of the water wave was analyzed numerically.

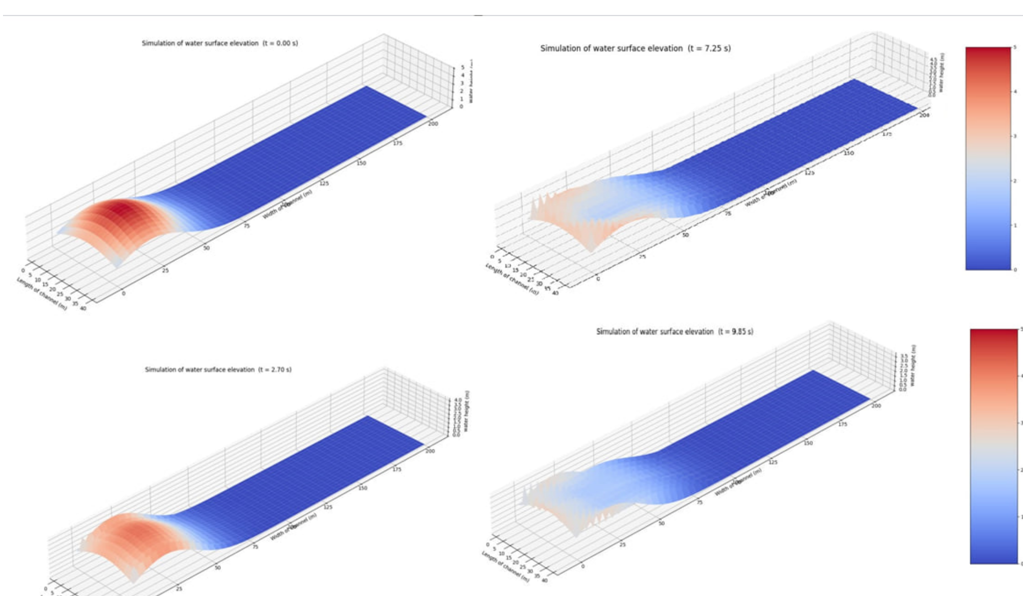

Figure 3. Simulation of water surface elevation; depth or height (m) at different time steps.

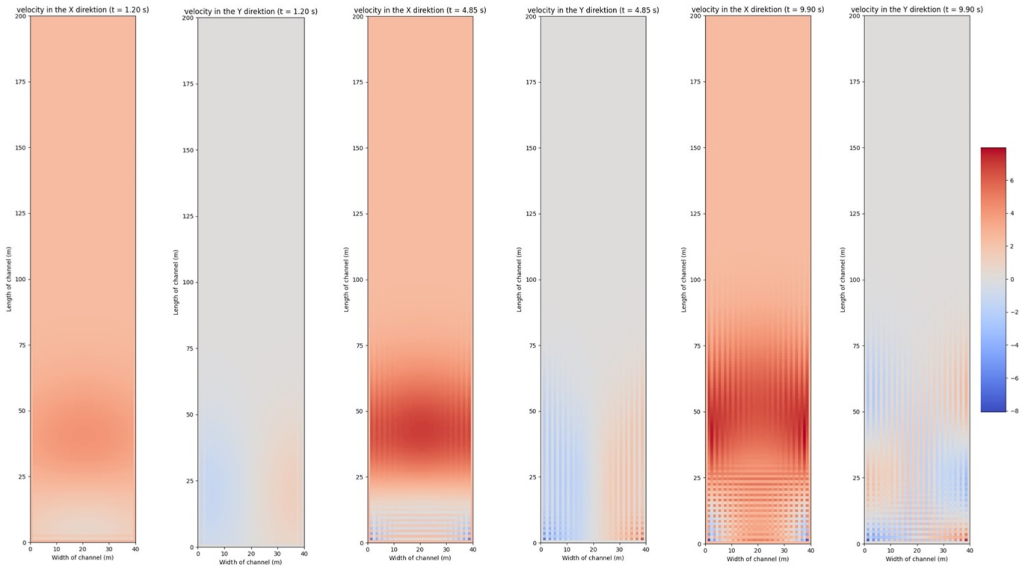

Figure 4. Water wave propagation speed parameters. x direction: u (m/s), y direction: v (m/s) at different time steps.

4. Conclusion

The two-dimensional Saint-Venant shallow water flow equations, which effectively model water movements in channels, enable the study and management of flood wave propagation. The selected equations for modeling were solved using the finite difference method. As an experiment, a wave was generated at the inlet of a rectangular channel (based on the wave height parameter) and analyzed. The simulation of water surface elevation and velocity parameters was developed and visualized at different time steps. Based on the initial simulation results and the models closeness to reality, the wave motion over time at various points in the channel was sufficiently understood, making it useful for early flood mitigation and management.

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On An Interpolation of a Function by Exponential-Trigonometric Natural Splines

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Abstract. One of the challenges in approximation is the interpolation problem. Splines are often used for function approximation. The theory of splines includes both algebraic and variational approaches. In both cases, the existence, uniqueness, and convergence of splines, along with algorithms for their construction, are studied based on their inherent properties. In this paper, we study the problem of constructing exponential, exponential-trigonometric natural splines in the Hilbert space $W_2^{(m,0)}(0,1)$. Here, using the Sobolev method, we present an algorithm for solving a system of linear algebraic equations for the coefficients of natural splines. For $m=1$ and $m=3$, we obtain explicit expressions for the coefficients of natural splines in the Hilbert spaces $W_2^{(1,0)}(0,1)$ and $W_2^{(3,0)}(0,1)$.

Keywords: Approximation, Interpolation, the extremal function, Hilbert space, coefficients, exponential natural spline, exponential - trigonometric natural spline.

MSC (2020): 65D05, 65D07

1. Introduction

One of the challenges in approximation is the interpolation problem. Traditionally, this is solved by constructing an interpolation polynomial. However, polynomial approximation becomes impractical when attempting to approximate functions with limited smoothness, which is common in many applications. As a result, splines are often used for function approximation. The theory of splines includes both algebraic and variational approaches. In the algebraic approach, splines are viewed as smooth piecewise polynomial functions. In the variational approach, splines are considered as elements of a Hilbert or Banach space that minimize specific functionals. In both cases, the existence, uniqueness, and convergence of splines, along with algorithms for their construction, are studied based on their inherent properties [1, 2, 3, 4, 5, 6].

In this context, alternative approximation methods that overcome the limitations of the previous ones have been actively developed. One such method, which has proven effective in both theoretical studies and practical applications, is the use of splines.

The theory of splines based on variational methods was studied and developed in the works [7, 8, 9, 10, 11, 12].

In the present work we study the problem of construction of a natural L-spline, $L \equiv d^m/dx^m + 1$ on the interval $[0,1]$ in the vector space $W_2^{(m,0)}(0,1)$. This space is defined as a

$$W_2^{(m,0)}(0,1) = \Big\{\varphi : [0,1] \to R \;\Big|\; \varphi^{(m-1)} \text{ is abs. cont. and } \varphi^{(m)} \in L_2(0,1)\Big\}.$$

The class $W_2^{(m,0)}(0,1)$, equipped with the semi-norm

$$\|\varphi\|_{W_2^{(m,0)}} = \left(\int_0^1 \left(\varphi^{(m)}(x) + \varphi(x)\right)^2 \mathrm{d}x\right)^{1/2}, \tag{1.1}$$

is a Hilbert space if we identify functions that differ by a solution of the equation $\varphi^{(m)}(x) + \varphi(x) = 0$.

Suppose, we are given the values y_β, $\beta = 0,1,...,N$ at points $x_\beta \in [0,1]$, $\beta = 0,1,...,N$.

Consider the following variational interpolation problem.

Problem 1. Among all functions $\varphi(x)$ in $W_2^{(m,0)}(0,1)$ satisfying the conditions

$$\varphi(x_\beta) = y_\beta, \quad \beta = 0,1,...,N, \tag{1.2}$$

find a function $S_m(x)$ which gives the minimum to the norm (1.1), where $x_\beta \in [0,1]$ are the nodes of interpolation.

The solution $S_m(x)$ of Problem 1 is a generalized spline and is uniquely defined with respect to mesh $\Delta: \ 0 = x_0 < x_1 < ... < x_N = 1$ on the interval $[0,1]$ as follows:

(i) $S_m(x)$ is a linear combination of exponential-trigonometric functions that satisfy the $L^*LS_m = 0$ equality on each open mesh interval $(x_\beta, x_{\beta+1})$, $\beta = 0,1,...,N-1$, where L^* is the formal adjoint of L;

(ii) $S_m(x)$ is a linear combination of exponential-trigonometric functions that satisfy the $LS_m = 0$ equality on intervals $(-\infty, 0)$ and $(1, \infty)$;

(iii) $S_m^{(\alpha)}(x_\beta^-) = S_m^{(\alpha)}(x_\beta^+)$, $\alpha = 0,1,...,m$ and $\beta = 1,2,...,N-1$;

(iv) $S_m(x_\beta) = y_\beta$, $\beta = 0,1,...,N$;

(v) $S_m(x)$ satisfies the following boundary conditions

$$\begin{array}{l} S_m^{(m+i)}(1) + S_m^{(i)}(1) = 0, \quad i = 0,1,\dots,m-2, \\ S_m^{(m+i)}(0) + S_m^{(i)}(0) = 0, \quad i = 0,1,\dots,m-2. \end{array}$$

We consider fundamental solution

$$G_m(x) = \frac{\mathrm{sgn} x}{2m} \cdot \left[\sinh(x) + \sum_{k=1}^{m-1} e^{x\cos\frac{\pi k}{m}} \cos\left(x \sin\left(\frac{\pi k}{m}\right) + \frac{\pi k}{m}\right)\right] \tag{1.3}$$

of the differential operator $\frac{d^{2m}}{dx^{2m}} - 1$ for odd natural numbers m, i.e., the solution of the equation

$$G_m^{(2m)}(x) - G_m(x) = \delta(x), \tag{1.4}$$

where $\delta(x)$ is Dirac's delta-function.

This work is a direct continuation of the work [13]. Following ([2], p.46, Theorem 2.2) we get

$$S_m(x) = \sum_{\gamma=0}^{N} C_\gamma G_m(x - x_\gamma) + Y_m(x), \tag{1.5}$$

where $C_\gamma, \gamma = 0,1,\dots,N, G_m(x)$ is defined by (1.3) and

$$Y_m(x) = d_0 e^{-x} + \sum_{k=1}^{\frac{m-1}{2}} e^{-x\cos\left(\frac{2\pi k}{m}\right)} \left[d_{1,k} \cos\left(x \sin\left(\frac{2\pi k}{m}\right)\right) + d_{2,k} \sin\left(x \sin\left(\frac{2\pi k}{m}\right)\right)\right] \tag{1.6}$$

where $d_0, d_{1,k}$ and $d_{2,k}$ are real numbers.

It is known that (see, for instance,[2]) the solution $S_m(x)$ of the form (1.5) of Problem 1 exists, is unique when $N+1 \geq m$ and coefficients C_γ, d_0, $d_{1,k}$ and $d_{2,k}$ of $S_m(x)$ are determined by the following system of $N+m+1$ linear equations

$$\sum_{\gamma=0}^{N} C_\gamma G_m(x_\beta - x_\gamma) + Y_m(x_\beta) = \varphi(x_\beta), \quad \beta = 0,1,...,N, \tag{1.7}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-x_\gamma} = 0, \tag{1.8}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-x_\gamma \cos\frac{2\pi k}{m}} \cos\left(x_\gamma \sin\frac{2\pi k}{m}\right) = 0, \quad k = \overline{1, \frac{m-1}{2}}, \tag{1.9}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-x_\gamma \cos\frac{2\pi k}{m}} \sin\left(x_\gamma \sin\frac{2\pi k}{m}\right) = 0, \quad k = \overline{1, \frac{m-1}{2}}. \tag{1.10}$$

It is easy to show that the spline $S_m(x)$, defined by equation (1.5) with coefficients C_γ, d_0, $d_{1,k}$ and $d_{2,k}(k = \overline{1, \frac{m-1}{2}})$, satisfies the conditions (i)-(v).

It should be noted that system (1.7) - (1.10) has a unique solution. The proof of the uniqueness of the solution of system (1.7) - (1.10) is similar to the proof of the uniqueness of the solution of the system for optimal coefficients in the space $L_2^{(m)}$ obtained in the works of Sobolev [14, 15].

Next, let's consider the case of equidistant nodes. Suppose that $x_\beta = h\beta, \beta = 0,1,2,...,N, h = \frac{1}{N}$.

Now we suppose that $C_\beta = 0$ when $\beta < 0$ and $\beta > N$. Then, using the convolution of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ (see. [14, 15]):

$$\varphi(h\beta) * \psi(h\beta) = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma),$$

we rewrite the system (1.7) - (1.10) in the following convolution form

$$C_\beta * G_m(h\beta) + Y_m(h\beta) = \varphi(h\beta), \quad \beta = 0,1,...,N, \tag{1.11}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-h\gamma} = 0, \tag{1.12}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-h\gamma \cos\frac{2\pi k}{m}} \cos\left(h\gamma \sin\frac{2\pi k}{m}\right) = 0, \quad k = \overline{1, \frac{m-1}{2}}, \tag{1.13}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-h\gamma \cos\frac{2\pi k}{m}} \sin\left(h\gamma \sin\frac{2\pi k}{m}\right) = 0, \quad k = \overline{1, \frac{m-1}{2}}. \tag{1.14}$$

To solve the system (1.11) - (1.14) by the Sobolev method, we need a discrete analogue of the differential operator $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - 1$ for odd m . The next section is devoted to the construction of this discrete operator $D_m(h\beta)$.

2. The discrete operator $D_m(h\beta)$

In this section, for an odd natural number m, a function $D_m(h\beta)$ of a discrete argument is given that satisfies the equation

$$D_m(h\beta) * G_m(h\beta) = \delta_{\mathrm{d}}(h\beta), \tag{2.1}$$

where

$$G_m(h\beta)=\frac{\mathrm{sgn}(h\beta)}{2m}\left(\sinh(h\beta)+\sum_{n=1}^{m-1}\mathrm{e}^{h\beta\cos\frac{\pi n}{m}}\cos\left(h\beta\sin\frac{\pi n}{m}+\frac{\pi n}{m}\right)\right), \tag{2.2}$$

$\delta_{\mathrm{d}}(h\beta)$ is the discrete delta function, i.e., $\delta_{\mathrm{d}}(h\beta)=\begin{cases}1, & \beta=0,\\ 0, & \beta\neq 0.\end{cases}$

The discrete function $D_m(h\beta)$ plays an important role in calculating the coefficients of exponential and exponential-trigonometric natural splines in the space $W_2^{(m,0)}(0,1)$. We note that equation (2.1) is a discrete analogue of the following equation

$$\left(\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}-1\right)G_m(x)=\delta(x), \tag{2.3}$$

where $G_m(x)$ is defined by equality (1.3), δ is the Dirac delta function.

The following is true:

Theorem 2.1. *The discrete analogue of the differential operator $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}-1$ satisfying equality (2.1), for odd m, has the form*

$$D_m(h\beta)=\frac{m}{K}\begin{cases}\sum\limits_{n=1}^{m-1}A_n\lambda_n^{|\beta|-1}, & |\beta|\geq 2,\\ 1+\sum\limits_{n=1}^{m-1}A_n, & |\beta|=1,\\ M_1-\frac{K_1}{K}+\sum\limits_{n=1}^{m-1}\frac{A_n}{\lambda_n}, & \beta=0,\end{cases} \tag{2.4}$$

where K, K_1, M_1, A_n and λ_n are defined in [16] .

Theorem 2.2. *The discrete analogue $D_m(h\beta)$ of the differential operator $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}-1$ satisfies the equalities*

1) $D_m(h\beta)*\mathrm{e}^{h\beta}=0$,

2) $D_m(h\beta)*\mathrm{e}^{-h\beta}=0$,

3) $D_m(h\beta)*\mathrm{e}^{h\beta\cos\left(\frac{2\pi k}{m}\right)}\cos\left(h\beta\sin\frac{2\pi k}{m}\right)=0$,

4) $D_m(h\beta)*\mathrm{e}^{-h\beta\cos\left(\frac{2\pi k}{m}\right)}\cos\left(h\beta\sin\frac{2\pi k}{m}\right)=0$,

5) $D_m(h\beta)*\mathrm{e}^{h\beta\cos\left(\frac{2\pi k}{m}\right)}\sin\left(h\beta\sin\frac{2\pi k}{m}\right)=0$,

6) $D_m(h\beta)*\mathrm{e}^{-h\beta\cos\left(\frac{2\pi k}{m}\right)}\sin\left(h\beta\sin\frac{2\pi k}{m}\right)=0$,

where $k=1,2,...,\frac{m-1}{2}$, m is an odd natural number.

The above theorems are proved in the paper [16].

3. Solution of the discrete system in convolutional form (1.11) - (1.14)

In this section we give an algorithm for finding the exact solution of the system (1.11) - (1.14) using the discrete operator $D_m(h\beta)$ obtained in the previous section.

Let's introduce the functions

$$\vartheta_m(h\beta)=C_\beta * G_m(h\beta) \tag{3.1}$$

and

$$u_m(h\beta)=\vartheta_m(h\beta)+Y_m(h\beta,d_0,d_{1,k},d_{2,k}), \tag{3.2}$$

where $Y_m(h\beta,d_0,d_{1,k},d_{2,k})$ is a discrete function of $Y_m(x)$ defined by (1.6).

