UZBEKISTAN ACADEMY OF SCIENCES
V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS

UZBEK MATHEMATICAL JOURNAL

Journal was founded in 1957. Until 1991 it was named by "Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk". Since 1991 it is known as "Uzbek Mathematical Journal". It has 4 issues annually.

Volume 69 Issue 2 2025

Uzbek Mathematical Journal is abstracting and indexing by

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MathSciNet **Zentralblatt** **Math VINITI**

TASHKENT - 2025

Academician of the Academy of Sciences of Uzbekistan Shavkat Arifjanovich Alimov on the 80th anniversary of the birth of

Shavkat Arifjanovich Alimov is a well-known scientist, a major specialist in mathematical physics and functional analysis, who made a great contribution to the spectral theory of differential operators, the theory of boundary value problems for equations of mathematical physics and harmonic analysis.

Sh. A. Alimov was born on March 2, 1945 in Nukus, the capital of Karakalpakstan, in the family of an employee. From 1952 to 1962 he studied at school in Tashkent. After graduating from Tashkent school No 88 with a gold medal in 1962, he entered the Faculty of Physics of the Lomonosov Moscow State University, where he graduated in 1968 with a degree in physics.

He graduated with honors from the Institute of Applied Mathematics of the USSR Academy of Sciences. From 1968 to 1970 he studied in the postgraduate course at the same department under the scientific supervision of Professor V. A. Il'in and in June 1970 at the Academic Council of the Institute of Applied Mathematics of the USSR Academy of Sciences under the chairmanship of Academician M. V. Keldysh he defended his PhD thesis on the theory of functions and functional analysis.

In May 1970, he began his career at the Lomonosov Moscow State University as an assistant at the newly opened Department of Computational Mathematics and Cybernetics. In May 1970, he started his career at the newly opened Faculty of Computational Mathematics and Cybernetics at Lomonosov Moscow State University as an assistant, then worked as an associate professor from 1972 to 1974.

In May 1973, at the age of 28, he defended his doctoral dissertation on the equations of mathematical physics at the Academic Council of the Faculty of Mathematical Physics of Moscow State University under the chairmanship of Academician A. N. Tikhonov. In 1973 he was awarded the highest youth prize of the country for his research on the spectral theory of equations of mathematical physics. In 1974, at the age of 29, he was elected by competition to the position of Professor at the Faculty of Computational Mathematics and Cybernetics of Moscow State University. For ten years - from 1974 to 1984 he worked as a professor of the Department of General Mathematics of the Faculty of Computational Mathematics and Cybernetics of MSU. During the same period he was a member of two specialized councils for the defense of doctoral dissertations: in the specialty 01.01.01 - "functional analysis and theory of functions" in the council chaired by A. N. Kolmogorov and in the specialty of 01.01.02 - "differential equations and mathematical physics" in the council chaired by A. N. Tikhonov. From 1970 to 1984 together with V. A. Il'in directed the Moscow University research seminar on functional methods of mathematical physics.

In September 1984 he moved to Tashkent State University as a professor, since January 1985 he worked as deputy director of the Institute of Mathematics of the Academy of Sciences of Uzbekistan.

From 1985 to 1987 worked as Rector of Samarkand State University, from 1987 to 1990 - Rector of Tashkent State University, from January 1990 to February 1992 - Minister of Higher and Secondary Specialized Education of the Republic of Uzbekistan. From 1992 to 1994, he headed the Department of Mathematical Physics of the Faculty of Applied Mathematics at Tashkent State University.