Then, taking into account (2.1), for the coefficients C_β we have

$$C_\beta = D_m(h\beta) * u_m\,(h\beta)\,. \tag{3.3}$$

So, if we find the function $u_m\,(h\beta)$, then the optimal coefficients are determined from the formula (3.3). In order to calculate the convolution (3.3), we need to find the representation of the function $u_m\,(h\beta)$ for all integer values of the argument β. From equality (1.11) it is clear that $u_m\,(h\beta) = \varphi(h\beta)$ for $h\beta \in [0,1]$.

$$\begin{aligned}\vartheta_m(h\beta) \;\; &= \;\; -\frac{1}{2m}\sum_{\gamma=0}^{N} C_\gamma\left[\frac{\mathrm{e}^{h\beta-h\gamma} - \mathrm{e}^{-h\beta+h\gamma}}{2}\right.\\ &\left.+\sum_{n=1}^{m-1}\mathrm{e}^{(h\beta-h\gamma)\cos\frac{\pi n}{m}}\cos\left((h\beta-h\gamma)\sin\frac{\pi n}{m}+\frac{\pi n}{m}\right)\right].\end{aligned} \tag{3.4}$$

From here, the internal sum over n, dividing by even and odd values of n, we obtain

$$\sum_{n=1}^{m-1}\mathrm{e}^{(h\beta-h\gamma)\cos\frac{\pi n}{m}}\cos\left((h\beta-h\gamma)\sin\frac{\pi n}{m}+\frac{\pi n}{m}\right) = S_1+S_2, \tag{3.5}$$

where

$$\begin{aligned} S_1 \;\; &= \;\; \sum_{k=1}^{\frac{m-1}{2}}\mathrm{e}^{(h\beta-h\gamma)\cos\frac{2\pi k}{m}}\cos\left((h\beta-h\gamma)\sin\frac{2\pi k}{m}+\frac{2\pi k}{m}\right),\\ S_2 \;\; &= \;\; \sum_{k=1}^{\frac{m-1}{2}}\mathrm{e}^{(h\beta-h\gamma)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\beta-h\gamma)\sin\frac{(2k-1)\pi}{m}+\frac{(2k-1)\pi}{m}\right).\end{aligned}$$

Assuming $2k = m+1-2j$ for S_2, we have

$$S_2 = -\sum_{j=1}^{\frac{m-1}{2}}\mathrm{e}^{-(h\beta-h\gamma)\cos\frac{2\pi j}{m}}\cos\left((h\gamma-h\beta)\sin\frac{2\pi j}{m}+\frac{2\pi j}{m}\right).$$

Hence,

$$S_2 = -\sum_{k=1}^{\frac{m-1}{2}}\mathrm{e}^{(h\gamma-h\beta)\cos\frac{2\pi k}{m}}\cos\left((h\gamma-h\beta)\sin\frac{2\pi k}{m}+\frac{2\pi k}{m}\right).$$

Now, using the formula $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, for S_1 and S_2 we get

$$\begin{aligned} S_1 \;\; &= \;\; \sum_{k=1}^{\frac{m-1}{2}}\left[\mathrm{e}^{-h\gamma\cos\frac{2\pi k}{m}}\cos\left(h\gamma\sin\frac{2\pi k}{m}\right)\mathrm{e}^{h\beta\cos\frac{2\pi k}{m}}\cos\left(h\beta\sin\frac{2\pi k}{m}+\frac{2\pi k}{m}\right)\right.\\ &\left.+\,\mathrm{e}^{-h\gamma\cos\frac{2\pi k}{m}}\sin\left(h\gamma\sin\frac{2\pi k}{m}\right)\mathrm{e}^{h\beta\cos\frac{2\pi k}{m}}\sin\left(h\beta\sin\frac{2\pi k}{m}+\frac{2\pi k}{m}\right)\right],\\ S_2 \;\; &= \;\; \sum_{k=1}^{\frac{m-1}{2}}\left[\mathrm{e}^{-h\beta\cos\frac{2\pi k}{m}}\cos\left(h\beta\sin\frac{2\pi k}{m}\right)\mathrm{e}^{h\gamma\cos\frac{2\pi k}{m}}\cos\left(h\gamma\sin\frac{2\pi k}{m}+\frac{2\pi k}{m}\right)\right.\\ &\left.+\,\mathrm{e}^{-h\beta\cos\frac{2\pi k}{m}}\sin\left(h\beta\sin\frac{2\pi k}{m}\right)\mathrm{e}^{h\gamma\cos\frac{2\pi k}{m}}\sin\left(h\gamma\sin\frac{2\pi k}{m}+\frac{2\pi k}{m}\right)\right].\end{aligned}$$

Substituting the sum (3.5) of the last expressions for S_1 and S_2 into (3.4) and using the equalities (1.12)-(1.14), after some simplifications for $\beta < 0$, we reduce the expression (3.4) to the form

$$\vartheta_m(h\beta) = -Y_m(h\beta, b_0, b_{1,k}, b_{2,k}),$$

where

$$\begin{aligned} Y_m(h\beta, b_0, b_{1,k}, b_{2,k}) &= b_0\, \mathrm{e}^{-h\beta} \\ &+ \sum_{k=1}^{\frac{m-1}{2}} \mathrm{e}^{-h\beta\cos\frac{2\pi k}{m}} \left[b_{1,k} \cos\left(h\beta \sin\frac{2\pi k}{m}\right) + b_{2,k} \sin\left(h\beta\sin\frac{2\pi k}{m}\right)\right], \end{aligned} \tag{3.6}$$

where $b_0, b_{1,k}, b_{2,k},\ k = 1, 2, ..., \frac{m-1}{2}$ are unknowns.

Calculations show that $\vartheta_m(h\beta)$ for $\beta > N$ has the form

$$\vartheta_m(h\beta) = Y_m(h\beta, b_0, b_{1,k}, b_{2,k}).$$

Using the last two expressions for $\vartheta_m(h\beta)$ and taking into account (3.2), we have

$$u_m(h\beta) = \begin{cases} Y_m(h\beta, d_0, d_{1,k}, d_{2,k}) - Y_m(h\beta, b_0, b_{1,k}, b_{2,k}), & \beta < 0, \\ Y_m(h\beta, d_0, d_{1,k}, d_{2,k}) + Y_m(h\beta, b_0, b_{1,k}, b_{2,k}), & \beta > N, \end{cases} \tag{3.7}$$

where $Y_m(h\beta, d_0, d_{1,k}, d_{2,k})$ is a discrete function of $Y_m(x)$ defined by (1.6) and $Y_m(h\beta, b_0, b_{1,k}, b_{2,k})$ defined by (3.6). Here $d_0, d_{1,k}, d_{2,k}$ and $b_0, b_{1,k}, b_{2,k}$ are unknowns.

Let us designate

$$\begin{aligned} &d_0^- = d_0 - b_0, \quad d_{1,k}^- = d_{1,k} - b_{1,k}, \quad d_{2,k}^- = d_{2,k} - b_{2,k},\ k = 1, 2, ..., \tfrac{m-1}{2}, \\ &d_0^+ = d_0 + b_0, \quad d_{1,k}^+ = d_{1,k} + b_{1,k}, \quad d_{2,k}^+ = d_{2,k} + b_{2,k},\ k = 1, 2, ..., \tfrac{m-1}{2}. \end{aligned} \tag{3.8}$$

If we find the unknowns $d_0^-, d_{1,k}^-, d_{2,k}^-$ and $d_0^+, d_{1,k}^+, d_{2,k}^+,\ k = 1, 2, ..., \frac{m-1}{2}$, then from (3.8) we obtain $d_0, d_{1,k}, d_{2,k}$ and $b_0, b_{1,k}, b_{2,k},\ k = 1, 2, ..., \frac{m-1}{2}$ as follows

$$\begin{aligned} &d_0 = \tfrac{1}{2}(d_0^+ + d_0^-), \quad d_{1,k} = \tfrac{1}{2}(d_{1,k}^+ + d_{1,k}^-), \quad d_{2,k} = \tfrac{1}{2}(d_{2,k}^+ + d_{2,k}^-),\ k = 1, 2, ..., \tfrac{m-1}{2}, \\ &b_0 = \tfrac{1}{2}(d_0^+ - d_0^-), \quad b_{1,k} = \tfrac{1}{2}(d_{1,k}^+ - d_{1,k}^-), \quad b_{2,k} = \tfrac{1}{2}(d_{2,k}^+ - d_{2,k}^-),\ k = 1, 2, ..., \tfrac{m-1}{2}. \end{aligned} \tag{3.9}$$

Since for $\beta < 0$ and $\beta > N$ the coefficients $C_\beta = 0$, then from (3.3) for $d_0^-, d_{1,k}^-, d_{2,k}^-$ and $d_0^+, d_{1,k}^+, d_{2,k}^+,\ k = 1, 2, ..., \frac{m-1}{2}$ we obtain the following system of $2m$ linear equations

$$D_m(h\beta) * u_m(h\beta) = 0 \text{ for } \beta = -1, -2, ..., -m \text{ and } \beta = N + 1, N + 2, ..., N + m.$$

Solving the last system taking into account (3.8) and (3.9) from equalities (1.11) and (3.7) we obtain the following explicit form for $u_m(h\beta)$:

$$u_m(h\beta) = \begin{cases} Y_m(h\beta, d_0^-, d_{1,k}^-, d_{2,k}^-), & \beta < 0, \\ \varphi(h\beta), & \beta = 0, 1, ..., N, \\ Y_m(h\beta, d_0^+, d_{1,k}^+, d_{2,k}^+), & \beta > N. \end{cases}$$

Then we obtain the coefficients C_β of the natural splines of the form (1.5) as follows

$$C_\beta = D_m(h\beta) * u_m(h\beta),\ \beta = 0, 1, ..., N.$$

Thus, the construction of natural splines of the form (1.2) in the space $W_2^{(m,0)}(0, 1)$ for odd natural numbers m is solved. In the next section we implement this algorithm in the cases $m = 1$ and $m = 3$.

4. Exponential-trigonometric natural splines

In this section we present the results of the implementation of the algorithm for constructing exponential-trigonometric natural splines (1.2) in the space $W_2^{(m,0)}(0,1)$ in the cases $m=1$ and $m=3$.

The case $m=1$: In this case Problem 1 is as follows.

Problem 2. Find the function $S_1(x) \in W_2^{(1,0)}(0,1)$ which minimizes the quantity

$$\int_0^1 (\varphi'(x)+\varphi(x))^2 \mathrm{d}x$$

and satisfies the interpolation conditions

$$S_1(h\beta)=\varphi(h\beta), \beta=0,1,...,N,$$

where $h=\frac{1}{N}, N=1,2,..., \varphi \in W_2^{(1,0)}(0,1)$.

The solution of Problem 2 is the exponential natural spline $S_1(x)$ and it has the following form

$$S_1(x)=\sum_{\gamma=0}^{N} C_\gamma \frac{\mathrm{sgn}(x-h\gamma)}{4}\left(e^{x-h\gamma}-e^{h\gamma-x}\right)+d_0 e^{-x}$$

and coefficients $C_\gamma, \gamma=0,1,...,N$ and d_0 of this spline satisfy the system

$$\sum_{\gamma=0}^{N} C_\gamma \frac{\mathrm{sgn}(h\beta-h\gamma)}{4}\left(e^{h\beta-h\gamma}-e^{h\gamma-h\beta}\right)+d_0 e^{-h\beta}=\varphi(h\beta), \quad \beta=0,1,...,N, \tag{4.1}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-h\gamma}=0. \tag{4.2}$$

In the paper [17] the system (4.1) - (4.2) was solved and the following theorem was proved.

Theorem 4.1. *The coefficients of the exponential natural spline $S_1(x)$ in the space $W_2^{(1,0)}(0,1)$ have the following form*

$$\begin{aligned}
C_0 &= \frac{2}{1-e^{2h}}\left(\varphi(0)-e^h\varphi(h)\right),\\
C_\beta &= \frac{2}{1-e^{2h}}\left((1+e^{2h})\varphi(h\beta)-e^h(\varphi(h\beta-h)+\varphi(h\beta+h))\right), \beta=1,2,...,N-1,\\
C_N &= \frac{2}{1-e^{2h}}\left(e^{2h}\varphi(1)-e^h\varphi(1-h)\right),\\
d_0 &= \frac{1}{2}\left(\varphi(0)+e\varphi(1)\right).
\end{aligned}$$

The case $m=3$: Then Problem 1 is as follows.

Problem 3. Find the function $S_3(x) \in W_2^{(3,0)}(0,1)$ which minimizes

$$\int_0^1 (\varphi'''(x)+\varphi(x))^2 \mathrm{d}x$$

and satisfies the interpolation conditions

$$S_3(h\beta)=\varphi(h\beta), \beta=0,1,...,N,$$

where $h=\frac{1}{N}, N=1,2,...,\varphi\in W_2^{(3,0)}(0,1)$.

The solution of Problem 3 is the exponential-trigonometric natural spline $S_3(x)$ and has the following form

$$S_3(x)=\sum_{\gamma=0}^{N}C_\gamma G_3(x-h\gamma)+d_0e^{-x}+d_{1,1}e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)+d_{2,1}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

and coefficients $C_\gamma, \gamma=0,1,...,N$ and $d_0, d_{1,1}, d_{2,1}$ of this spline satisfy the system

$$\sum_{\gamma=0}^{N}C_\gamma G_3(h\beta-h\gamma)+d_0e^{-h\beta}+d_{1,1}e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)$$

$$+d_{2,1}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)=\varphi(h\beta),\quad \beta=0,1,...,N, \tag{4.3}$$

$$\sum_{\gamma=0}^{N}C_\gamma e^{-h\gamma}=0, \tag{4.4}$$

$$\sum_{\gamma=0}^{N}C_\gamma e^{\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right)=0, \tag{4.5}$$

$$\sum_{\gamma=0}^{N}C_\gamma e^{\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)=0. \tag{4.6}$$

In the paper [18] the system (4.3) - (4.6) was solved and the following theorem was proved.

Theorem 4.2. *The coefficients of the exponential-trigonometric natural spline spline $S_3(x)$ in the space $W_2^{(3,0)}(0,1)$ have the following form*

$$C[0]=\frac{3}{K}\Bigg[\varphi(0)\left(K_3-K_1\right)+\varphi(h)+d_0^-e^h+d_{1,1}^-e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-d_{2,1}^-e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)$$
$$+\sum_{k=1}^{2}\frac{A_k}{\lambda_k}\left(\sum_{\gamma=0}^{N}\lambda_k^\gamma\varphi(h\gamma)+M_k+\lambda_k^N\cdot N_k\right)\Bigg],$$

$$C[\beta]=\frac{3}{K}\Bigg[\varphi(h\beta)\left(K_3-K_1\right)+\varphi(h(\beta-1))+\varphi(h(\beta+1))$$
$$+\sum_{k=1}^{2}\frac{A_k}{\lambda_k}\left(\sum_{\gamma=0}^{N}\lambda_k^{|\beta-\gamma|}\varphi(h\gamma)+\lambda_k^\beta\cdot M_k+\lambda_k^{N-\beta}\cdot N_k\right)\Bigg],\beta=1,2,...,N-1,$$

$$C[N]=\frac{3}{K}\Bigg[\varphi(1)\left(K_3-K_1\right)+\varphi(1-h)+d_0^+e^{-(1+h)}+d_{1,1}^+e^{\frac{1+h}{2}}\cos\left(\frac{\sqrt{3}}{2}(1+h)\right)$$
$$+d_{2,1}^+e^{\frac{1+h}{2}}\sin\left(\frac{\sqrt{3}}{2}(1+h)\right)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k}\left(\sum_{\gamma=0}^{N}\lambda_k^{N-\gamma}\varphi(h\gamma)+\lambda_k^N\cdot M_k+N_k\right)\Bigg],$$

where $K_1, K_3, d_0^-, d_0^+, A_k, \lambda_k, d_{k,1}^-, d_{k,1}^+, M_k$ and $N_k (k=1,2)$ are given in [18] .

5. Conclusion

Thus, in this paper, using the Sobolev method, we presented an algorithm for solving a system of algebraic equations for the coefficients of natural splines of the form (1.2). Solving the system (1.11)-(1.14) in the cases $m = 1$ and $m = 3$, we obtained explicit expressions for the coefficients C_β and using these coefficients we constructed exponential-trigonometric natural splines of the form $S_1(x)$ and $S_3(x)$ in the spaces $W_2^{(1,0)}(0,1)$ and $W_2^{(3,0)}(0,1)$, respectively.

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Results on Best Proximity Point in Orthogonal Partially Ordered Metric Space

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Abstract. This research paper delves into the application of orthogonal partially ordered metric spaces within fixed-point theory, particularly focusing on introducing the concept of Best Proximity Point (BPP). The study establishes theorems related to BPP, employing orthogonal partially ordered contractive mappings. Moreover, the paper explores the implications of these theorems, considering both self-mappings and non-self mappings that share the same parameter set. It also includes examples to illustrate the practical relevance of the established theorems and corollaries in various mathematical contexts. Through introducing this innovative concept, the study provides a theoretical framework for analyzing mappings in diverse scenarios.

Keywords: Best Proximity Point, orthogonal set, partially ordered set, orthogonal partially ordered metric space, orthogonal proximal contraction mapping.

MSC (2020): 46N20, 46T20.

1. Introduction

Over the past century, fixed point theory has been a vibrant research field due to its significant applications. Fixed-point theory plays a crucial role in the study of nonlinear analysis across diverse fields such as biology, chemistry, engineering, and physics. It provides a powerful framework for solving differential and integral equations, optimization problems, and variational inequalities. With its wide range of applications, fixed-point theory has become fundamental to understanding and analyzing complex systems and structures. Concurrently, in functional analysis, the concept of proximity pairs for two sets has been explored briefly. Researchers have extensively studied the conditions under which the best proximity points for sets can be found. While some researchers focused on fixed point results, others examined scenarios where exact solutions to the equation $T(x) = x$ are not available. Pioneers like Ky Fan[11], Segal, Singh[31], and Prolla[26] have made substantial contributions to best approximation theory, providing insights into situations lacking fixed points. Under specific smooth conditions, they demonstrated the existence of approximate solutions to such equations.

Notably, Ky Fan[11] proved the existence of the best approximation for a continuous function on a compact convex subset of a normed space. Later, in 1989, Segal et al. [31] established the existence of the best approximation for an approximately compact subset of a normed space. Prolla[26] further extended this concept to multifunctions. Towards the late 1990s and early 2000s, researchers began unifying fixed point and best approximation results using the idea of the best proximity point for mappings[8, 9, 17]. This line of research led to numerous generalizations by many researchers[2, 23, 24, 7, 1, 30, 32, 38, 29].

Simultaneously, the Banach contraction principle emerged as a fundamental discovery in fixed point theory. It has been generalized and applied to various metric spaces, including semi-metrics, quasi-metrics, pseudo-metrics, fuzzy metric spaces, and partial metric spaces and several other generalizations[24, 7, 18, 5, 12, 20, 19, 35, 34, 33, 13].

In 2017, Gordji et al.[14, 10] introduced a novel type of metric space called an orthogonal metric space and proved fixed point results within this framework. They demonstrated the application of these results to establish the existence and uniqueness of solutions for first-order ordinary differential equations, where the Banach contraction mapping principle is not applicable. In 2023, Poonguzali et al. [22] introduced the conception of best proximity points in orthogonal metric spaces. Further, advancements in the theory, with subsequent generalizations were proposed by various authors[15, 16, 21, 37, 22, 36, 3]. Notably, Prakasam[25] initiated the concept of orthogonal partially ordered metric spaces in 2023. The absence of the concept of best proximity points within orthogonal partially ordered metric spaces has left a gap in our understanding. Our research addresses this gap by introducing the notion of best proximity points within orthogonal partially ordered metric spaces and establishing related results.

This novel approach employs orthogonal partially ordered contractive mappings and introduces a new inequality involving rational contractions and a control function for $\alpha-$proximal admissible mappings. By doing so, we derive optimality results that enhance our comprehension of best proximity points for non-self mappings within orthogonal partially ordered metric spaces. This innovative contribution broadens the scope of best proximity point theory and paves the way for further exploration in this promising research domain.

2. Preliminaries

The following are some fundamental concepts for the writing of the article:
Let us assume that $\mathcal{U}$ and $\mathcal{V}$ are non-empty subsets of an orthogonal partially ordered metric space $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$. We define

$$\begin{aligned}
\mathfrak{d}(\mathcal{U}, \mathcal{V}) &= \inf \{\mathfrak{d}(\mathfrak{u}, \mathfrak{v}) : \mathfrak{u} \in \mathcal{U} \text{ and } \mathfrak{v} \in \mathcal{V}\}, \\
\mathcal{U}_0 &= \{\mathfrak{u} \in \mathcal{U} : \mathfrak{d}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) \text{ for some } \mathfrak{v} \in \mathcal{V}\}, \\
\mathcal{V}_0 &= \{\mathfrak{v} \in \mathcal{V} : \mathfrak{d}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) \text{ for some } \mathfrak{u} \in \mathcal{U}\}.
\end{aligned}$$

It is important to note that for every $\mathfrak{u} \in \mathcal{U}_0$, there exists $\mathfrak{v} \in \mathcal{V}_0$ such that $\mathfrak{d}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{d}(\mathcal{U}, \mathcal{V})$. Also, for every $\mathfrak{v} \in \mathcal{V}_0$, there exists $\mathfrak{u} \in \mathcal{U}_0$ such that $\mathfrak{d}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{d}(\mathcal{U}, \mathcal{V})$.

Definition 2.1. [28, 4] The triple $(\mathfrak{Y}, \mathfrak{d} \preceq)$ is called partially ordered metric space, if $(\mathfrak{Y}, \preceq)$ is a partially ordered set together with $(\mathfrak{Y}, \mathfrak{d})$ is a metric space.

Definition 2.2. [6] In the context of a partially ordered metric space $(\mathfrak{Y}, \mathfrak{d}, \preceq)$, let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of $\mathfrak{Y}$. A mapping $\mathfrak{T} : \mathcal{U} \to \mathcal{V}$ is said to be a proximally increasing mapping if and only if for any $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{v}_1, \mathfrak{v}_2 \in \mathcal{U}$,

$$\left.\begin{aligned}
\mathfrak{v}_1 &\preceq \mathfrak{v}_2 \\
\mathfrak{d}(\mathfrak{u}_1, \mathfrak{T}\mathfrak{v}_1) &= \mathfrak{d}(\mathcal{U}, \mathcal{V}) \\
\mathfrak{d}(\mathfrak{u}_2, \mathfrak{T}\mathfrak{v}_2) &= \mathfrak{d}(\mathcal{U}, \mathcal{V})
\end{aligned}\right\} \Rightarrow \mathfrak{u}_1 \preceq \mathfrak{u}_2$$

Definition 2.3. [27] Suppose $\mathcal{U}$ and $\mathcal{V}$ two non-empty subsets of a metric space $(\mathfrak{Y}, \mathfrak{d})$ such that $\mathcal{U}_0 \neq \emptyset$. The pair $(\mathcal{U}, \mathcal{V})$ satisfies P-property if and only if $\mathfrak{d}(\mathfrak{u}_1, \mathfrak{v}_1) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) = \mathfrak{d}(\mathfrak{u}_2, \mathfrak{v}_2)$ implies $\mathfrak{d}(\mathfrak{u}_1, \mathfrak{u}_2) = \mathfrak{d}(\mathfrak{v}_1, \mathfrak{v}_2)$, for all $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{U}$ and $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathcal{V}$.

Definition 2.4. [14] A set $(\mathfrak{Y}, \perp)$ is said to be orthogonal if there exists an element $\mathfrak{u}_0 \in \mathfrak{Y}$ such that either $\mathfrak{u}_0 \perp \mathfrak{u}$ or $\mathfrak{u} \perp \mathfrak{u}_0$ holds for every $\mathfrak{u} \in \mathfrak{Y}$, where $\perp$ is a binary relation defined over $\mathfrak{Y}$.

Definition 2.5. [14] In a metric space $(\mathfrak{Y}, \mathfrak{d})$ with a binary relation $\perp$ defined over $\mathfrak{Y}$, an ordered triplet $(\mathfrak{Y}, \mathfrak{d}, \perp)$ is considered to be an orthogonal metric space if there exists an element $\mathfrak{u}_0 \in \mathfrak{Y}$ such that $\mathfrak{u}_0 \perp \mathfrak{u}$ holds for every $\mathfrak{u} \in \mathfrak{Y}$.

We now state some neccessary concepts which are important for establishment of main results.

Definition 2.6. A partially ordered metric space $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$ is called an orthogonal partially ordered metric space if $\mathfrak{Y}$ is an orthogonal set with respect to the binary relation $\perp_{\preceq}$. We denote it by $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$, where $\perp_{\preceq}$ is called orthogonal partial ordering relation.

Definition 2.7. Consider an orthogonal partially ordered metric space $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$. Then:
(1) A sequence $\{\mathfrak{u}_m\}_{m\in\mathbb{N}}$ in $\mathfrak{Y}$ is termed as an OPO-sequence in $\mathfrak{Y}$ if $\mathfrak{u}_m \perp_{\preceq} \mathfrak{u}_{m+1}$, for all $m \in \mathbb{N}$.
(2) An OPO-sequence $\{\mathfrak{u}_m\}_{m\in\mathbb{N}}$ is said to converge to a point $\mathfrak{u} \in \mathfrak{Y}$ if $\lim_{m\to\infty} \mathfrak{d}(\mathfrak{u}_m, \mathfrak{u}) = 0$.
(3) A self-mapping $\mathfrak{T}$ on $\mathfrak{Y}$ is considered to be $\perp_{\preceq}$ −continuous at a point $\mathfrak{u} \in \mathfrak{Y}$ if for any OPO-sequence $\{\mathfrak{u}_m\}_{m\in\mathbb{N}}$ of $\mathfrak{Y}$,

$$\lim_{m\to\infty} \mathfrak{d}(\mathfrak{u}_m, \mathfrak{u}) = 0 \implies \lim_{m\to\infty} \mathfrak{d}(\mathfrak{T}\mathfrak{u}_m, \mathfrak{T}\mathfrak{u}) = 0.$$

Moreover, $\mathfrak{T}$ is said to be $\perp_{\preceq}$ −continuous on $\mathfrak{Y}$ if $\mathfrak{T}$ is $\perp_{\preceq}$ −continuous at each point of $\mathfrak{Y}$.
(4) A self-mapping $\mathfrak{T}$ on $\mathfrak{Y}$ is referred to as $\perp_{\preceq}$ −preserving if for $\mathfrak{u}, \mathfrak{v} \in \mathfrak{Y}$,

$$\mathfrak{u} \perp_{\preceq} \mathfrak{v} \implies \mathfrak{T}\mathfrak{u} \perp_{\preceq} \mathfrak{T}\mathfrak{v}.$$

(5) An O-sequence $\{\mathfrak{u}_m\}_{m\in\mathbb{N}}$ in $\mathfrak{Y}$ is described as a Cauchy OPO-sequence if for any $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ so that $\mathfrak{d}(\mathfrak{u}_p, \mathfrak{u}_q) < \epsilon$, for all $p, q \geq m_0$.
(6) $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$ is purported to be an orthogonally complete (O-complete) partiallly ordered metric space if every Cauchy OPO-sequence in $\mathfrak{Y}$ converges to a point in $\mathfrak{Y}$.

3. Main Results

The following are some essential definitions for the results:

Definition 3.1. In the context of an orthogonal partially ordered metric space $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$, let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of $\mathfrak{Y}$. An element $\mathfrak{u}^* \in \mathcal{U}$ is said to be a best proximity point of the mapping $\mathfrak{T} : \mathcal{U} \longrightarrow \mathcal{V}$ if and only if $\mathfrak{d}(\mathfrak{u}^*, \mathfrak{T}\mathfrak{u}^*) = \mathfrak{d}(\mathcal{U}, \mathcal{V})$.

Definition 3.2. In the context of an orthogonal partially ordered metric space $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$, let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of $\mathfrak{Y}$ and let $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty)$. Then, $\mathfrak{T}$ is said to be a α-proximal admissible mapping if $\alpha(\mathfrak{a}_0, \mathfrak{b}_0) \geq 1$ and $\mathfrak{d}(\mathfrak{a}_1, \mathfrak{T}\mathfrak{a}_0) = \mathfrak{d}(\mathfrak{b}_1, \mathfrak{T}\mathfrak{b}_0) = \mathfrak{d}(\mathcal{U}, \mathcal{V})$ implies $\alpha(\mathfrak{a}_1, \mathfrak{b}_1) \geq 1$, for all $\mathfrak{a}_1, \mathfrak{a}_0, \mathfrak{b}_1, \mathfrak{b}_0 \in \mathcal{U}$.

Definition 3.3. In the context of an orthogonal partially ordered metric space $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$, let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of $\mathfrak{Y}$. A mapping $\mathfrak{T} : \mathcal{U} \to \mathcal{V}$ is said to be a proximally $\perp_{\preceq}$-preserving monotone mapping if and only if for any $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{v}_1, \mathfrak{v}_2 \in \mathcal{U}$,

$$\left.\begin{array}{r} \mathfrak{v}_1 \perp_{\preceq} \mathfrak{v}_2 \\ \mathfrak{d}(\mathfrak{u}_1, \mathfrak{T}\mathfrak{v}_1) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) \\ \mathfrak{d}(\mathfrak{u}_2, \mathfrak{T}\mathfrak{v}_2) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) \end{array}\right\} \Rightarrow \mathfrak{u}_1 \perp_{\preceq} \mathfrak{u}_2.$$

Now, we present our results.

Theorem 3.4. *Consider $\mathcal{U}$ and $\mathcal{V}$ as two non-empty closed subsets of an O-complete partially ordered metric space $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$ such that $\mathcal{U}_0 \neq \emptyset$. Consider a mapping $\mathfrak{T} : \mathcal{U} \to \mathcal{V}$ such that for some $0 < \eta < 1$ and $\mathfrak{u} \perp_{\preceq} \mathfrak{v}$*

$$\mathfrak{d}(\mathfrak{T}(\mathfrak{u}), \mathfrak{T}(\mathfrak{v})) \leq \eta\ \mathfrak{d}(\mathfrak{u}, \mathfrak{v}),\ \textit{for all}\ \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U},$$

satisfying the following:
i) $\mathfrak{T}(\mathcal{U}_0) \subseteq \mathcal{V}_0$ and $(\mathcal{U}, \mathcal{V})$ satisfies P-property.
ii) There exist $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{U}_0$ such that

$$\mathfrak{d}(\mathfrak{u}_1, \mathfrak{T}(\mathfrak{u}_0)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}).$$
$$\textit{and} \qquad \mathfrak{u}_0 \perp_{\preceq} \mathfrak{u}_1.$$

iii) $\mathfrak{T}$ is $\perp_{\preceq}$ −continuous and proximally $\perp_{\preceq}$ − preserving monotone.
Then, $\mathfrak{T}$ has a unique best proximity point in $\mathcal{U}$.

Proof. From condition (ii), there exist $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{U}_0$ such that

$$\begin{aligned} \mathfrak{d}(\mathfrak{u}_1, \mathfrak{T}(\mathfrak{u}_0)) &= \mathfrak{d}(\mathcal{U}, \mathcal{V}), \\ \text{and} \qquad \mathfrak{u}_0 &\perp_{\preceq} \mathfrak{u}_1. \end{aligned}$$

Since, $\mathfrak{T}(\mathcal{U}_0) \subseteq \mathcal{V}_0$, there exists $\mathfrak{u}_2 \in \mathcal{U}_0$ such that

$$\mathfrak{d}(\mathfrak{u}_2, \mathfrak{T}(\mathfrak{u}_1)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}).$$

By proximally $\perp_{\preceq} -$ preserving condition of $\mathfrak{T}$, we obtain,

$$\mathfrak{u}_1 \perp_{\preceq} \mathfrak{u}_2.$$

Proceeding this way, we obtain a sequence $\{\mathfrak{u}_n\} \in \mathcal{U}_0$ such that

$$\begin{aligned} \mathfrak{d}(\mathfrak{u}_{n+1}, \mathfrak{T}(\mathfrak{u}_n)) &= \mathfrak{d}(\mathcal{U}, \mathcal{V}) \text{ for all } n \in \mathbb{N}, \\ \text{and} \qquad \mathfrak{u}_n &\perp_{\preceq} \mathfrak{u}_{n+1}. \end{aligned}$$

By P-property,

$$\begin{aligned} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &= \mathfrak{d}(\mathfrak{T}(\mathfrak{u}_{n-1}), \mathfrak{T}(\mathfrak{u}_n)) \\ &\leq \eta \ \mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n) \\ &. \\ &. \\ &. \\ &\leq \eta^n \ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1). \end{aligned}$$

Therefore, $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0$.
Now, for $m > n$,

$$\begin{aligned} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_m) &\leq \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) + \mathfrak{d}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) + \cdots + \mathfrak{d}(\mathfrak{u}_{m-1}, \mathfrak{u}_m) \\ &\leq \eta^n \ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1) + \eta^{n+1} \ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1) + \cdots + \eta^{m-1} \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1). \end{aligned}$$

Letting $m, n \to \infty$, $\mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_m) \to 0$.
Thus, $\{\mathfrak{u}_n\}$ is a Cauchy OPO-sequence.
Since, $\mathfrak{Y}$ is O-complete partially ordered metric space and $\mathcal{U}$ is a closed subset in $\mathfrak{Y}$, there exists $\mathfrak{u}^* \in \mathcal{U}$ such that $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}^*) = 0$ i.e. $\mathfrak{u}_n = \mathfrak{u}^*$.
Now, $\mathfrak{T}$ is $\perp_{\preceq}$ $-$continuous.
Since, $\mathfrak{u}_n \to \mathfrak{u}^*$, we have $\mathfrak{T}(\mathfrak{u}_n) \to \mathfrak{T}(\mathfrak{u}^*)$ as $n \to \infty$.
Therefore

$$\mathfrak{d}(\mathcal{U}, \mathcal{V}) = \mathfrak{d}(\mathfrak{u}_{n+1}, \mathfrak{T}(\mathfrak{u}_n)) \to \mathfrak{d}(\mathfrak{u}^*, \mathfrak{T}(\mathfrak{u}^*)), \quad \text{as} \quad n \to \infty.$$

Thus, $\mathfrak{u}^* \in \mathcal{U}$ is a best proximity point of $\mathfrak{T}$.
Assume that $\mathfrak{v}^* \in \mathcal{U}$ is another best proximity point of $\mathfrak{T}$ such that $\mathfrak{u}^* \perp_{\preceq} \mathfrak{v}^*$.
Then,

$$\mathfrak{d}(\mathfrak{u}^*, \mathfrak{T}(\mathfrak{u}^*)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) = \mathfrak{d}(\mathfrak{v}^*, \mathfrak{T}(\mathfrak{v}^*)).$$

Using P-property,

$$\begin{aligned}\mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*) &= \mathfrak{d}(\mathfrak{T}(\mathfrak{u}^*), \mathfrak{T}(\mathfrak{v}^*)) \\ &\leq \eta\, \mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*) \\ &< \mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*),\end{aligned}$$

which is a contradiction, i.e. $\mathfrak{u}^* = \mathfrak{v}^*$.
Hence, $\mathfrak{T}$ has a unique best proximity point. □

We define the following:

Definition 3.5. Φ, Ψ be the family of functions $\phi, \psi : [0, \infty) \to [0, \infty)$ where
i) ϕ and ψ are increasing.
ii) Both must attain continuity.
iii) $\phi(t) < t$ and $\phi(0) = 0$.

Theorem 3.6. *Consider $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$ as an O-complete partially ordered metric space, and $\mathcal{U}$ and $\mathcal{V}$ as two non-empty closed subsets of $\mathfrak{Y}$ such that $\mathcal{U}_0 \neq \emptyset$. Consider $\mathfrak{T} : \mathcal{U} \to \mathcal{V}$ as an α-proximal admissible mapping such that for some $\mathfrak{p} \in (0, \frac{\pi}{2})$,*

$$\alpha(\mathfrak{u}, \mathfrak{v})\ \mathfrak{d}(\mathfrak{T}(\mathfrak{u}), \mathfrak{T}(\mathfrak{v})) \leq sin\ \mathfrak{p}\ \phi(\mathfrak{d}(\mathfrak{u}, \mathfrak{v})) - \psi(\mathfrak{d}(\mathfrak{u}, \mathfrak{v})),\ \textit{for all}\ \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U},$$

with $\mathfrak{u} \perp_{\preceq} \mathfrak{v}$ satisfying the following:
i) $\mathfrak{T}(\mathcal{U}_0) \subseteq \mathcal{V}_0$ and $(\mathcal{U}, \mathcal{V})$ satisfies P-property.
ii) There exist $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{U}_0$ such that

$$\begin{aligned}&\mathfrak{d}(\mathfrak{u}_1, \mathfrak{T}(\mathfrak{u}_0)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}), \\ &\textit{and} \qquad \mathfrak{u}_0 \perp_{\preceq} \mathfrak{u}_1.\end{aligned}$$

iii) $\mathfrak{T}$ is $\perp_{\preceq}$-continuous and proximally $\perp_{\preceq}$-preserving monotone.
Then, $\mathfrak{T}$ has a unique best proximity point in $\mathcal{U}$.