From 1994 to 1995, he served as Deputy Minister of Foreign Affairs of the Republic of Uzbekistan. From November 1995 to August 1998, he was Ambassador Extraordinary and Plenipotentiary of the Republic of Uzbekistan to the People's Republic of China. From August 1998 to January 2003, he worked as Vice-Rector for Research at the University of World Economy and Diplomacy (Tashkent). From September 2000 to June 2001, he worked as a Visiting Researcher at the California Institute of Technology (CalTech), USA. After returning Tashkent until 2012, he worked as a professor at the Department of Mathematical Physics of the National University of Uzbekistan. At the same time, from the first days of opening the Tashkent branch of Lomonosov Moscow State University in 2006, he worked as a professor at the Department of Applied Mathematics of the said branch. From 2012 to 2017, he headed the Mathematical Modeling Laboratory of the Malaysian Institute of Microelectronic Systems (MIMOS) in Kuala Lumpur, being at the same time Chief Scientist of this Institute.

From 2017 to 2019, he worked as a professor at the Department of Differential Equations and Mathematical Physics, National University of Uzbekistan

In the early seventies of the 20th century, Sh.A. Alimov studied the convergence and summability of spectral expansions associated with elliptic operators of arbitrary order with smooth coefficients

In the late seventies, Sh. A. Alimov, at the suggestion of A. V. Bitsadze, studied the degenerate boundary value problem with sloping derivative for elliptic equations of the second order. He found, in particular, the exact order of loss of smoothness of the solution depending on the degree of degeneracy of the vector field defining the boundary conditions. Somewhat later, a similar result was obtained by the Swedish mathematician Bent Wintzel.

In the early eighties, elliptic equations with singular coefficients were studied by Sh.A. Alimov. This class of equations includes the Schrdinger equation with a potential having singularities not only at individual points but also on manifolds that can go to infinity.

At the same time, Sh.A. Alimov studied the spectral properties of nonlocal boundary value problems in which the boundary conditions relate the values of the desired function at some boundary segment to its values at certain interior points of the region. For the first time the problem of this type was formulated and investigated by A.V. Bitsadze and A.A. Samarskii. The peculiarity of such problems is that they are non-self-conjugate, and therefore their spectrum may have a more complicated character than in the classical case. For a number of nonlocal boundary value problems, Sh.A. Alimov succeeded in proving the existence of a complete set of eigenfunctions and the basis of the corresponding system.

Since 2005, Sh.A. Alimov has obtained important results in the theory of boundary control of the heat exchange process. In particular, conditions were found to ensure obtaining a given temperature in a limited volume for a certain time and an estimate of the minimum time required for this depending on the power and location of heat or cold sources was given.

At present, Sh.A. Alimov is conducting research on mathematical problems of peridynamics related to the theory of hypersingular integrals. The methods of spectral theory developed by him earlier turned out to be an effective tool for studying properties of hypersingular integral equations of peridynamics, allowing to find conditions of solvability of these equations. Scientific merits of Sh.A. Alimov were widely recognized. In 1984 he was elected a corresponding member of the Academy of Sciences of Uzbekistan, in 1991 - academician of the International Academy of Higher School Sciences, since 2000 Sh.A. Alimov is academician of the Academy of Sciences of the Republic of Uzbekistan. In 1985, Sh.A. Alimov was awarded the title of laureate of the State Prize of Uzbekistan named after Biruni for his research in the field of mathematical physics. In 2019 he was awarded the Order "Mehnat Shukhrati" for achievements in science and education.

In 2023, Sh.A. Alimov was awarded the sign "Excellent Worker of Higher Education" for conscientious work in the field of higher education, selfless labor, giving an example of quality education to the younger generation, worthy contribution to the development of the system.

In 2024, for many years of conscientious and fruitful work in the sphere of higher education Sh.A. Alimov was awarded the title "Hero of Labor" of the I degree.

Sh.A. Alimov has more than 150 published scientific and a large number of educational and methodical works. Among his students are 10 doctors and more than 20 candidates of sciences working in universities of Uzbekistan, Russia, USA, Finland, Malaysia, universities of other countries.

Sh.A. Alimov traveled to universities in the USA, Japan, Germany, Hungary, Poland and other states to conduct scientific research and give lectures. As a head or a member of official delegations Sh.A. Alimov visited many cities and countries such as Washington (USA), London (Great Britain), Paris (France), Rome (Italy), Brussels (Belgium), Vienna (Austria), Prague (Czech Republic), Bratislava (Slovakia), Sofia (Bulgaria), Skopje (Macedonia), Tokyo (Japan), Cairo (Egypt), Delhi (India), Jakarta (Indonesia) and others.