Proof. From condition (ii), there exist $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{U}_0$ such that

$$\begin{aligned}&\mathfrak{d}(\mathfrak{u}_1, \mathfrak{T}(\mathfrak{u}_0)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}), \\ &\text{and} \qquad \mathfrak{u}_0 \perp_{\preceq} \mathfrak{u}_1.\end{aligned}$$

Since, $\mathfrak{T}(\mathcal{U}_0) \subseteq \mathcal{V}_0$, there exists $\mathfrak{u}_2 \in \mathcal{U}_0$ such that

$$\mathfrak{d}(\mathfrak{u}_2, \mathfrak{T}(\mathfrak{u}_1)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}).$$

By proximally $\perp_{\preceq}$-preserving condition of $\mathfrak{T}$, we obtain,

$$\mathfrak{u}_1 \perp_{\preceq} \mathfrak{u}_2.$$

Proceeding this way, we obtain a sequence $\{\mathfrak{u}_n\} \in \mathcal{U}_0$ such that

$$\begin{aligned}&\mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) \text{ for all } n \in \mathbb{N}, \\ \text{and} \qquad &\mathfrak{u}_n \perp_{\preceq} \mathfrak{u}_{n+1}.\end{aligned}$$

By P-property,

$$\begin{aligned}\mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &= \mathfrak{d}(\mathfrak{T}(\mathfrak{u}_{n-1}), \mathfrak{T}(\mathfrak{u}_n)) \\ &\leq \alpha(\mathfrak{u}_{n-1}, \mathfrak{u}_n)\ \mathfrak{d}(\mathfrak{T}(\mathfrak{u}_{n-1}), \mathfrak{T}(\mathfrak{u}_n)) \\ &\leq sin\ \mathfrak{p}\ \phi(\mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n)) - \psi(\mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n)) \\ &\leq sin\ \mathfrak{p}\ \phi(\mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n)) \\ &< sin\ \mathfrak{p}\ \mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n) \\ &. \\ &. \\ &. \\ &< (sin\ \mathfrak{p})^n\ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0.$$

Now, for $m > n$,

$$\begin{aligned}\mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_m) &\leq \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) + \mathfrak{d}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) + \cdots + \mathfrak{d}(\mathfrak{u}_{m-1}, \mathfrak{u}_m) \\ &< (sin\ \mathfrak{p})^n\ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1) + (sin\ \mathfrak{p})^{n+1}\ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1) + \cdots + (sin\ \mathfrak{p})^{m-1}\mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1).\end{aligned}$$

Letting $m, n \to \infty$, $\mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_m) \to 0$.
Thus, $\{\mathfrak{u}_n\}$ is a Cauchy OPO-sequence.
Since, $\mathfrak{Y}$ is O-complete partially ordered metric space and $\mathcal{U}$ is a closed subset in $\mathfrak{Y}$, there exists $\mathfrak{u}^* \in \mathcal{U}$ such that $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}^*) = 0$ i.e. $\mathfrak{u}_n = \mathfrak{u}^*$.
Now, $\mathfrak{T}$ is $\perp_{\preceq}$ $-$continuous.
Since, $\mathfrak{u}_n \to \mathfrak{u}^*$, we have $\mathfrak{T}(\mathfrak{u}_n) \to \mathfrak{T}(\mathfrak{u}^*)$ as $n \to \infty$.
Therefore

$$\mathfrak{d}(\mathcal{U}, \mathcal{V}) = \mathfrak{d}(\mathfrak{u}_{n+1}, \mathfrak{T}(\mathfrak{u}_n)) \to \mathfrak{d}(\mathfrak{u}^*, \mathfrak{T}(\mathfrak{u}^*)), \quad \text{as} \quad n \to \infty.$$

Thus, $\mathfrak{u}^* \in \mathcal{U}$ is a best proximity point of $\mathfrak{T}$.
Assume that $\mathfrak{v}^* \in \mathcal{U}$ is another best proximity point of $\mathfrak{T}$ such that $\mathfrak{u}^* \perp_{\preceq} \mathfrak{v}^*$.
Then,

$$\mathfrak{d}(\mathfrak{u}^*, \mathfrak{T}(\mathfrak{u}^*)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) = \mathfrak{d}(\mathfrak{v}^*, \mathfrak{T}(\mathfrak{v}^*)).$$

Now, by using P-property, we have

$$\begin{aligned}\mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*) &= \mathfrak{d}(\mathfrak{T}(\mathfrak{u}^*), \mathfrak{T}(\mathfrak{v}^*)) \\ &\leq \alpha(\mathfrak{u}^*, \mathfrak{v}^*)\ \mathfrak{d}(\mathfrak{T}(\mathfrak{u}^*), \mathfrak{T}(\mathfrak{v}^*)) \\ &\leq sin\ \mathfrak{p}\ \phi(\mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*)) - \psi(\mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*)) \\ &\leq sin\ \mathfrak{p}\ \phi(\mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*)) \\ &< \mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*).\end{aligned}$$

which is a contradiction, i.e. $\mathfrak{u}^* = \mathfrak{v}^*$.
Hence, $\mathfrak{T}$ has a unique best proximity point. □

Theorem 3.7. *Consider* $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$ *as an O-complete partially ordered metric space, and* $\mathcal{U}$ *and* $\mathcal{V}$ *as two non-empty closed subsets of* $\mathfrak{Y}$ *such that* $\mathcal{U}_0 \neq \emptyset$. *Consider* $\mathfrak{T} : \mathcal{U} \to \mathcal{V}$ *as an* α*-proximal admissible mapping such that for some* $\aleph \in (-\infty, \infty)$, $\eta \in (0, 1)$,

$$\alpha(\mathfrak{u}, \mathfrak{v})\ \mathfrak{d}(\mathfrak{T}(\mathfrak{u}), \mathfrak{T}(\mathfrak{v})) \leq \frac{1}{cosh(\aleph)} \phi\bigg(\frac{\eta\ \mathfrak{d}(\mathfrak{u}, \mathfrak{v})}{1 + \mathfrak{d}(\mathfrak{u}, \mathfrak{v})}\bigg) - \psi(\mathfrak{d}(\mathfrak{T}(\mathfrak{u}), \mathfrak{T}(\mathfrak{v}))), \forall\ \mathfrak{u}, \mathfrak{v} \in \mathcal{U},$$

with $\mathfrak{u} \perp_{\preceq} \mathfrak{v}$ *satisfying the following:*
i) $\mathfrak{T}(\mathcal{U}_0) \subseteq \mathcal{V}_0$ *and* $(\mathcal{U}, \mathcal{V})$ *satisfies* P*-property.*
ii) There exist $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{U}_0$ *such that*

$$\mathfrak{d}(\mathfrak{u}_1, \mathfrak{T}(\mathfrak{u}_0)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}),$$
$$\textit{and}\ \ \mathfrak{u}_0 \perp_{\preceq} \mathfrak{u}_1.$$

iii) $\mathfrak{T}$ *is* $\perp_{\preceq}$ *−continuous and proximally* $\perp_{\preceq}$ *−preserving monotone.*
Then, $\mathfrak{T}$ *has a unique best proximity point in* $\mathcal{U}$.

Proof. From condition (ii), there exist $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{U}_0$ such that

$$\mathfrak{d}(\mathfrak{u}_1, \mathfrak{T}(\mathfrak{u}_0)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}),$$
$$\text{and} \qquad \mathfrak{u}_0 \perp_{\preceq} \mathfrak{u}_1.$$

Since, $\mathfrak{T}(\mathcal{U}_0) \subseteq \mathcal{V}_0$, there exists $\mathfrak{u}_2 \in \mathcal{U}_0$ such that

$$\mathfrak{d}(\mathfrak{u}_2, \mathfrak{T}(\mathfrak{u}_1)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}).$$

By proximally $\perp_{\preceq}$ −preserving condition of $\mathfrak{T}$, we obtain,

$$\mathfrak{u}_1 \perp_{\preceq} \mathfrak{u}_2.$$

Proceeding this way, we obtain a sequence $\{\mathfrak{u}_n\} \in \mathcal{U}_0$ such that

$$\mathfrak{d}(\mathfrak{u}_{n+1}, \mathfrak{T}(\mathfrak{u}_n)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) \text{ for all } n \in \mathbb{N},$$
$$\text{and} \qquad \mathfrak{u}_n \perp_{\preceq} \mathfrak{u}_{n+1}.$$

By using P-property, we get

$$\begin{aligned}
\mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &= \mathfrak{d}(\mathfrak{T}(\mathfrak{u}_{n-1}), \mathfrak{T}(\mathfrak{u}_n)) \\
&\leq \alpha(\mathfrak{u}_{n-1}, \mathfrak{u}_n)\ \mathfrak{d}(\mathfrak{T}(\mathfrak{u}_{n-1}), \mathfrak{T}(\mathfrak{u}_n)) \\
&\leq \frac{1}{cosh(\aleph)}\ \phi\bigg(\frac{\eta\ \mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n)}{1 + \mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n)}\bigg) - \psi(\mathfrak{d}(\mathfrak{T}(\mathfrak{u}_{n-1}), \mathfrak{T}(\mathfrak{u}_n))) \\
&\leq \frac{1}{cosh(\aleph)}\ \phi\bigg(\frac{\eta\ \mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n)}{1 + \mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n)}\bigg) \\
&< \eta\ \mathfrak{d}(\mathfrak{u}_{n-1}, \mathfrak{u}_n) \\
&. \\
&. \\
&. \\
&< \eta^n\ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1).
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0.$$

Now, for $m > n$,

$$\begin{aligned} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_m) &\leq \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) + \mathfrak{d}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) + \cdots + \mathfrak{d}(\mathfrak{u}_{m-1}, \mathfrak{u}_m) \\ &< \eta^n \ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1) + \eta^{n+1} \ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1) + \cdots + \eta^{m-1} \ \mathfrak{d}(\mathfrak{u}_0, \mathfrak{u}_1). \end{aligned}$$

Letting $m, n \to \infty$, $\mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}_m) \to 0$.
Thus, $\{\mathfrak{u}_n\}$ is a Cauchy OPO-sequence.
Since, $\mathfrak{Y}$ is O-complete partially ordered metric space and $\mathcal{U}$ is a closed subset in $\mathfrak{Y}$, there exists $\mathfrak{u}^* \in \mathcal{U}$ such that $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{u}_n, \mathfrak{u}^*) = 0$ i.e. $\mathfrak{u}_n = \mathfrak{u}^*$.
Now, $\mathfrak{T}$ is $\perp_{\preceq}$ $-$continuous.
Since, $\mathfrak{u}_n \to \mathfrak{u}^*$, we have $\mathfrak{T}(\mathfrak{u}_n) \to \mathfrak{T}(\mathfrak{u}^*)$ as $n \to \infty$.
Therefore

$$\mathfrak{d}(\mathcal{U}, \mathcal{V}) = \mathfrak{d}(\mathfrak{u}_{n+1}, \mathfrak{T}(\mathfrak{u}_n)) \to \mathfrak{d}(\mathfrak{u}^*, \mathfrak{T}(\mathfrak{u}^*)), \quad \text{as} \quad n \to \infty.$$

Thus, $\mathfrak{u}^* \in \mathcal{U}$ is a best proximity point of $\mathfrak{T}$.
Assume that $\mathfrak{v}^* \in \mathcal{U}$ is another best proximity point of $\mathfrak{T}$ such that $\mathfrak{u}^* \perp_{\preceq} \mathfrak{v}^*$.
Then,

$$\mathfrak{d}(\mathfrak{u}^*, \mathfrak{T}(\mathfrak{u}^*)) = \mathfrak{d}(\mathcal{U}, \mathcal{V}) = \mathfrak{d}(\mathfrak{v}^*, \mathfrak{T}(\mathfrak{v}^*)).$$

Now, by using P-property, we have

$$\begin{aligned} \mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*) &= \mathfrak{d}(\mathfrak{T}(\mathfrak{u}^*), \mathfrak{T}(\mathfrak{v}^*)) \\ &\leq \alpha(\mathfrak{u}^*, \mathfrak{v}^*) \ \mathfrak{d}(\mathfrak{T}(\mathfrak{u}^*), \mathfrak{T}(\mathfrak{v}^*)) \\ &\leq \frac{1}{cosh(\aleph)} \ \phi\left(\frac{\eta \ \mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*)}{1 + \mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*)}\right) - \psi(\mathfrak{d}(\mathfrak{T}(\mathfrak{u}^*), \mathfrak{T}(\mathfrak{v}^*))) \\ &< \ \eta \ \mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*) \\ &< \mathfrak{d}(\mathfrak{u}^*, \mathfrak{v}^*), \end{aligned}$$

which is a contradiction, i.e. $\mathfrak{u}^* = \mathfrak{v}^*$.
Hence, $\mathfrak{T}$ has a unique best proximity point. □

4. Illustration

Example 4.1. Assume $\mathfrak{Y} = \mathbb{R}$ and $\mathfrak{d} : \mathfrak{Y} \times \mathfrak{Y} \to [0, \infty)$ be the standard metric defined as

$$\mathfrak{d}(\mathfrak{s}, \mathfrak{u}) = |\mathfrak{s} - \mathfrak{u}|, \ \forall \ \mathfrak{s}, \mathfrak{u} \in \mathfrak{Y}.$$

We define the binary relation $\perp_{\preceq}$ as $\mathfrak{s} \perp_{\preceq} \mathfrak{u} \Leftrightarrow \mathfrak{s}|\mathfrak{u}$ ($\mathfrak{s}$ divides $\mathfrak{u}$), for all $\mathfrak{s}, \mathfrak{u} \in \mathfrak{Y}$.
Now, consider $\mathcal{U} = \{1, 2, 3, 6, 12, 24\}$ and $\mathcal{V} = \{-2.4, -1.2, -0.6, -0.3, -0.2, -0.1\}$.
We define $\mathfrak{T} : \mathcal{U} \to \mathcal{V}$ as

$$\mathfrak{T}\mathfrak{u} = -\frac{\mathfrak{u}}{10} \ \forall \ \mathfrak{u} \in \mathcal{U}.$$

So, $\mathfrak{T}$ is $\perp_{\preceq}$-continuous and proximally $\perp_{\preceq}$-preserving.
We have

$$\mathfrak{d}(1, \mathfrak{T}(1)) = \mathfrak{d}(1, -0.1) = 1.1 = \mathfrak{d}(\mathcal{U}, \mathcal{V}).$$

Now, it can be easily verified that for all $\eta \geq 0.1$, the following holds:

$$\mathfrak{d}(\mathfrak{T}\mathfrak{s}, \mathfrak{T}\mathfrak{u}) \leq \eta \, \mathfrak{d}(\mathfrak{s}, \mathfrak{u}), \quad \forall \, \mathfrak{s}, \mathfrak{u} \in \mathcal{U}.$$

Thus, all the assumptions of Theorem 3.4 are satisfied and 1 is the unique best proximity point of $\mathfrak{T}$.

Example 4.2. Assume $\mathfrak{Y} = \mathbb{R}$ and $\mathfrak{d} : \mathfrak{Y} \times \mathfrak{Y} \to [0, \infty)$ be the standard metric defined as

$$\mathfrak{d}(\mathfrak{s}, \mathfrak{u}) = |\mathfrak{s} - \mathfrak{u}| \; \forall \mathfrak{s}, \mathfrak{u} \in \mathfrak{Y}.$$

We define the binary relation $\perp_{\preceq}$ as $\mathfrak{s} \perp_{\preceq} \mathfrak{u} \Leftrightarrow \mathfrak{s} \leq \mathfrak{u}$, for all $\mathfrak{s}, \mathfrak{u} \in \mathfrak{Y}$.
Now, consider $\mathcal{U} = [-1, 0]$ and $\mathcal{V} = [3, 4]$.
We define $\mathfrak{T} : \mathcal{U} \to \mathcal{V}$ as

$$\mathfrak{T}(\mathfrak{u}) = \begin{cases} 3; & \mathfrak{u} = 0 \\ 4 + \frac{\mathfrak{u}}{10}; & \mathfrak{u} \neq 0 \end{cases}$$

So, $\mathfrak{T}$ is $\perp_{\preceq}$-continuous and proximally $\perp_{\preceq}$-preserving.
We have

$$\mathfrak{d}(0, \mathfrak{T}(0)) = \mathfrak{d}(0, 3) = 3 = \mathfrak{d}(\mathcal{U}, \mathcal{V}).$$

Consider $\phi, \psi : [0, \infty) \to [0, \infty)$ be defined as

$$\phi(\mathfrak{t}) = \frac{993\mathfrak{t}}{1000}$$
$$\psi(\mathfrak{t}) = \frac{\mathfrak{t}}{5145}$$

Now, taking $\alpha = \frac{1003}{1000}$ and $\wp = \frac{89\pi}{180}$, we can verify the following:

$$\alpha(\mathfrak{s}, \mathfrak{u}) \; \mathfrak{d}(\mathfrak{T}\mathfrak{s}, \mathfrak{T}\mathfrak{u}) \leq (sin \; \mathfrak{p}) \; \phi(\mathfrak{d}(\mathfrak{s}, \mathfrak{u})) - \psi(\mathfrak{d}(\mathfrak{s}, \mathfrak{u})), \quad \forall \, \mathfrak{s}, \mathfrak{u} \in \mathcal{U}.$$

Thus, all the assumptions of Theorem 3.6 are satisfied and 0 is the unique best proximity point of $\mathfrak{T}$.

Example 4.3. Assume $\mathfrak{Y} = \mathbb{R} \times \mathbb{R}$ and $\mathfrak{d} : \mathfrak{Y} \times \mathfrak{Y} \to [0, \infty)$ be the Euclidean metric.
We define the binary relation $\perp_{\preceq}$ as $(\mathfrak{s}_1, \mathfrak{s}_2) \perp_{\preceq} (\mathfrak{u}_1, \mathfrak{u}_2) \Leftrightarrow \mathfrak{s}_1 \leq \mathfrak{u}_1$ and $\mathfrak{s}_2 \leq \mathfrak{u}_2$, for all $(\mathfrak{s}_1, \mathfrak{s}_2), (\mathfrak{u}_1, \mathfrak{u}_2) \in \mathfrak{Y}$.
Now, consider $\mathcal{U} = (\mathfrak{a}_1, \mathfrak{a}_2)$ such that $\mathfrak{a}_1, \mathfrak{a}_2 \in [-1, 0]$
and $\mathcal{V} = (\mathfrak{b}_1, \mathfrak{b}_2)$ such that $\mathfrak{b}_1, \mathfrak{b}_2 \in [1, 2]$.
We define $\mathfrak{T} : \mathcal{U} \to \mathcal{V}$ as

$$\mathfrak{T}(\mathfrak{s}, \mathfrak{u}) = \begin{cases} (1, 1); & (\mathfrak{s}, \mathfrak{u}) = (0, 0) \\ (\frac{\mathfrak{s}}{10} + 2, \frac{\mathfrak{u}}{10} + 2); & \text{otherwise} \end{cases}$$

So, $\mathfrak{T}$ is $\perp_{\preceq}$-continuous and proximally $\perp_{\preceq}$-preserving.
We have

$$\mathfrak{d}((0, 0), \mathfrak{T}(0, 0)) = \mathfrak{d}((0, 0), (1, 1)) = \sqrt{2} = \mathfrak{d}(\mathcal{U}, \mathcal{V}).$$

Consider $\phi, \psi : [0, \infty) \to [0, \infty)$ be defined as

$$\phi(\mathfrak{t}) = \frac{997\mathfrak{t}}{1000}$$
$$\psi(\mathfrak{t}) = \frac{\mathfrak{t}}{6455}$$

Now, taking $\alpha = \frac{1001}{1000}, \eta = \frac{97}{100}$ and $\aleph = \frac{1}{1230}$, we can verify the following:

$$\alpha(\mathfrak{s},\mathfrak{u})\mathfrak{d}(\mathfrak{T}\mathfrak{s},\mathfrak{T}\mathfrak{u}) \leq \frac{1}{cosh(\aleph)}\ \phi\left(\frac{\eta\ \mathfrak{d}(\mathfrak{s},\mathfrak{u})}{1+\mathfrak{d}(\mathfrak{s},\mathfrak{u})}\right) - \psi(\mathfrak{d}(\mathfrak{T}\mathfrak{s},\mathfrak{T}\mathfrak{u})), \quad \forall\ \mathfrak{s},\mathfrak{u} \in \mathcal{U}.$$

Thus, all the assumptions of Theorem 3.7 are satisfied and $(0,0)$ is the unique best proximity point of $\mathfrak{T}$.

5. Consequences

Corollary 5.1. *Suppose $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$ be an O-complete partially ordered metric space. Consider a $\perp_{\preceq}$ −continuous and $\perp_{\preceq}$ −preserving monotone mapping $\mathfrak{T} : \mathfrak{Y} \to \mathfrak{Y}$ such that for some $\eta \in (0,1)$,*

$$\mathfrak{d}(\mathfrak{T}(\mathfrak{u}), \mathfrak{T}(\mathfrak{v})) \leq \eta\ \mathfrak{d}(\mathfrak{u},\mathfrak{v}),\ \textit{for all}\ \ \mathfrak{u},\mathfrak{v} \in \mathfrak{Y}$$

with $\mathfrak{u} \perp_{\preceq} \mathfrak{v}$. Then, $\mathfrak{T}$ has a unique fixed point.

Corollary 5.2. *Suppose $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$ be an O-complete partially ordered metric space. Consider an $\alpha-$admissible, $\perp_{\preceq}$ −continuous and $\perp_{\preceq}$ −preserving monotone mapping $\mathfrak{T} : \mathfrak{Y} \to \mathfrak{Y}$ such that for some $\mathfrak{p} \in (0, \frac{\pi}{2})$,*

$$\alpha(\mathfrak{u},\mathfrak{v})\ \mathfrak{d}(\mathfrak{T}(\mathfrak{u}), \mathfrak{T}(\mathfrak{v})) \leq |sin\ \mathfrak{p}|\ \phi(\mathfrak{d}(\mathfrak{u},\mathfrak{v})) - \psi(\mathfrak{d}(\mathfrak{u},\mathfrak{v}),\ \textit{for all}\ \ \mathfrak{u},\mathfrak{v} \in \mathfrak{Y}$$

with $\mathfrak{u} \perp_{\preceq} \mathfrak{v}$. Then, $\mathfrak{T}$ has a unique fixed point.

Corollary 5.3. *Suppose $(\mathfrak{Y}, \mathfrak{d}, \perp_{\preceq})$ be an O-complete partially ordered metric space. Consider an $\alpha-$admissible, $\perp_{\preceq}$ −continuous and $\perp_{\preceq}$ −preserving monotone mapping $\mathfrak{T} : \mathfrak{Y} \to \mathfrak{Y}$ such that for some $\eta \in (0,1)$ and $\aleph \neq 0$,*

$$\alpha(\mathfrak{u},\mathfrak{v})\ \mathfrak{d}(\mathfrak{T}(\mathfrak{u}), \mathfrak{T}(\mathfrak{v})) \leq \frac{1}{cosh(\aleph)}\ \phi\left(\frac{\eta\ \mathfrak{d}(\mathfrak{u},\mathfrak{v})}{1+\mathfrak{d}(\mathfrak{u},\mathfrak{v})}\right) - \psi(\mathfrak{d}(\mathfrak{T}(\mathfrak{u}), \mathfrak{T}(\mathfrak{v}))),$$

for all $\mathfrak{u},\mathfrak{v} \in \mathfrak{Y}$ with $\mathfrak{u} \perp_{\preceq} \mathfrak{v}$. Then, $\mathfrak{T}$ has a unique fixed point.

6. Conclusion

This research paper has presented a comprehensive exploration of the application of orthogonal partially ordered metric spaces within the realm of fixed-point theory, with a particular focus on introducing the concept of Best Proximity Point (BPP). The study has successfully established theorems concerning BPP by utilizing orthogonal fuzzy contractive mappings. Furthermore, the implications of these theorems have been thoroughly examined, encompassing both self-mappings and non-self mappings sharing the same parameter set. Through the inclusion of illustrative examples, the practical relevance of the established theorems and corollaries in various mathematical contexts has been demonstrated.

By introducing innovative concepts such as BPP, this research provides a robust theoretical framework for analyzing mappings in diverse scenarios. The findings presented in this paper not only contribute to advancing the theoretical understanding of extending partial ordered metric spaces but also broaden the applicability of fixed-point theory across different domains. In essence, this research serves as a valuable resource for mathematicians and practitioners alike, offering insights into the intricacies of orthogonal partially ordered metric spaces and their implications in fixed-point theory. It opens avenues for further exploration and application in solving real-world problems, thus paving the way for future research endeavours in this field.

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On a mathematical model with a free boundary for the dynamics of the Tumor-Immune system interaction

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Abstract. Mathematical models of tumor growth are essential for accurately monitoring and predicting the spatiotemporal dynamics of tumor development. In this work, we present a mathematical model designed to describe the early stages of interaction between immune cells and tumor cells through a system of partial differential equations. The model is structured in terms of size and space, incorporating the migration of tumor antigen-specific cytotoxic effector cells toward the tumor microenvironment via a chemotactic mechanism.

Keywords: Mathematical model, customs clearance, linear optimization, objective function with variable coefficients, customs risk.

MSC (2020): 92B05,35R35, 35B40

1. Introduction

Some local tissue cells in biological systems lose their normal regulation or become infected with specific viruses during growth, subsequently transforming into tumor cells.

Cancer development results from the accumulation of mutations that disrupt a limited number of key pathways, sufficient to initiate and sustain tumor progression. Tumors grow not only due to genetic and epigenetic changes that confer growth advantages but also under the influence of immune cells within the tumor microenvironment [1, 2]. Experimental and clinical evidence suggests that the immune system plays a critical role in preventing and eradicating tumors (see, for example, [1, 3, 4]).

Mathematical models can help elucidate the interaction between tumor growth and immune responses [5, 6, 7, 8, 9]. These models can also be extended to describe and optimize the effects of chemotherapeutic interventions and strategies to enhance immune responses [10]. However, most existing models rely on complex systems of ordinary differential equations (ODE) and do not account for spatial heterogeneity or the mobility of immune cells. Additionally, many models do not provide a detailed examination of uncontrolled cell division during the early stages of tumor growth.To address these gaps, we propose a model based on size and spatially structured interacting cell populations. This model is specifically designed to describe the early stages of tumor growth, with immune cell migration regulated by chemotaxis in response to signals emitted by the tumor. The construction of the system of coupled partial differential equations (PDE) is based on a set of modeling assumptions. While these simplifying assumptions may be questioned, they aim to preserve the most significant mechanisms while keeping the system of equations as simple as possible. The modeling discussion is driven by the following considerations: 1) feasibility for numerical simulation, ensuring the model is computationally efficient, 2) parameter reduction: introducing additional parameters for more complex phenomena might obscure the analysis due to limited knowledge of their effective values and difficulties in obtaining measurements [11].

Many unanswered questions remain regarding how the immune system interacts with a growing tumor and which immune system components play crucial roles in response to immunotherapy. Mathematical models provide an analytical framework to address these questions and can be used both descriptively and predictively. Developing tumor growth models that incorporate the immune response is essential. The ultimate goal is to create models capable of simulating system responses to novel biological treatments, such as vaccine therapy.

Recent studies highlight the immune system's critical role in recognizing and destroying tumor cells. The immune response to tumor cells is typically mediated by natural killer (NK) cells and CD8+ cytotoxic T-lymphocytes (CTL). As the first line of defense, NK cells eliminate tumor cells through various mechanisms.

Mathematical models, employing both ODE [12] and PDE [13], have been widely used to investigate the complex dynamics of tumor-immune system interactions. These models have yielded valuable insights with significant theoretical and clinical implications for cancer research. For instance, in [14], the authors present an analytical approach for describing and solving systems of porous medium equations, demonstrating applications in invasive biological dynamics. Furthermore, intriguing analytical results related to invasive systems with nonlinear diffusion and advection have also been reported.

2. Preliminary results

Mathematical models of tumor-immune interactions provide an analytical framework for studying tumor-immune dynamics. In [15], the authors present a mathematical model comprising three ordinary differential equations to describe interactions between tumor and immune cells, with a particular focus on the role of natural killer (NK) cells and CD8+ cytotoxic T-lymphocytes (CTL) in immune surveillance.

We consider two interacting cell populations:

- cytotoxic effector cells specific to tumor antigens, including CD8+ T-cells and natural killer (NK) cells;
- tumor cells.

The specific biological assumptions underlying the model construction are based on the behavior of effector cells within the microenvironment of the growing tumor and the key phenomena governing tumor cell growth:

The proposed model in [15] is built on the following fundamental assumptions:

1. NK-cells are constantly present and active in the host's body, even in the absence of tumor cells.
2. CTL appear in significant numbers only when tumor cells are present in the host.
3. Both NK-cells and CTL have the ability to kill tumor cells. However, CTL play the dominant role in tumor destruction as part of adaptive immunity.
4. When the immune system encounters tumor cells, some NK-cells and CTL become inactivated and do not cause further damage to cells.

This model provides valuable insights into the dynamics of tumor-immune interactions, highlighting the distinct roles of innate and adaptive immunity in tumor control and offering a framework for further investigation and optimization of immunotherapeutic strategies.

The variables $N(t), L(t)$ and $T(t)$ are used to denote the quantities of NK-cells, CTL, and tumor cells, respectively. The model describing the growth, death, and interactions of these populations is defined as follows:

$$\begin{cases} N'(t) = aN(t)(1 - bN(t)) - \alpha_1 N(t)T(t), \\ L'(t) = rN(t)T(t) - \mu L(t) - \beta_1 L(t)T(t), \\ T'(t) = cT(t)(1 - dT(t)) - \alpha_2 N(t)T(t) - \beta_2 L(t)T(t), \end{cases} \tag{2.1}$$

with initial conditions $N(0) = N_0 \geq 0, L(0) = L_0 \geq 0$ and $T(0) = T_0 \geq 0$.

All parameters $a, b, c, d, r, \mu, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive constants.

A model with a free boundary.

Based on (2.1), we propose the following model, which governs the spatiotemporal evolution of the system as well as the free boundaries. We use $u(t,x), v(t,x)$ and $w(t,x)$ to denote the concentrations of NK-cells, CTL and tumor cells at time t respectively. The free-boundary model for the system of parabolic equations is defined as follows: $(t,x) \in Q$, $(t,x) \in D$

$$\begin{cases} u_t - u_{xx} = au(t,x)(1-bu(t,x)) - \alpha_1 u(t,x)w(t,x), \\ v_t = ru(t,x)w(t,x) - \mu v(t,x) - \beta_1 v(t,x)w(t,x) \\ w_t - w_{xx} = cw(t,x)(1-dw(t,x)) - \alpha_2 u(t,x)w(t,x) - \beta_2 v(t,x)w(t,x), \\ v(t,x) = w(t,x) = 0, t > 0, \quad x \notin (g(t) < x < h(t)), \\ u_x(t,-l) = u_x(t,l) = 0, \\ g'(t) = -\beta w_x(t,g(t)), h'(t) = -\beta w_x(t,h(t)), \quad t > 0, \\ u(0,x) = u_0(x), -l < x < l, v(0,x) = v_0(x), w(0,x) = w_0(x), -h_0 < x < h_0, \\ h(0) = -g(0) = h_0. \end{cases} \tag{2.2}$$

Here $Q = (t;x) : 0 < t, -l < x < l, D = (t;x) : 0 < t, g(t) < x < h(t)$, all initial functions and parameters are positive; $x = g(t)$ and $x = h(t)$ are moving unknown boundaries representing the tumor's spreading fronts, which are determined jointly with $u(t,x), v(t,x), w(t,x)$.

We assume that u_0, v_0 and w_0 satisfy the following conditions:

A)$u_0(x) \in C_2[-l,l], v_0(x), w_0(x) \in C_2([-h_0,h_0])$;

B)$u_0 > 0$ in $(-l,l), v_0, w_0 > 0$ in $(-h_0,h_0)$;

C) The appropriate compatibility conditions at the corner points are satisfied.

3. A priori estimates

The results of local existence and uniqueness hold for any quasilinear parabolic equation, provided that the given functions are sufficiently smooth, without any restrictions on the growth behavior of these functions with respect to u and u_x (see, for example, [16, 17]). Such conditions are necessary when considering the global solvability of boundary value problems.

Lemma 3.1. *Suppose that functions $(u, v, g(t), h(t))$ give a solution of problem (2.2). Then there exist positive constants M_i, $i = \overline{1,5}$ independent of T for which the estimates*

$$0 < u\,(t,x) \leq M_1 = \max\{\|u_0\|, \frac{1}{b}\}, \quad in \quad Q_T = \{(t,x) : 0 < t \leq T, -l < x < l\},$$

$$0 < v\,(t,x) \leq M_2 = M_1 + \|v_0\|, \quad in \quad D_T = \{(t,x) : 0 < t \leq T, g(t) < x < h(t)\},$$

$$0 < w\,(t,x) \leq M_3, \quad in \quad D_T,$$

$$0 < -g'(t) \leq M_4, \quad 0 < h'(t) \leq M_5, \quad 0 < t \leq T$$

are true.

The proof of the lemma is carried out using the maximum principle and comparison theorems.

We will establish Hölder norm bounds $|\cdot|_{1+\alpha}$ and $|\cdot|_{2+\alpha}$ in $\overline{Q}_T$ and $\overline{D}_T$. For each equation of the system, we formulate the corresponding problems:

$$\begin{cases} u_{xx} + a_1(u,w) - u_t = 0, \quad \text{in} \quad Q_T, \\ u(0,x) = u_0(x), \quad -l \leq x \leq l, \\ u_x(t,-l) = u_x(t,l) = 0, \quad 0 \leq t \leq T, \end{cases} \tag{3.1}$$

where $a_1(u,w) = au(t,x)(1-bu(t,x)) - \alpha_1 u(t,x)w(t,x)$,

$$\begin{cases} a_2(v,u,w) - v_t = 0, \text{ in } D_T, \\ v(0,x) = v_0(x), \quad -h_0 \leq x \leq h_0, \\ v(t,g(t)) = v(t,h(t)) = 0, \quad 0 \leq t \leq T, \end{cases} \tag{3.2}$$

where $a_2(u,v,w) = ru(t,x)w(t,x) - \mu v(t,x) - \beta_1 v(t,x)w(t,x)$,

$$\begin{cases} w_{xx} + a_3(u,v,w) - w_t = 0, \quad \text{in} \quad D_T, \\ w(0,x) = w_0(x), \quad -h_0 \le x \le h_0, \\ w(t,g(t)) = w(t,h(t)) = 0, \quad 0 \le t \le T, \\ g'(t) = -\beta w_x(t,g(t)), \quad h'(t) = -\beta w_x(t,h(t)), \quad 0 \le t \le T, \end{cases} \tag{3.3}$$

where $a_3(u,v,w) = cw(t,x)(1 - dw(t,x)) - \alpha_2 u(t,x)w(t,x) - \beta_2 v(t,x)w(t,x)$.