Sh.A. Alimov meets his eightieth birthday in the prime of creative forces and we heartily congratulate him on his jubilee and wish him strong health, new successes in scientific and pedagogical activity, family welfare and long life.

Students and Editorial Board

Professor Ravshan Radjabovich Ashurov on his 70th birthday

Ravshan Radjabovich Ashurov is a prominent scientist-mathematician, a major specialist in the field of differential equations and harmonic analysis, as well as the author of scientific works recognized by scientists all over the world in the field of fractional order equations.

R. R. Ashurov was born on March 19, 1955 in Tashkent. In 1972 graduated from school No 90 of Tashkent with a gold medal and entered the Faculty of Applied Mathematics and Mechanics of Tashkent State University (Tashkent State University, now Mirzo Ulugbek National University of Uzbekistan). In 1976, R. Ashurov transferred to the Faculty of Computational Mathematics and Cybernetics of the Lomonosov Moscow State University (MSU CM&C), where he graduated in 1978 with a diploma with honors. In the same year he entered the postgraduate program of the Faculty of CM&C of MSU. Under the scientific supervision of Professor (now Academician) Sh. A. Alimov, he began research on the spectral theory of elliptic differential operators. In October 1981, at the Academic Council of the Faculty of CM&C of MSU under the chairmanship of Academician A. N. Tikhonov, he defended his Ph. 01.01.02 - differential equations and mathematical physics. In October 1981, he returned to Tashkent State University and began his career at the Department of Differential Equations of the Faculty of Mathematics as an assistant, senior lecturer and then as an associate professor.

In 1984, R. R. Ashurov was sent to Great Britain and underwent a one-year internship at the University of Birmingham under the supervision of the famous mathematician, Professor V. N. Everitt. During the internship period he prepared and published a number of papers on the theory of quasi-differential operators.

In 1989, he entered the doctoral program of Tashkent State University and was seconded to the Faculty of CM&C of Moscow State University. Professor Sh. A. Alimov was his scientific adviser in the doctoral program. In December 1992, he defended his doctoral thesis on specialty 01.01.02 - differential equations at the Academic Council of the Faculty of CM&C of MSU under the chairmanship of Academician A. N. Tikhonov.

Since December 1992, R. R. Ashurov continued his work at the Faculty of Mechanics and Mathematics of Tashkent State University as Associate Professor, Professor and then Head of the Department of Mathematical Physics. From 2001 to February 2003 he was Dean of this faculty.

From September to December 2002, he was an intern at Bowling Green State University in the United States under an IREX grant. The internship was devoted to studying the issue of work organization in US universities.

From February 2003 to October 2004, he served as Rector of the Tashkent Regional State Pedagogical Institute; from 2004 to 2006, he served as Rector of the Mirzo Ulugbek National University of Uzbekistan. From July 2006 to February 2009, he worked as a leading researcher at the Institute of Mathematics of the Academy of Sciences of Uzbekistan. At the same time, from 2007 to 2009, he worked as a professor at the Tashkent branch of the Lomonosov Moscow State University.

In February 2009, he was invited by University Putra Malaysia (Kuala Lumpur) to work as a Lead Researcher at the Institute of Advanced Technology of that university, where he worked until February 2012.

From 2012 to 2018, he continued his research activities as a leading researcher at the Institute of Mathematics at the National University of Uzbekistan (Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan).

In 2018, he was appointed to the position of the head of the laboratory "Differential Equations and their Applications" of the Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan, where he still holds the position of the head of this laboratory.

R.R. Ashurov visited China, Iran, Germany and Russia as Rector of the National University of Uzbekistan. He continues to regularly visit Russia, the United Kingdom, the United States, Japan, Bulgaria, Germany, Italy, Hong Kong, Malaysia, the United Arab Emirates, Brazil, Finland, Turkey, China, Belgium and other countries to conduct research and give lectures.

R.R. Ashurov has published more than 150 scientific articles, textbooks and teaching aids (including more than 100 articles published in prestigious foreign journals). One of them is the textbook "Mathematical Analysis", which was twice reprinted, published jointly with Academician of the Academy of Sciences of Uzbekistan Sh.A. Alimov. Moreover, R.R. Ashurov is the author of several monographs published in the Republic of Uzbekistan and leading foreign countries.

R.R. Ashurov is a member of the editorial board of the following scientific journals in mathematics: "Uzbek Mathematical Journal Bulletin of the Institute of Mathematics", "Vestnik KRAUNTS", "Fractional Differential Equations Zagreb, Croatia", "Computational Mathematics and Modeling".

Among R.R. Ashurov's students are 5 doctors of sciences, 8 candidates of sciences, who work in Uzbekistan, Malaysia, Canada and the Arab Republic of Egypt. Currently, 5 of his graduate students are on the verge of defending their PhD dissertation. R.R. Ashurov has also done significant work to attract young people to science at the National University of Uzbekistan, where he constantly organizes scientific seminars and discusses scientific papers with graduate and undergraduate students.

We sincerely congratulate Ravshan Rajabovich on his jubilee, wish him good health, great success in scientific and pedagogical activities, family happiness and long life.

Students and Editorial Board

Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.10-22
DOI: 10.29229/uzmj.2025-2-1

On the solvability of the Cauchy problem in Gevrey classes for the equation with Weyl fractional derivative

Alimov Sh.

Dedicated to the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. A new representation of fractional-order Weyl derivatives is given. The Cauchy problem is studied for partial differential equations containing Weyl derivatives. The conditions under which this problem has solutions from the Gevrey classes are found.

Keywords: Weyl derivatives of fractional order, Cauchy problem, Gevrey classes
MSC (2020): 35R11, 34A12

1. Introduction

1.1. **Fractional Weyl derivative.** Consider the Hilbert space H^0 of 2π-periodic functions orthogonal to unity:

$$H^0 \;=\; \{f \in L_2[-\pi,\pi] \;:\; \int_{-\pi}^{\pi} f(x)\,dx = 0\},$$

with the usual inner product

$$(f,g) \;=\; \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

Let the Fourier series of the function $f \in H^0$ have the form

$$f(x) \;=\; \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx). \tag{1.1}$$

For any $\alpha > 0$, the symbol H^α denotes the Sobolev class of functions $f \in H^0$ for which the norm

$$\|f\|_\alpha^2 \;=\; \sum_{k=1}^{\infty} k^{2\alpha}(a_k^2 + b_k^2)$$

is finite.

For any $\alpha \in \mathbb{R}$ and any function $f \in H^\beta$, where $\beta = \max\{\alpha, 0\}$, we define the Weyl fractional differentiation operator ∂^α by the equality

$$\partial^\alpha f(x) \;=\; \sum_{k=1}^{\infty} k^\alpha \left[a_k\cos\left(kx + \frac{\pi\alpha}{2}\right) + b_k\sin\left(kx + \frac{\pi\alpha}{2}\right)\right] \tag{1.2}$$

(see [19], and also [14], Chapter. 4, §19).

Note that for any $m \in \mathbb{N}$, the equality holds

$$\partial^m f(x) \;=\; f^{(m)}(x),$$

understood in the sense of $L_2[-\pi,\pi]$ metrics.

1.2. **The Cauchy problem.** Let $0 < \alpha \le 2$. Consider equation

$$u_{tt}(x,t) \ + \ \partial^\alpha u(x,t) \ = \ 0, \quad -\pi \le x \le \pi, \quad t > 0, \tag{1.3}$$

with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = 0. \tag{1.4}$$

For $0 < \alpha < 1$, equation (1.3) is related to the inverse problem for the subdiffusion equation, which, according to the authors of numerous works (see, for example, [9, 12, 10, 3]), more adequately describes various evolutionary processes, including the type of pandemic.