Theorem 3.2. *Assume that $u(t,x)$ and $u_x(t,x)$ are continuous in Q_T and suppose that $u(t,x)$, is a solution for problem (3.1). Then*

$$|u_x(t,x)| \le C_1(M_1, M_2) \quad in \quad Q_T. \tag{3.4}$$

Moreover, if the weak second derivatives u_{xx}, u_{tx} in $L_2(Q_T)$, then there exists $\alpha = \alpha(M_1)$, such that

$$|u|_{1+\alpha}^{Q_T} \le C_2(M_1, C_1). \tag{3.5}$$

Additionally, assume that, $u(t,x)$ satisfying (3.1) in Q_T , is continuous with its derivatives u_t, u_x, u_{xx} and $|u|_{2+\alpha}^{Q_T} < \infty$. Then

$$|u|_{2+\alpha}^{Q_T} \le C_3(M_1, C_1, C_2). \tag{3.6}$$

Proof. The estimates (3.4)-(3.6) for $(t,x) \in Q_T$ are immediate consequences of the results of [16]. For (3.2) and (3.3) a priori estimates are constructed as follows. Interior estimates in (t,x): $0 \le t \le T, -h_0 \le x \le h_0$ are established, just like in [16]. In order to get bounds near unknown curves, we perform the transformation

$$\tau = t, \quad y = \frac{(x - g(t)) - (h(t) - x))}{g(t) - h(t)}.$$

and straighten our the boundary. Then $\Omega = \{(\tau, y) : 0 < t < T, -1 < y < 1\}$ corresponds to domain D_T, and for the required functions the corresponding parabolic problems with bounded coefficients and the right-hand side are obtained. The rest of the proof is completed by following the steps described in [16]. In doing so, we first establish estimates for $|(\cdot)_x|$ and then for $|\cdot|_{1+\alpha}$. Estimates of the highest derivatives are obtained from the results for linear equations [17]. □

Theorem 3.3. *Let (u,v,w,g,h) be the unique global solution of (2.2). Then there exists a constant $C > 0$ such that*

$$|w(t,\cdot)|_{C^1([g(t),h(t)])} \le C, \quad |g', h'|_{C^{\alpha/2}([1,\infty))} \le C, \quad \forall t > 1. \tag{3.7}$$

Proof. Inequality (3.7) can be proved using a similar technique from ([19],theorem 2.1) for the case $h_\infty - g_\infty < \infty$ and from ([20], theorem 2.2) for the case $h_\infty - g_\infty = \infty$. Therefore, a detailed proof is omitted.

□

4. The existence and uniqueness of the solution

4.1. **The uniqueness of a solution.** We rewrite the equation for $w(t,x)$ in the form

$$w_t - w_{xx} = f(u,v,w) \quad \text{in} \quad D \tag{4.1}$$

where $f(u,v,w) = \frac{cw}{1-dw} - \alpha_2 uw - \beta_2 vw$.

Integrating (4.1) over $D_T = \{(\eta, \xi) : 0 < \eta \le t, g(\eta) < \xi < h(\eta)$, we obtain

$$\int\limits_{\partial D_t} (w_x d\eta + w d\xi) = \iint\limits_{D_t} f(u,v,w) d\eta d\xi.$$

or

$$h(t) - g(t) = 2h_0 + \beta \int\limits_{-h_0}^{h_0} w_0(\xi)\, d\xi - \beta \int\limits_{g(t)}^{h(t)} w(t,\xi)\, d\xi + \beta \iint\limits_{D_t} f(u,v,w) d\eta d\xi. \qquad (4.2)$$

Theorem 4.1. *Suppose that the assumptions of Lemma 3.1 and Theorem 2.2 holds. Then the solution to problem* (2.2) *is unique if it exists.*

Proof. Suppose that the functions u_i, v_i, w_i, $g_i(t)$, $h_i(t)$, $i = \overline{1,2}$ solve system (2.2) and $y_1(t) = \min(g_1(t), g_2(t))$, $y_2(t) = \max(g_1(t), g_2(t))$,
$\chi_1(t) = \min(h_1(t), h_2(t))$, $\chi_2(t) = \max(h_1(t), h_2(t))$.

Let us consider the problem for the difference between these two groups of solutions and, taking into account (4.2), we find

$$|h_1(t) - h_2(t)| + |g_1(t) - g_2(t)| \le C_1 \left(\int\limits_{y_1(t)}^{y_2(t)} |w_j(t,\xi)|\, d\xi + \int\limits_{y_2(t)}^{\chi_1(t)} |w_1(t,\xi) - w_2(t,\xi)|\, d\xi \right)$$

$$+ C_1 \left(\int\limits_{\chi_1(t)}^{\chi_2(t)} |w_j(t,\xi)|\, d\xi + \int\limits_0^t d\eta \int\limits_{y_1(\eta)}^{y_2(\eta)} |f(u_j, v_j, w_j)|\, d\xi + \int\limits_0^t d\eta \int\limits_{\chi_1(\eta)}^{\chi_2(\eta)} |f(u_j, v_j, w_j)|\, d\xi \right)$$

$$+ C_1 \int\limits_0^t d\eta \int\limits_{y_2(\eta)}^{\chi_1(\eta)} |f(u_1, v_1, w_1) - f(u_2, v_2, w_2)|\, d\xi$$

where C_1 is a positive constant,

$$(u_i, v_i, w_i) = \begin{cases} (u_1, v_1, w_1), & \text{if} \quad g_1(t) < g_2(t), \\ (u_2, v_2, w_2), & \text{if} \quad g_2(t) < g_1(t), \end{cases}$$

$$(u_i, v_i, w_i, z_i) = \begin{cases} (u_1, v_1, w_1), & \text{if} \quad h_2(t) < h_1(t), \\ (u_2, v_2, w_2), & \text{if} \quad h_1(t) < h_2(t). \end{cases}$$

Further, using the ideas and result of [22] the proof of the theorem is completed. □

4.2. **On the existence of a solution.** Since all the necessary assessments it has been established that by applying the idea and results of the work ([21], Theorem 2), it is possible to complete the proof of the theorem.

Theorem 4.2. *There is a* T $(0 < t < \infty)$ *such that problem* (2.2) *admits a unique solution*

$$(u, v, w, g, h) \in C^{1+\alpha, 2+\alpha}(Q_T) \times [C^{1+\alpha, 2+\alpha}(D_T)]^3 \times [C^{1+\alpha}([0,T])]^2$$

satisfying

$$|u|_{C^{1+\alpha,2+\alpha}(Q_T)} + |v|_{C^{1+\alpha,\alpha}(D_T)} + |w|_{C^{1+\alpha,2+\alpha}(D_T)} + |h|_{C^{1+\alpha}([0,T])} + |g|_{C^{1+\alpha}([0,T])} \le C,$$

where $\alpha \in (0,1)$, C *depend on* h_0, α, $|u_0, v_0, w_0|_{C^2}$.

Proof. To prove the solvability of a nonlinear problem, you can use various theorems from the theory of nonlinear equations, remembering that the uniqueness theorem of the classical solution is valid for it. Let us use the Leray-Schauder principle [17], established by a priori estimates $|\cdot|_{1+\alpha}$ for all possible solutions of nonlinear problems, and the solvability theorem in Hölder classes for linear problems. In this case, the existence theorems for systems are the same as for the case of one equation, since each of the equations can be considered as a linear equation with continuous Hölder coefficients. A more detailed exposition of the technique can be found, for example, in (Section VI, [17]; Section VII, [18]).

□

5. Behavior of the solution with an unlimited increase in time

Now, let us prove that the unknown boundaries $x = g(t)$ and $x = h(t)$ do not intersect the lateral boundaries $x = -l$ and $x = l$, respectively, ensuring that the tumor does not spread throughout the entire considered domain. On the other hand, this condition is necessary for the correctness of the problem for arbitrary time intervals.

From Lemma 3.1, it follows that $x = h(t)$ and $x = g(t)$ monotonically increases (or decreases), respectively. Therefore, there exist $h_\infty \in (0, \infty]$ and $g_\infty \in [-\infty, 0)$ such that

$$\lim_{t\to\infty} g(t) = g_\infty, \quad \lim_{t\to\infty} h(t) = h_\infty.$$

The case $h_\infty = -g_\infty = \infty$ is referred to as the spreading scenario, while the case $h_\infty - g_\infty < \infty$ is called the vanishing scenario.

Using the established a priori estimates from problem (2.2) and applying Lemma 2.2 from [19], a lower bound for $u(t, x)$ can be derived as $u(t, x) \geq u^* > 0$.

Lemma 5.1. *Let (u, v, w, g, h) be the unique solution of (2.2) and*

$$rc < \alpha_2, \mu c \leq (\beta_2 + c\beta_1)v_0(x)e^{-(\beta_2+c\beta_1)t}, t > 0, -g_0 < x < h_0.$$

$$2h_0 + \int_{g_0}^{h_0} [\mu w_0(x) + cv_0(x)]\, dx < l.$$

Then $0 < h(t) - g(t) < l$ and $0 < h_\infty - g_\infty < l$.

Proof. We have

$$\frac{d}{dt}\int_{g(t)}^{h(t)} (\mu w + cv)dx = \int_{g(t)}^{h(t)} (\mu w_t + cv_t)dx = \int_{g(t)}^{h(t)} \mu(w_{xx} + cw(1 - dw) - \alpha_2 uw - \beta_2 vw)dx$$

$$+ \int_{g(t)}^{h(t)} c(ruw - \mu v - \beta_1 vw)dx \leq \mu(w_x(t, h(t)) - w_x(t, g(t)) + \int_{g(t)}^{h(t)} [(\mu cw - \alpha_2 uw - \beta_2 vw$$

$$-d\mu cw^2 + rcuw - c\mu v - c\beta_1 vw)]dx \leq \mu(-\frac{1}{\beta}(h'(t) + \frac{1}{\beta}g'(t)) \leq -\frac{\mu}{\beta}(h'(t) - g'(t))$$

$$+ \int_{g(t)}^{h(t)} (rc - \alpha_2)uw + w(\mu c - \beta_2 v - c\beta_1 v) + \mu c(w - v - dw^2)dx$$

$$\leq -\frac{\mu}{\beta}(h'(t) - g'(t)) + \int_{g(t)}^{h(t)} (rc - \alpha_2)uw + w(\mu c - (\beta_2 + c\beta_1)v)v_0 e^{-(\beta_2+c\beta_1)t}$$

$$+\mu c(w(1 - dw) - v_0 e^{-(\beta_2+c\beta_1)t})dx$$

then

$$\frac{d}{dt}\int_{g(t)}^{h(t)}(\mu w+cv)dx \le -\frac{\mu}{\beta}(h'(t)-g'(t)). \tag{5.1}$$

Integrating from 0 to t gives

$$h(t)-g(t) \le 2h_0+\beta\int_{g_0}^{h_0}(\mu w_0(x)+cv_0(x))dx.$$

Letting $t\to\infty$, we have $h_\infty - g_\infty < l$.

The proof is complete.

□

Lemma 5.2. *Suppose that the assumptions of Lemma 3.1 and Theorem 2.2 holds. Then*

$$\lim_{t\to\infty}|w(t,\cdot)|_{C([g(t),h(t)])}=0.$$

6. Conclusion

We present a new mathematical model to describe the interactions between tumor and immune cells, with particular emphasis on the role of natural killer (NK) cells and CD8+ cytotoxic T-lymphocytes (CTL) in immune surveillance. The model captures the interaction between tumor and immune cells through a system of differential equations.

The existence, uniqueness, and uniform estimates of the global solution have been established, along with the behavior of the solution components and the unknown boundary over large time intervals. Using the basic reproduction number R_0 conditions for tumor spread or regression have been analyzed.

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On ℓ-Catch Problem in Differential Games with Inertial Players and Non-Stationary Geometric Constraints on Controls

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Abstract. In this paper, we consider the ℓ-catch problems in a differential game of one pursuer and one evader in $\mathbb{R}^n$. The movements of the players are carried out by the same type of differential equations. Non-stationary geometric constraints are imposed on the control functions of the players. We say that the ℓ-catch is completed if the state of the pursuer approaches that of the evader at ℓ distance. To solve the problems, the pursuer's strategy, which is represented as a generalization of the Π-strategy, plays a crucial role. We derive the necessary and sufficient conditions for the completion of the differential game. This work continues and extends the contributions of Isaacs, Petrosyan, Pshenichnyi, the authors of this article, and other researchers.

Keywords: Differential game, ℓ-catch, pursuer, evader, strategy.
MSC (2020): 49N70,49N75,91A23,91A24

1. Introduction

Differential game theory is a branch of modern mathematics that analyzes strategic interactions between multiple players in dynamic systems described by differential equations. The concept of "Differential game" was introduced by American mathematician R. Isaacs who is widely regarded as the founder of differential game theory. Foundations of differential games is typically traced to the works of R. Isaacs [1], L. S. Pontryagin [2], N. N. Krasovskii [3], L. D. Berkovitz [4], A. I. Subbotin [5], L. A. Petrosyan [6], B. N. Pshenichnyi [8], N. Yu. Satimov [9], A. Azamov [12] and many other researchers.

Pursuit-evasion problems are considered with specific interest in differential game theory. In particular, many methods developed for solving pursuit problems further enhance the optimality of the game. One of such key methods is the parallel pursuit strategy (in briefly, Π-strategy) that was proved highly effective in several works (see references [6, 13, 15, 16, 23]). Additionally, it can be said that the primary example named "the game with a life-line" in reference [1] was analyzed by L. A. Petrosyan using the Π-strategy [6].

In practical mathematical models, achieving an exact capture is often unattainable. Therefore, it is more realistic to study ℓ-approach problem or ℓ-catch problem, where the first player only needs to enter the ℓ-neighborhood of the second player to complete the pursuit. Analytical methods for constructing a strategy of best approach (a generalization of the Π-strategy) in the case of ℓ-catch under geometric and integral constraints on the players' controls are explored in [9]–[11],[15]–[16]. Additionally, various methods for solving such problems with different levels of information available to the players are proposed in [17]–[19]. Notably, in the work of L.A. Petrosyan and V.G. Dutkevich [7], a game with a "life-line" is solved using a generalization of the Π-strategy in the context of ℓ-catch.

The main purpose of our work is to analyze ℓ-catch problems in a differential game with inertial players under non-stationary geometric constraints on controls when dissimilar requirements are imposed on the initial conditions. First of all, we suppose that the differences of the initial velocity vectors of the players and the differences of their initial position vectors are linearly dependent. In the next case, we assume that these vectors are considered to be linearly independent. We solve each problem using a generalization of the Π-strategy and derive the sufficient conditions for the solvability of the pursuit problem.

2. Preliminaries

We study the ℓ-catch problem of the differential game between two inertial players, Pursuer P and Evader E, whose objectives are diametrically opposed, in the space $\mathbb{R}^n$.

Let a parameter $x \in \mathbb{R}^n$ (a parameter $y \in \mathbb{R}^n$) designate the position of the Pursuer (the Evader). Then, given initial states $(x_0, x_1) \in \mathbb{R}^n$ and $(y_0, y_1) \in \mathbb{R}^n$, the following system of differential equations describes the motions of players:

$$\ddot{x} = u, \quad x(0) = x_0, \quad \dot{x}(0) = x_1, \tag{2.1}$$

$$\ddot{y} = v, \quad y(0) = y_0, \quad \dot{y}(0) = y_1, \tag{2.2}$$

where x, y, u, $v \in \mathbb{R}^n$; x_0 and y_0 are the initial positions of the players which are regarded as $|x_0 - y_0| > \ell$ for some $\ell > 0$ at the time $t = 0$; x_1 and y_1 are their initial velocity vectors. The controls u and v are considered as the acceleration vectors of the players, which are required to become measurable functions $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ and $v(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$, obeying the *geometric constraints* (for brevity, the *G-constraints*, [20]–[24])

$$|u(t)| \leq \alpha(t) \ \text{ for almost all } \ t \geq 0, \tag{2.3}$$

$$|v(t)| \leq \beta(t) \ \text{ for almost all } \ t \geq 0, \tag{2.4}$$

where $\alpha(t)$ and $\beta(t)$ are non-negative integrable functions representing the maximum acceleration values of players P and E at each current time $t \geq 0$.

Let the class of all admissible controls $u(\cdot)$ (resp., $v(\cdot)$) corresponding to (2.3) (resp., (2.4)) be denoted by $\mathbf{U}_P$ for the Pursuer (resp., by $\mathbf{V}_E$ for the Evader).

By equations (2.1) and (2.2), for controls $u(\cdot) \in \mathbf{U}_P$ and $v(\cdot) \in \mathbf{V}_E$, every triplet $(x_0, x_1, u(\cdot))$ and $(y_0, y_1, v(\cdot))$ generates the trajectories

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)u(s)ds, \qquad y(t) = y_0 + y_1 t + \int_0^t (t-s)v(s)ds,$$

of the Pursuer and the Evader, respectively.

The Pursuer's primary goal is to get inside an ℓ-radius vicinity of the Evader, that is, to achieve the relation $|x(t^*) - y(t^*)| \leq \ell$ at some finite time $t^* > 0$. The Evader's goal, on the other hand, is to prevent the Pursuer from getting to this ℓ-radius zone, i.e., to maintain the inequality $|x(t) - y(t)| > \ell$ for all $t \geq 0$.

If new denotations $z = x - y$, $z_0 = x_0 - y_0$, $z_1 = x_1 - y_1$ are used in order to make our calculations convenient, then systems (2.1) and (2.2) reduce to the following form:

$$\ddot{z} = u - v, \quad z(0) = z_0, \quad \dot{z}(0) = z_1. \tag{2.5}$$

Given any admissible controls $u(\cdot) \in \mathbf{U}_P$ and $v(\cdot) \in \mathbf{V}_E$, the solution of (2.5) will be characterized as follows:

$$z(t) = z_0 + z_1 t + \int_0^t (t-s)\left(u(s) - v(s)\right) ds. \tag{2.6}$$

Definition 2.1. A map $\mathbf{u} : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbf{V}_E \to \mathbf{U}_P$ is said to be a strategy of the Pursuer if:

1^o. (Admissibility.) the inclusion $\mathbf{u}[z_0, t, v(\cdot)] \in \mathbf{U}_P$ is met for each control $v(\cdot) \in \mathbf{V}_E$ and for each $t \geq 0$;

2^o. (Volterranianity.) For $v_1(\cdot)$, $v_2(\cdot) \in \mathbf{V}_E$ and for every t, $t \geq 0$, the equality $v_1(s) = v_2(s)$ a.e. on $[0, t]$ implies $u_1(s) = u_2(s)$ a.e. on $[0, t]$ with $u_i(\cdot) = \mathbf{u}[z_0, t, v_i(\cdot)]$.