Note that for $\alpha = 2$, problem (1.3)-(1.4) is a Cauchy problem for the Laplace equation. Although according to the famous example of J. Hadamard (see [5], [7]), this problem is ill-posed, nevertheless it finds wide application in solving important applied problems (see [15]).

Let $T > 0$. For any $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we define the space of $C^m(H^\alpha, T)$ functions

$$u : H^\alpha \times (0,T) \to \mathbb{R},$$

such that $u(x,t)$ has derivatives with respect to t of the order of m inside the interval $(0,T)$, continuous according to the norm of the space H^α.

As a solution to the Cauchy problem (1.3)-(1.4) on the interval $(0,T)$, we call the function $u(x,t)$ from the class $C^2(H^\alpha,T)$ satisfying equation (1.3) and the initial conditions (1.4) in the following sense:

$$\lim_{t\to 0} \int_{-\pi}^{\pi} |u(x,t) - f(x)|^2 \, dx \ = \ 0, \tag{1.5}$$

and

$$\lim_{t\to 0} \int_{-\pi}^{\pi} \left| \frac{\partial u(x,t)}{\partial t} \right|^2 dx \ = \ 0. \tag{1.6}$$

The purpose of this paper is to find out the class whose membership of the initial function $f(x)$ ensures the existence of a solution to the Cauchy problem (1.3)-(1.4).

1.3. **Gevrey classes.**

Definition 1.1. Let $\sigma \ge 1$ and $p \ge 1$. We say that the function $f \in C^\infty[-\pi,\pi]$ belongs to the class Gevrey $G_p^\sigma[-\pi,\pi]$ if there exists a constant $M = M(f)$ such that for any $\alpha > 0$ the estimate

$$\|\partial^\alpha f\|_{L_p[-\pi,\pi]} \ \le \ M^\alpha [\Gamma(\alpha+1)]^\sigma \tag{1.7}$$

is valid.

On the right side of (1.7) is the Euler gamma function.

If we limit ourselves to the requirement of performing estimate (1.7) for all $\alpha \in \mathbb{N}$ only, then the corresponding class will coincide with the Gevrey class $J_p^\sigma[-\pi,\pi]$, which was considered in [16, 17, 18] (see also [8], Chapter VI, §4, and [6], Chapter 8, Sec. 8.4.) It is clear that $G_p^\sigma[-\pi,\pi] \subset J_p^\sigma[-\pi,\pi]$.

In this paper, we obtain the following result on the solvability in the Gevrey classes $G_2^\sigma[-\pi,\pi]$ of the Cauchy problem (1.3)-(1.4) on the half-line $t > 0$.

Theorem 1.2. *Let $\sigma\alpha < 2$. Then for any function $f \in G_2^\sigma[-\pi,\pi]$, a solution to the Cauchy problem* (1.3)-(1.4) *exists on the half-line $t > 0$ and is unique.*

The next result relates to the critical case when the Gevrey class exponent σ and the order α of the fractional differentiation operator are related by the equality $\sigma\alpha = 2$. In this case, it is possible to prove only local solvability.

Theorem 1.3. *Let $\sigma\alpha = 2$. Let the function f belong to the class Gevrey $G_2^\sigma[-\pi, \pi]$. Then the solution of the Cauchy problem* (1.3)-(1.4) *exists on the interval* $0 \le t < T(\alpha)$, *where*

$$T(\alpha) \; = \; \frac{2M^{-\alpha/2}}{\alpha \sin \frac{\pi\alpha}{4}}. \tag{1.8}$$

Note that in the case of $\alpha = 2$, when $T(\alpha) = 1/M$, a similar result for analytical functions was obtained in [2]. This paper also shows that the found interval for the existence of a solution cannot be increased.

2. Elementary properties of the Weyl derivative

We introduce a two-dimensional vector of the trigonometric system

$$T_k(x) \; = \; \begin{pmatrix} \cos kx \\ \sin kx \end{pmatrix}.$$

The vector of the Fourier coefficients of the function (1.1) is

$$f_k \; = \; \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) T_k(x)\, dx \; = \; \begin{pmatrix} a_k \\ b_k \end{pmatrix}.$$

The decomposition of the function $f \in H^0$ into a Fourier series can be written as follows:

$$f(x) \; = \; \sum_{k=1}^{\infty} f_k T_k(x).$$

The Parseval equality takes the form

$$\|f\|^2 \; = \; \pi \sum_{k=1}^{\infty} |f_k|^2. \tag{2.1}$$

In what follows, the main role is played by (2×2)-matrix of clockwise rotation by the angle $\pi\alpha/2$:

$$J^\alpha \; = \; \left\| \begin{matrix} \cos \frac{\pi\alpha}{2} & \sin \frac{\pi\alpha}{2} \\ -\sin \frac{\pi\alpha}{2} & \cos \frac{\pi\alpha}{2} \end{matrix} \right\|. \tag{2.2}$$

Let's introduce the "imaginary" component of this matrix:

$$J \; = \; \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\|.$$

Note that the matrix J is related to the Pauli matrix σ_y by the relation $J = i\sigma_y$ (see, for example, [11], Ch. 8, §55, formula (55.7)). The matrix (2.2) can be written as follows:

$$J^\alpha \; = \; \cos \frac{\pi\alpha}{2}\, I \; + \; \sin \frac{\pi\alpha}{2}\, J, \tag{2.3}$$

where I is the identity matrix. In particular, $J^1 = J$, and $J^2 = -I$.

Using the matrix J^α, the fractional Weyl derivative of the order of α can be written as follows:

$$\partial^\alpha f(x) \; = \; \sum_{k=1}^{\infty} k^\alpha (J^\alpha f_k) T_k(x). \tag{2.4}$$

The coincidence of the right-hand sides of formulas (2.4) and (1.2) is checked by direct calculation.

Indeed, by opening the parentheses in definition (1.2) and grouping the coefficients for $\cos kx$ and $\sin kx$, we get

$$\partial^\alpha f(x) = \sum_{k=1}^{\infty} k^\alpha \left[\left(a_k \cos\frac{\pi\alpha}{2} + b_k \sin\frac{\pi\alpha}{2}\right)\cos kx + \left(b_k \cos\frac{\pi\alpha}{2} - a_k \sin\frac{\pi\alpha}{2}\right)\sin kx\right].$$

Taking into account the definition of the matrix (2.2), from here we obtain the required equality (2.4). Thus, equality (2.4) can be considered as an equivalent definition of the Weyl derivative of fractional order α.

For an arbitrary number sequence $\{c_k\}$, we define the operator

$$Bf(x) \ = \ \sum_{k=1}^{\infty} c_k \cdot (J^\alpha f_k) T_k(x)$$

with a natural domain of definition

$$D(B) \ = \ \{f \in H^0 \ : \ \sum_{k=1}^{\infty} |c_k|^2 |J^\alpha f_k|^2 \ < \ +\infty\}.$$

Below we will need the following simple statement.

Proposition 2.1. *Equality*

$$\|Bf\|^2 \ = \ \pi \sum_{k=1}^{\infty} |c_k|^2 |f_k|^2$$

holds for any function $f \in D(B)$.

The validity of this equality follows directly from the fact that the matrix J^α is orthogonal.

It follows from proposition 1 and from (2.4) that for any function $f \in H^\alpha$ the equality

$$\|\partial^\alpha f\|^2 \ = \ \pi \sum_{k=1}^{\infty} k^{2\alpha} |f_k|^2 \tag{2.5}$$

holds.