In the current work, we focus on discussing the ℓ-catch problems with the dynamics of players (2.1)–(2.2) under the constraints (2.3)–(2.4) in the following two cases:

Case 1. The vectors z_0 and z_1 are linearly dependent, so there exists a specific finite scalar k, $k \in \mathbb{R}$, for which $z_1 = kz_0$.

Case 2. The vectors z_0 and z_1 are linearly independent.

3. The ℓ-catch problem in *Case 1.*

In this section, the Pursuer's strategy will be formulated as a dependence on its acceleration $\alpha(t)$, the current time t, the initially given constants z_0, ℓ and the current values of the control $v(t)$ at each current moment $t \geq 0$.

Let

$$\gamma_\ell(z_0, t, v) = \frac{1}{h} \cdot \left[\langle v, z_0\rangle + \alpha(t)\ell + \sqrt{(\langle v, z_0\rangle + \alpha(t)\ell)^2 + h(\alpha^2(t) - |v|^2)}\right], \tag{3.1}$$

where $\langle v, z_0\rangle$ is referred as the scalar product of v and z_0 in $\mathbb{R}^n$, and let

$$\mu(z_0, t, v) = -\ell \frac{v - \gamma_\ell(z_0, t, v)z_0}{|v - \gamma_\ell(z_0, t, v)z_0|}, \quad h = |z_0|^2 - \ell^2.$$

The function defined in (3.1) is referred to as the resolving function ([14, 24]).

Definition 3.1. The control function

$$\mathbf{u}_\ell(z_0, t, v) = v + \gamma_\ell(z_0, t, v)\big(\mu(z_0, t, v) - z_0\big) \tag{3.2}$$

is said to be Pursuer's strategy or for brevity, Π_ℓ-strategy.

Assumption 3.2. Let one of the conditions be valid:

a) $\alpha(t) > \beta(t)$ and $k \in \mathbb{R}$;

b) $\alpha(t) = \beta(t)$ and $k < 0$

for all $t \geq 0$. Then the equation $\Gamma(t) = 0$ has the least one positive root with respect to t that will be denoted by T_ℓ, where

$$\Gamma(t) = 1 - \frac{1}{|z_0| - \ell} \int_0^t (t - s)(\alpha(s) - \beta(s))ds + \frac{|z_0|k}{|z_0| - \ell}t.$$

Definition 3.3. It is said that Π_ℓ-strategy (3.2) guarantees a win for the Pursuer on time interval $[0, T_\ell]$ if for any control $v(\cdot) \in \mathbf{V}_E$ of the Evader

a) an inclusion $\mathbf{u}_\ell(z_0, t, v(\cdot)) \in \mathbf{U}_P$ is valid on the time interval $[0, T_\ell]$;

b) there exists some time moment t^*, $t^* \in [0, T_\ell]$ at which the solution $z(t)$ of the equation

$$\ddot{z} = \mathbf{u}_\ell(z_0, t, v(t)) - v(t), \quad z(0) = z_0, \quad \dot{z}(0) = z_1 \tag{3.3}$$

satisfies the inequality $|z(t^*)| \leq \ell$.

Here we say that the number T_ℓ is a guaranteed time of ℓ-catch in *Case 1.*

Theorem 3.4. *Let Assumption 3.2 be satisfied. Then Π_ℓ-strategy (3.2) guarantees a win for the Pursuer in the ℓ-catch problem of Case 1 on the time interval $[0, T_\ell]$.*

Proof. Assume that the Evader begins its motion by choosing an admissible control $v(\cdot) \in \mathbf{V}_E$ and the Pursuer applies Π_ℓ-strategy (3.2). By virtue of the equality $z_1 = kz_0$, the solution of equation (3.3) takes the form

$$z(t) = z_0 + z_0kt + \int_0^t (t - s)\big(\mathbf{u}_\ell(z_0, s, v(s)) - v(s)\big)ds.$$

From (3.2), we form

$$z(t) = z_0\left(1 + kt - \int\limits_0^t (t-s)\gamma_\ell(z_0, s, v(s))ds\right) + \int\limits_0^t (t-s)\gamma_\ell(z_0, s, v(s))\mu(z_0, s, v(s))ds.$$

From here, and taking account (2.4) and the lemma about minimum-maximum in [25], we obtain the following estimates:

$$|z(t)| - \ell \le |z_0|kt + (|z_0| - \ell)\left(1 - \int\limits_0^t (t-s)\gamma_\ell(z_0, s, v(s))ds\right) \le$$

$$\le |z_0|kt + (|z_0| - \ell)\left(1 - \min_{v(\cdot)\in \mathbf{V}_E}\int\limits_0^t (t-s)\gamma_\ell(z_0, s, v(s))ds\right) \le$$

$$\le |z_0|kt + (|z_0| - \ell)\left(1 - \frac{1}{h}\int\limits_0^t (t-s)\Big[\alpha(s)\ell - |v(s)||z_0| + \alpha(s)|z_0| - |v(s)|\ell\Big]ds\right) \le$$

$$\le (|z_{10}| - \ell)\Gamma(t).$$

On the basis of Assumption 3.2, the inequality $|z(T_\ell)| \le \ell$, i.e., $\big|x\left(T_\ell\right) - y\left(T_\ell\right)\big| \le \ell$ holds at the time $t = T_\ell$. Thus, conclude that in the time interval $[0, T_\ell]$, there exists the time t^* such that the ℓ-catch occurs even earlier than $t = T_\ell$. □

4. The ℓ-catch problem in *Case 2*

In this section, we assume that the vectors z_0 and z_1 are linearly independent. To solve the ℓ-catch problem in this case, we divide the problem into two subcases.

Subcase 2.1. In this part of *Case 2*, the Pursuer attempts to match its velocity with that of the Evader by applying a Π-strategy (see Definition 4.1), that is,

$$\dot{x}(\theta) = \dot{y}(\theta), \tag{4.1}$$

where θ is some finite time.

Subcase 2.2. Using the Π_ℓ-strategy (see Definition 4.5), the Pursuer attempts to enter an ℓ-radius vicinity of the Evader during the time interval $[\theta, +\infty]$.

4.1. Applying the Π-strategy in *Subcase 2.1.*

Let's introduce new variables $x_* = \dot{x}$, $y_* = \dot{y}$. Then from (2.1)–(2.2), we form the following the first order differential game:

$$\dot{x}_* = u, \qquad x_*(0) = x_1, \tag{4.2}$$

$$\dot{y}_* = v, \qquad y_*(0) = y_1, \tag{4.3}$$

respectively, where u and v are the rates of change of the vectors x_*, y_* with respect to time t. x_1, y_1 are the initial states of the vectors x_*, y_* and $x_1 \ne y_1$. On the other hand, we return to *Case 1.*

Definition 4.1. For *Subcase 2.1*, the function

$$\mathbf{u}(z_1, t, v) = v - \lambda(z_1, t, v)\xi_1 \tag{4.4}$$

is said to be the Π-strategy of the Pursuer, where

$$\lambda(z_1, t, v) = \langle \xi_1, v\rangle + \sqrt{\langle \xi_1, v\rangle^2 + \alpha^2(t) - |v|^2}, \quad \xi_1 = z_1/|z_1|, \quad z_1 = x_1 - y_1,$$

and $\mathbf{u}(z_1, t, v(t))$, $t \ge 0$ – its realization for each $v(\cdot) \in \mathbf{V}_E$.

The existence of time θ is shown by the following theorem.

Theorem 4.2. *Let $\alpha(t) \geq \beta(t)$ in Subcase 2.1, and the Pursuer apply Π-strategy (4.4). Then for any control $v(\cdot) \in \mathbf{V}_E$ of the Evader the equality (4.1) can be achieved in some time $\theta \in [T_1, T_2]$, where T_1 is the first positive root of equation $\Lambda_1(z_1, t) = 1 - \int_0^t (\alpha(s) + \beta(s))/|z_1| ds = 0$ and T_2 is the first positive root of equation $\Lambda_2(z_1, t) = 1 - \int_0^t (\alpha(s) - \beta(s))/|z_1| ds = 0$*

Proof. Let player E select some control $v(\cdot) \in \mathbf{V}_E$, and player P implement Π-strategy (4.4) and let be $z_* = x_* - y_*$. Then from (4.2)–(4.3), we identify the solution of the resulting Cauchy problem as follows:

$$z_*(t) = z_1 - \int_0^t \lambda(z_1, s, v(s))\xi_1 ds = z_1 \Lambda(z_1, t, v_t(\cdot)), \tag{4.5}$$

where

$$\Lambda(z_1, t, v_t(\cdot)) = 1 - \frac{1}{|z_1|} \int_0^t \lambda(z_1, s, v(s)) ds. \tag{4.6}$$

and $\Lambda(z_1, t, v_t(\cdot))$ is named the function of convergence of the velocities of the players.

Using the view of function $\lambda(z_1, t, v)$ in Definition 4.1, for $|v(t)| \leq \beta(t)$ we have the inequality

$$\alpha(t) - \beta(t) \leq \lambda(z_1, t, v(t)) \leq \alpha(t) + \beta(t).$$

According to this, and from (4.6) we obtain the relations

$$\Lambda_1(z_1, t) \leq \Lambda(z_1, t, v_t(\cdot)) \leq \Lambda_2(z_1, t),$$

where $\Lambda_1(z_1, t) = 1 - \int_0^t (\alpha(s) + \beta(s))/|z_1| ds$ and $\Lambda_2(z_1, t) = 1 - \int_0^t (\alpha(s) - \beta(s))/|z_1| ds$. Thus, we get that the function $\Lambda(z_1, t, v_t(\cdot))$ vanishes on the time interval $[T_1, T_2]$, i.e., there exists $\theta \in [T_1, T_2]$, such that $\Lambda(z_1, \theta, v_\theta(\cdot)) = 0$. Therefore, from (4.5) we get $z_*(\theta) = 0$ or $\dot{x}(\theta) = \dot{y}(\theta)$, which completes the proof of the Theorem 4.2. $\square$

It was demonstrated in [23] that for simple pursuit game of the form (4.2)–(4.3), the Π-strategy of the type (4.4) ensures both the optimal approach of the players and the optimal time of their coincidence. Therefore, in this case as well, the strategy (4.4) guarantees the optimal convergence of the players' velocities.

4.2. **Estimating the distance between the players in $[0, \theta]$.** In the present subsection, we will consider an analysis of the inter-player distance during the time interval $[0, \theta]$ to answer whether the distance between players approaches the value ℓ within this specific time interval.

Let $v(\cdot) \in \mathbf{V}_E$ be a control selected by player E, and let player P employ Π-strategy (4.4) in the time interval $[0, \theta]$. Then by (2.6) and (4.4), we arrive the result

$$z(t) = z_0 + z_1 t - \int_0^t (t - s)\lambda(z_1, s, v(s))\xi_1 ds.$$

It follows that

$$z(t) = z_0 + z_1 \overline{\Lambda}(z_1, t, v_t(\cdot)), \tag{4.7}$$

where

$$\overline{\Lambda}(z_1, t, v_t(\cdot)) = t - \frac{1}{|z_1|}\int\limits_0^t (t-s)\lambda\left(z_1, s, v(s)\right) ds. \tag{4.8}$$

Therefore, from (4.6) and (4.8) we find $d\overline{\Lambda}(z_1, t, v_t(\cdot))/dt = \Lambda(z_1, t, v_t(\cdot))$. By virtue of $\Lambda(z_1, t, v_t(\cdot)) \geq 0$ in $[0, \theta]$, as a result, the final equality reveals that $\overline{\Lambda}(z_1, t, v_t(\cdot))$ is monotonically increasing function during the time $t \in [0, \theta]$. From this, in light of the equality $\lambda\left(z_1, t, v(t)\right) \leq \alpha(t) - \beta(t)$ for $|v(t)| \leq \beta(t)$ and from $\theta \in [T_1, T_2]$, we form the following relations:

$$\overline{\Lambda}(z_1, t, v_t(\cdot)) \leq \overline{\Lambda}(z_1, \theta, v_\theta(\cdot)) \leq \theta - \frac{1}{|z_1|}\int\limits_0^\theta (\theta - s)\left(\alpha(s) - \beta(s)\right) ds \leq \Delta,$$

where

$$\Delta = T_2 - \frac{1}{|z_1|}\int\limits_0^{T_2} (T_2 - s)\left(\alpha(s) - \beta(s)\right) ds.$$

Consequently, for any arbitrary control $v(\cdot) \in \mathbf{V}_E$ on $[0, \theta]$, the following estimate hold:

$$0 \leq \overline{\Lambda}(z_1, t, v_t(\cdot)) \leq \Delta. \tag{4.9}$$

Now, consider the function $\omega : \mathbb{R}_+ \to \mathbb{R}^n$ defined by $\omega(\tau) = z_0 + \tau z_1$, where $\tau \in [0, \Delta]$, when z_0, z_1 are non-collinear vectors. It is evident that the image of $\omega(\tau)$ for $\tau \in [0, \Delta]$ is a finite line segment in $\mathbb{R}^n$ with direction vector z_1. Due to the functional form of $\omega(\tau)$, the following crucial property can be readily verified.

Proposition 4.3. *a) If $\langle z_0, z_1\rangle > 0$, then the function $|\omega(\tau)|$ is increasing in $\tau \in [0, \Delta]$, and it is approximated as:*

$$|z_0| \leq |\omega(\tau)| \leq |z_0 + z_1\Delta|;$$

b) If $-|z_1|\sqrt{h} < \langle z_0, z_1\rangle < 0$, then for each $\tau \in [0, \Delta]$:

$$\ell < \left| z_0 - \frac{\langle z_0, z_1\rangle}{|z_1|^2} z_1 \right| \leq |\omega(\tau)| \leq \max\left\{|z_0|, |z_0 + z_1\Delta|\right\},$$

where $h = |z_0|^2 - \ell^2$.

Consequently, from Property 4.3 and considering equations (4.7) and (4.9), we deduce the subsequent conclusion within the time interval $[0, \theta]$:

$$\ell < L_1 \leq |z(t)| \leq L_2, \tag{4.10}$$

where

$$L_1 = \begin{cases} |z_0| & \text{if } \langle z_0, z_1\rangle > 0, \\ \left| z_0 - \frac{\langle z_0, z_1\rangle}{|z_1|^2} z_1 \right| & \text{if } -|z_1|\sqrt{h} < \langle z_0, z_1\rangle < 0, \end{cases}$$

$$L_2 = \begin{cases} |z_0 + z_1\Delta| & \text{if } \langle z_0, z_1\rangle > 0, \\ \max\left\{|z_0|, |z_0 + z_1\Delta|\right\} & \text{if } -|z_1|\sqrt{h} < \langle z_0, z_1\rangle < 0. \end{cases}$$

Theorem 4.4. *If $\alpha(t) \geq \beta(t)$ and the Pursuer uses Π-strategy (4.4) under the conditions of Property 4.3, then estimate (4.10) holds for any control $v(\cdot) \in \mathbf{V}_E$ during $[0, \theta]$.*

Proof. The proof of Theorem 4.4 directly follows from Property 4.3. □

4.3. **Solution of the ℓ-catch problem for the case $t \geq \theta$.** According to Theorem 4.4, during the time interval $[0, \theta]$, the players cannot close the distance between them by ℓ. However, θ is the time of coincidence of their velocities. From this point forward, we consider the following game:

$$\ddot{x} = u, \quad x(\theta) = x_0^*, \quad \dot{x}(\theta) = x_1^*, \tag{4.11}$$

$$\ddot{y} = v, \quad y(\theta) = y_0^*, \quad \dot{y}(\theta) = y_1^*, \tag{4.12}$$

where x_0^*, y_0^* are the states of the players, and x_1^*, y_1^* are the states of their velocities at the time θ. Theorem 4.4 shows that $|x_0^* - y_0^*| > \ell$. However, in (4.11)–(4.12), equality (4.1) is met in the form $x_1^* = y_1^*$, i.e., the velocities coincide at $t = \theta$, and the game reduces to *Case 1*, when $k = 0$.

Definition 4.5. For *Subcase 2.2*, the function

$$\mathbf{u}_\ell^*(z_0^*, t, v) = v + \gamma_\ell^*(z_0^*, t, v)\left(\mu^*(z_0^*, t, v) - z_0^*\right) \tag{4.13}$$

is said to be the Π_ℓ-strategy of the Pursuer, where

$$\gamma_\ell^*(z_0^*, t, v) = \frac{1}{h^*}\left[\langle z_0^*, v\rangle + \alpha(t)\ell + \sqrt{(\langle z_0^*, v\rangle + \alpha(t)\ell)^2 + (\alpha^2(t) - |v|^2)h^*}\right],$$

$h^* = |z_0^*|^2 - \ell^2$, $\mu^*(z_0^*, t, v) = -(v - \gamma^*(z_0^*, t, v)z_0^*)\ell / |v - \gamma_\ell^*(z_0^*, t, v)z_0^*|$ and $z_0^* = x_0^* - y_0^*$.

Theorem 4.6. *For Subcase 2.2, under the conditions of Property 4.3, let $\alpha(t) \geq \beta(t)$. Then Π_ℓ-strategy (4.13) provides to win for the Pursuer in the ℓ-catch game of Case 2 on the time interval $[\theta, \overline{T}_\ell]$, where $\overline{T}_\ell = \overline{t}^*(z_0^*) + \theta$ and $\overline{t}^*(z_0^*)$ is the first positive root of the following equation:*

$$\int_0^t (t - s)\left(\alpha(s) - \beta(s)\right) ds = z_0^* - \ell.$$

Proof. The proof of Theorem 4.6 is carried out similarly to Theorem 4.2. Only here the strategy (4.13) is implemented. □

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Cauchy problem for differential equation with regularized Prabhakar fractional derivative

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Abstract. In this short note, we want to give the main parts of the verification of the ordered Prabhakar fractional order differential operator and the Cauchy problem for an equation with singular coefficients. That is, we present the exact solution of the Cauchy problem for the Prabhakar fractional derivative partial differential equation. Using the Fourier method, we find the general solution of the given equation. The unknown coefficient is found using the Hankel transformation. We have also presented important statements the bivariate Mittag-Leffler function $E_2(x, y)$ and the Bessel function. The main objective is the Cauchy problem for a singular coefficient equation involving ordered Prabhakar derivatives. We present here the proof algorithm and a brief description of the main steps.