3. Fourier coefficients of functions from the Gevrey classes

Let $0 < \rho < 1$. Consider the function

$$\Phi_\rho(t) \ = \ t^{1/\rho - 1} e^{\rho t^{1/\rho}}, \quad t > 0. \tag{3.1}$$

Lemma 3.1. *Let $\mu > 0$, $\beta > 0$, and $\sigma \geq 1$. If $\beta\sigma < 1$, then for any $\theta > (\beta/2)^{\beta\sigma}$ and for an arbitrary function $f \in G_2^\sigma[-\pi, \pi]$ the estimate*

$$\sum_{k=1}^{\infty} f_k^2 e^{\mu k^\beta} \ \leq \ C_\theta \Phi_{1-\beta\sigma}(\theta \mu M^\beta(f)) \tag{3.2}$$

is valid, where $M = M(f)$ is a constant included in the definition (1.7).

Proof. Decomposing the exponent into a power series and taking into account equality (2.5), we obtain

$$\sum_{k=1}^{\infty} f_k^2 e^{\mu k^\beta} = \sum_{k=1}^{\infty} f_k^2 \sum_{n=0}^{\infty} \frac{\mu^n k^{\beta n}}{n!} = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \sum_{k=1}^{\infty} f_k^2 \, k^{\beta n} =$$

$$= \sum_{n=0}^{\infty} \frac{\mu^n}{\Gamma(n+1)} \, \|\partial^{n\beta/2} f\|^2.$$

Then, according to (1.7), for $f \in G_2^\sigma[-\pi, \pi]$ we get

$$\sum_{k=1}^{\infty} f_k^2 e^{\mu k^\beta} \le \sum_{n=0}^{\infty} \frac{\mu^n}{\Gamma(n+1)} \, M^{\beta n} [\Gamma(n\beta/2+1)]^{2\sigma}. \tag{3.3}$$

Next, applying Stirling's formula (see [20], §12.33), for each term B_n of the series (3.3), we obtain

$$B_n = \frac{\mu^n M^{\beta n} [\Gamma(n\beta/2+1)]^{2\sigma}}{\Gamma(n+1)} \simeq \frac{\mu^n M^{n\beta}}{\sqrt{2\pi n}} \cdot \frac{e^n}{n^n} \cdot \left[\sqrt{\pi n \beta} \left(\frac{n\beta}{2e} \right)^{n\beta/2} \right]^{2\sigma}.$$

Here and everywhere below, the notation $B_n \simeq A_n$ means that equality

$$B_n = A_n \left(1 + \frac{O(1)}{n} \right)$$

holds.

Therefore,

$$B_n \simeq \frac{\mu^n e^n M^{n\beta}}{\sqrt{2\pi n}} \cdot (\pi n \beta)^\sigma \left(\frac{\beta}{2e} \right)^{n\beta\sigma} \cdot n^{(\beta\sigma - 1)n}.$$

Thus,

$$B_n \simeq C_1 \frac{n^{\sigma - 1/2}}{n^{(1-\beta\sigma)n}} \left[\mu e M^\beta \left(\frac{\beta}{2e} \right)^{\beta\sigma} \right]^n, \tag{3.4}$$

where

$$C_1 = \frac{(\pi\beta)^\sigma}{\sqrt{2\pi}}.$$

Note that for $\rho \in \mathbb{R}$

$$n^{\rho n} = \frac{e^{\rho n}}{(\sqrt{2\pi n})^\rho} \left(\sqrt{2\pi n} \cdot \frac{n^n}{e^n} \right)^\rho \simeq \frac{e^{\rho n}}{(\sqrt{2\pi n})^\rho} (n!)^\rho.$$

Hence,

$$n^{(1-\beta\sigma)n} = \simeq \frac{e^{(1-\beta\sigma)n}}{(\sqrt{2\pi n})^{1-\beta\sigma}} (n!)^{1-\beta\sigma} \,. \tag{3.5}$$

Substituting estimate (3.5) into (3.4), we obtain the following important asymptotic equality:

$$B_n \simeq C_1 \frac{(\sqrt{2\pi})^{1-\beta\sigma} \cdot n^{\sigma(1-\beta/2)}}{(n!)^{1-\beta\sigma}} \left[\mu M^\beta \left(\frac{\beta}{2} \right)^{\beta\sigma} \right]^n. \tag{3.6}$$

Note that for any $p > 1$ the inequality holds

$$n^{\sigma(1-\beta/2)} \cdot p^{-n} \le \text{const}, \quad n \in \mathbb{N},$$

where the constant does not depend on n.