Keywords: Cauchy problem, Prabhakar fractional operator, Hankel transform, bivariate Mittag-Leffler function, Bessel function

MSC (2020):33C10, 33E12, 35A22, 35B30, 35R11

1. Introduction

It is well-known that many special functions appear in solutions for differential equations. For instance, hypergeometric functions are key part of solutions for singular elliptic equations and many other degenerate partial differential equations. In fractional calculus, so called Mittag-Leffler type functions play crucial role. Multivariable analogs of such functions are also important and they are linked with multi-term fractional differential equations. In this regard, we note that a general solution of the differential equation involving Caputo–Dzherbashyan derivative with the Mittag-Leffler function in a right-hand side was represented via bivariate Mittag-Leffler type function [13].

In some recent investigations, solutions of certain Cauchy problems were represented via infinite series of Mittag-Leffler type functions [14, 15, 16, 17]. Namely, in [14], the solution to the Cauchy problem for differential equation with the regularized Prabhakar fractional derivative.

In this paper, we study the Cauchy problem for a time-fractional order equation with Riemann-Liouville fractional derivatives and the Bessel operator. The solution to the problem under consideration is solved by the Hankel transformation method [1, 2, 3].

Hankel transformation and other transformations are used to solve problems in mechanics, elasticity theory, thermal conductivity, electrodynamics and other branches of theoretical physics. More detailed information about the Hankel transform can be found in [4, 5, 6, 7, 8, 9, 10, 11].

2. Definition of the Hankel Transform

The Hankel transform is an integral transform and was first developed by the mathematician Hermann Hankel. It is also known as the Fourier–Bessel transform. Just as the Fourier transform for an infinite interval is related to the Fourier series over a finite interval, so the Hankel transform over an infinite interval is related to the Fourier–Bessel series over a finite interval. The Hankel transform expresses any given function $f(r)$ as the weighted sum of an infinite number of Bessel functions of the first kind $J_\nu(\eta r)$. The Bessel functions in the sum are all of the same order ν, but differ in a scaling factor η along the $r$axis. The necessary coefficient

F_ν of each Bessel function in the sum, as a function of the scaling factor η constitutes the transformed function.

Definition 2.1. Let $f(r)$ be a function defined for $r \geq 0$. The υth order Hankel transform of $f(r)$is defined as

$$F_\nu(\eta) = \int_0^\infty r f(r) J_\nu(\eta r)\, dr, \left(\nu > -\frac{1}{2}\right), \tag{2.1}$$

where $J_\nu(x)$ is well known Bessel function of the first kind and defined as

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\nu+1)\, n!} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{2.2}$$

Theorem 2.2. *If the functionf (x) piecewise continuous in any finite interval (has bounded variation) belonging to the interval $(0, \infty)$ and integral converges*

$$\int_0^\infty |f(\xi)| \sqrt{\xi} d\xi,$$

then the Hankel transform exists and the inversion of the Hankel transform is given by the following formula

$$f(r) = \int_0^\infty \eta F_\nu(\eta) J_\nu(\eta r)\, d\eta. \tag{2.3}$$

Formulas (2.1) and (2.3) can be written in the following form

$$f(r) = \int_0^\infty J_\nu(r\xi)\, \xi d\xi \int_0^\infty f(\zeta) J_\nu(\xi\zeta)\, \zeta d\zeta = \int_0^\infty f(\zeta)\, \zeta d\zeta \int_0^\infty J_\nu(\xi\zeta) J_\nu(r\xi)\, \xi d\xi. \tag{2.4}$$

From relation (2.4), it follows

$$f(\eta) = \int_0^\infty a(\xi)\, \xi J_\nu(\eta\xi)\, d\xi, \tag{2.5}$$

where

$$a(\xi) = \int_0^\infty f(\rho) J_\nu(\rho\xi)\, \rho d\rho, \tag{2.6}$$

formulas (2.1), (2.3) and (2.4) given in monographs [1, 2, 3].

From the properties of the Dirac delta function [7, 8], it follows

$$\int_0^\infty \delta(x-a)\, \Phi(x)\, dx = \Phi(a), \tag{2.7}$$

$$\delta(x-a) = x \int_0^\infty t J_\mu(xt) J_\mu(at)\, dt, \ |\mu| < \frac{1}{2}, \tag{2.8}$$

As can be seen from the above properties, we can conclude that the function $f(r)$ is compatible with Dirac delta function.

3. Solving the Cauchy problem for the time-fractional diffusion equation

Let us consider the following time-fractional diffusion equation

$$ {}^{PC}\mathrm{D}_{0t}^{\alpha,\beta,\gamma,\delta}u(x,t) - u_{xx}(x,t) - \frac{\nu}{x}u_x(x,t) = 0, \quad 0<\nu<1, \tag{3.1}$$

in a domain $\Omega = \{(x,t): \ x>0, \ 0<t<\infty\}$. Here $\alpha,\beta \in \mathrm{R}^+$, $\gamma,\delta \in \mathrm{R}$, $m=[\beta]+1$, $m-1 \le \beta < m$ and

$$ {}^{PC}\mathrm{D}_{0t}^{\alpha,\beta,\gamma,\delta}y(t) = {}^{P}I_{0t}^{\alpha,m-\beta,-\gamma,\delta}\frac{d^m}{dt^m}y(t), \tag{3.2}$$

represents regularized Prabhakar fractional derivative [18] and

$$ {}^{P}I_{0t}^{\alpha,\beta,\gamma,\delta}y(t) = \int_0^t (t-\xi)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left[\delta(t-\xi)^{\alpha}\right]y(\xi)\,d\xi, \ \ t>0, \tag{3.3}$$

represents Prabhakar fractional integral [20]. Here $E_{\alpha,\beta}^{\gamma}(z)$ is Prabhakar function

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)\,n!}.$$

We note that above-given definitions are valid for $\alpha,\beta,\gamma,\delta \in \mathrm{C}$ such that $\Re(\alpha)>0$ and $m-1<\Re(\beta)<m$, $m\in\mathrm{N}$. We formulate the Cauchy problem for time-fractional diffusion equation in the case of $0<\beta<1$.

Cauchy problem. It is required to find in the domain a solution to equation (3.1) satisfying the initial condition

$$u(x,t)|_{t=0} = \psi(x), \quad 0<x<\infty, \tag{3.4}$$

and for any fixed t we have

$$\lim_{x\to\infty} u(x,t) = 0, \tag{3.5}$$

where

$$\psi(x) \in \mathbf{C}^2, \ \ \int_0^{\infty}|\psi(x)|\,x^{\frac{\nu}{2}}dx < c = const,$$

in addition to this

$$\psi(0)=0, \qquad \lim_{x\to\infty}\psi(x)=0. \tag{3.6}$$

Solution. We look for the solution of equation (3.1) using the Fourier method

$$u(x,t) = X(x)\,T(t). \tag{3.7}$$

Substituting (3.7) in the equation (3.1), we have two equations

$$X_{xx} + \frac{\nu}{x}X_x + \lambda^2 X = 0, \tag{3.8}$$

$$ {}^{PC}\mathrm{D}_{0t}^{\alpha,\beta,\gamma,\delta}T(t) + \lambda^2 T(t) = 0, \ \ (\lambda \in \mathbf{R}\setminus\{0\}). \tag{3.9}$$

By substituting the product $X = x^{\frac{1-\nu}{2}}\theta(x\lambda)$ into (3.8), we get the following Bessel equation

$$x^2\lambda^2\theta_{xx}(x\lambda) + x\lambda\theta_x(x\lambda) + \left(x^2\lambda^2 - \frac{(\nu-1)^2}{4}\right)\theta(x\lambda) = 0, \ \ (0<x<\infty, \ \ \theta(x\lambda) \in \mathbf{C}^2),$$

We know that, when $\frac{\nu-1}{2}$ is not integer the functions $J_{\frac{\nu-1}{2}}(\lambda x)$ and $J_{\frac{1-\nu}{2}}(\lambda x)$ are linear independent solutions of above Bessel equation [12]. From conditions (3.6), we use only the function $J_{\frac{1-\nu}{2}}(\lambda x)$. Then the solution of equation (3.8) has the following form

$$X = c(\lambda)\, x^{\frac{1-\nu}{2}} J_{\frac{1-\nu}{2}}(\lambda x). \tag{3.10}$$

In [14], the solution of equation (3.9) is represented by an infinite series. Then, E. Karimov and
A. Hasanov [16] expressed the infinite series solution of equation (3.9) as a function $E_2(\cdot)$

$$T(t) = a(\lambda)\left[1 - \lambda^2 t^\beta \Gamma(\gamma)\, E_2\left(\begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta+1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} -\lambda^2 t^\beta \\ \delta t^\alpha \end{matrix} \right)\right]. \tag{3.11}$$

Here $E_2(\cdot)$ is a bi-variate Mittag-Leffler function [19]

$$E_2\left(\begin{matrix} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2; \\ \delta_1, \alpha_3, \beta_2; \delta_2, \alpha_4; \delta_3, \beta_3; \end{matrix} \;\middle|\; \begin{matrix} x \\ y \end{matrix} \right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m + \beta_1 n} (\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1 + \alpha_3 m + \beta_2 n)} \frac{x^m}{\Gamma(\delta_2 + \alpha_4 m)} \frac{y^n}{\Gamma(\delta_3 + \beta_3 n)}.$$

Using the solutions (3.10) and (3.11), we can write the solution of the form (3.7) in the following form

$$u(x,t) = x^{\frac{1-\nu}{2}} \int_0^\infty C(\lambda)\, J_{\frac{1-\nu}{2}}(\lambda x) \left(1 - \lambda^2 t^\beta \Gamma(\gamma)\, E_2\left(\begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta+1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} -\lambda^2 t^\beta \\ \delta t^\alpha \end{matrix} \right)\right) d\lambda. \tag{3.12}$$

By (3.12) we determine the unknown function $C(\lambda)$. Considering the initial condition (3.4), we have

$$u(x,t)|_{t=0} = x^{\frac{1-\nu}{2}} \int_0^\infty C(\lambda)\, J_{\frac{1-\nu}{2}}(\lambda x)\, d\lambda = \psi(x),$$

Using equalities (2.5) and (2.6), we determine unknown function

$$\frac{x^{\frac{\nu-1}{2}}}{2} \psi(x) = \frac{1}{2} \int_0^\infty \frac{C(\lambda)}{\lambda} \lambda J_{\frac{1-\nu}{2}}(\lambda x)\, d\lambda,$$

$$C(\lambda) = \frac{\lambda}{2} \int_0^\infty \rho^{\frac{\nu-1}{2}} \psi(\rho)\, J_{\frac{1-\nu}{2}}(\lambda\rho)\, \rho d\rho. \tag{3.13}$$

Substituting (3.13) in (3.12), we get the following solution of the Cauchy problem

$$u(x,t) = \frac{1}{2} x^{\frac{1-\nu}{2}} \int_0^\infty \rho^{\frac{\nu-1}{2}} \psi(\rho)\, G(x,t,\rho)\, \rho d\rho, \tag{3.14}$$

where

$$G(x,t,\rho) = \int_0^\infty J_{\frac{1-\nu}{2}}(\lambda\rho)\, J_{\frac{1-\nu}{2}}(\lambda x) \left(1 - \lambda^2 t^\beta \Gamma(\gamma)\, E_2\left(\begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta+1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} -\lambda^2 t^\beta \\ \delta t^\alpha \end{matrix} \right)\right) \lambda d\lambda.$$

First, let us show that the constructed function (3.14) satisfies equation (3.1). From the definition of regularized Prabhakar fractional derivative operator (3.2), it is clear that

$$\begin{aligned} {}^{PC}\mathrm{D}_{0t}^{\alpha,\beta,\gamma,\delta} u(x,t) = \tfrac{1}{2} x^{\frac{1-\nu}{2}} \int_0^\infty \int_0^\infty \rho^{\frac{\nu-1}{2}} \psi(\rho)\, J_{\frac{1-\nu}{2}}(\lambda\rho)\, J_{\frac{1-\nu}{2}}(\lambda x) \\ \times {}^{PC}\mathrm{D}_{0t}^{\alpha,\beta,\gamma,\delta} \left(1 - \lambda^2 t^\beta \Gamma(\gamma)\, E_2\left(\begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta+1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} -\lambda^2 t^\beta \\ \delta t^\alpha \end{matrix} \right)\right) \lambda d\lambda \rho d\rho. \end{aligned} \tag{3.15}$$

Next, we calculate the derivatives with respect to x of function (3.14), which are involved in equation (3.1).

$$u_x(x,t) = \tfrac{1}{2}x^{\frac{1-\nu}{2}}\int_0^\infty \rho^{\frac{\nu-1}{2}}\psi(\rho)\left(1-\lambda^2 t^\beta\Gamma(\gamma)E_2\left(\begin{array}{c}\gamma,\gamma,1;1,0\\ \beta+1,\beta,\alpha;\gamma,\gamma;1,1\end{array}\middle|\begin{array}{c}-\lambda^2t^\beta\\ \delta t^\alpha\end{array}\right)\right)\lambda\rho$$
$$\times\int_0^\infty\left[\frac{1-\nu}{2}x^{-1}J_{\frac{1-\nu}{2}}(\rho\lambda)J_{\frac{1-\nu}{2}}(\lambda x)+\lambda J_{\frac{1-\nu}{2}}(\rho\lambda)J'_{\frac{1-\nu}{2}}(\lambda x)\right]d\rho d\lambda\,. \tag{3.16}$$

Then, we calculate second derivative of the function (3.14) with respect to the variable x.

$$u_{xx}(x,t) = \tfrac{1}{2}x^{\frac{1-\nu}{2}}\int_0^\infty \rho^{\frac{\nu-1}{2}}\psi(\rho)\left(1-\lambda^2 t^\beta\Gamma(\gamma)E_2\left(\begin{array}{c}\gamma,\gamma,1;1,0\\ \beta+1,\beta,\alpha;\gamma,\gamma;1,1\end{array}\middle|\begin{array}{c}-\lambda^2t^\beta\\ \delta t^\alpha\end{array}\right)\right)\lambda\rho$$
$$\times\int_0^\infty J_{\frac{1-\nu}{2}}(\rho\lambda)\left(\lambda^2 J''_{\frac{1-\nu}{2}}(\lambda x)+x^{-1}\lambda J'_{\frac{1-\nu}{2}}(\lambda x)-\nu x^{-1}\lambda J'_{\frac{1-\nu}{2}}(\lambda x)-\left(\frac{1-\nu^2}{4x^2}\right)J_{\frac{1-\nu}{2}}(\lambda x)\right)d\rho d\lambda\,.$$

We can make a slight simplification by using the fact that the function $J_{\frac{1-\nu}{2}}(\lambda x)$ is linear independent solution of Bessel equation.

$$u_{xx}(x,t) = \frac{1}{2}x^{\frac{1-\nu}{2}}\int_0^\infty \rho^{\frac{\nu-1}{2}}\psi(\rho)\rho$$
$$\times\int_0^\infty\left[\left(\frac{\nu^2-\nu}{2x^2}-\lambda^2\right)J_{\frac{1-\nu}{2}}(\rho\lambda)J_{\frac{1-\nu}{2}}(\lambda x)-\nu x^{-1}\lambda J_{\frac{1-\nu}{2}}(\rho\lambda)J'_{\frac{1-\nu}{2}}(\lambda x)\right]$$
$$\times\left(1-\lambda^2 t^\beta\Gamma(\gamma)E_2\left(\begin{array}{c}\gamma,\gamma,1;1,0\\ \beta+1,\beta,\alpha;\gamma,\gamma;1,1\end{array}\middle|\begin{array}{c}-\lambda^2t^\beta\\ \delta t^\alpha\end{array}\right)\right)\lambda d\lambda d\rho. \tag{3.17}$$

We substitute the obtained derivatives (3.15), (3.16) and (3.17) in equation (3.1). After some simplifications, we get

$$\int_0^\infty\int_0^\infty \rho^{\frac{\nu-1}{2}}\psi(\rho)\left(J_{\frac{\nu-1}{2}}(\lambda\rho)J_{\frac{\nu-1}{2}}(\lambda x)+J_{\frac{1-\nu}{2}}(\lambda\rho)J_{\frac{1-\nu}{2}}(\lambda x)\right)$$
$$\times\left[{}^{PC}\mathtt{D}_{0t}^{\alpha,\beta,\gamma,\delta}T(t)+\lambda^2T(t)\right]\lambda d\lambda\rho d\rho=0,$$

where $T(t)$ is given, as before, by (3.11). We consider that the right-hand side of the expression (3.9) is zero.Thus, the solution (3.14) satisfies equation (3.1).

Let us show that the solution (3.14) satisfies the initial condition (3.4). Using condition (3.4) and the properties of the Dirac delta function (2.7), (2.8), we obtain

$$u(x,t)|_{t=0} = \tfrac{1}{2}x^{\frac{1-\nu}{2}}\int_0^\infty\int_0^\infty \rho^{\frac{\nu-1}{2}}\psi(\rho)J_{\frac{1-\nu}{2}}(\lambda\rho)J_{\frac{1-\nu}{2}}(\lambda x)\lambda\rho d\lambda d\rho$$
$$=\tfrac{1}{2}x^{\frac{1-\nu}{2}}2\int_0^\infty \rho^{\frac{\nu-1}{2}}\psi(\rho)\delta(x-\rho)d\rho = x^{\frac{1-\nu}{2}}x^{\frac{\nu-1}{2}}\psi(x)=\psi(x).$$

Consequently, the solution (3.14) satisfies the initial condition.

Now we show that function (3.14) satisfies condition (3.5). For any fixed t, function $T(t)$ is bounded. Further, taking into account the asymptotic representation of the Bessel functions [12] for $z\to\infty$, with $|\arg(z)|<\pi$

$$J_\mu(z)\sim\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\pi\mu}{2}-\frac{\pi}{4}\right),$$

then $X \sim c(\lambda)x^{-\frac{\nu}{2}}$.And this means that on the basis of (3.7) we are convinced that the condition (3.5) is satisfied.

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