Applying this inequality to the right-hand side of (3.6), we obtain the following estimate:

$$B_n \le \frac{C}{(n!)^{1-\beta\sigma}} \left[p\mu M^\beta \left(\frac{\beta}{2}\right)^{\beta\sigma} \right]^n .$$

Set $\theta = p\left(\frac{\beta}{2}\right)^{\beta\sigma}$. Then from (3.3) we get

$$\sum_{k=1}^{\infty} f_k^2 e^{\mu k^\beta} \le C \sum_{n=0}^{\infty} \frac{\left(\theta\mu M^\beta\right)^n}{(n!)^{1-\beta\sigma}}. \tag{3.7}$$

Consider for $0 < \rho \le 1$ the function

$$F_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\rho}, \quad z \in \mathbb{C}.$$

For this function, the following estimate is valid on a real half-line (see [1]):

$$F_\rho(z) \le \frac{12}{\rho^2} z^{1/\rho-1} \cdot e^{\rho z^{1/\rho}}, \quad z \ge 1. \tag{3.8}$$

Set $\rho = 1 - \beta\sigma$ and $z = \theta\mu M^\beta$. Substituting these values in (3.8), from (3.7) and (3.1) we obtain the required estimate (3.2). □

Corollary 3.2. *Under the conditions of Lemma 2.1, for any $s > 0$ and any $\delta > 0$, the estimate is valid*

$$\sum_{k=1}^{\infty} k^s f_k^2 e^{\mu k^\beta} \le C_\theta(s,\delta)\Phi_{1-\beta\sigma}(\theta(\mu+\delta)M^\beta(f)).$$

Proof. Indeed, for any $s > 0$ and any $\delta > 0$, the inequality holds

$$k^s e^{-\delta k^\beta} \le \text{const}, \quad k \in \mathbb{N}, \tag{3.9}$$

where the constant does not depend on k.

Therefore, according to (3.2),

$$\sum_{k=1}^{\infty} k^s f_k^2 e^{\mu k^\beta} = \sum_{k=1}^{\infty} k^s e^{-\delta k^\beta} f_k^2 e^{(\mu+\delta)k^\beta} \le C \sum_{k=1}^{\infty} f_k^2 e^{(\mu+\delta)k^\beta} \le$$
$$\le C\Phi_{1-\beta\sigma}(\theta(\mu+\delta)M^\beta(f)).$$

□

Set

$$\mu_\beta = \frac{2}{\beta M^\beta} . \tag{3.10}$$

Lemma 3.3. *Let $\mu > 0$, $\beta > 0$, and $\sigma \ge 1$. If $\beta\sigma = 1$ then for any function $f \in H^\sigma$ if condition*

$$\mu < \mu_\beta$$

is fulfilled, the following estimate

$$\sum_{k=1}^{\infty} f_k^2 e^{\mu k^\beta} \le C\left(1 - \frac{\mu}{\mu_\beta}\right)^{-\sigma-1/2} \tag{3.11}$$

is valid.

Proof. Note that inequality (3.3) is also valid for $\beta\sigma = 1$. In this case, we can use the relation (3.4), according to which

$$B_n \;\simeq\; C_1 n^{\sigma-1/2}\left[\frac{\beta\mu M^\beta}{2}\right]^n.$$

Denoting $q = \frac{\beta\mu M^\beta}{2}$, from (3.3) we obtain

$$\sum_{k=1}^{\infty} f_k^2 e^{\mu k^\beta} \;\le\; C\sum_{n=1}^{\infty} n^{\sigma-1/2} q^n \;\le\; \frac{C}{(1-q)^{\sigma+1/2}}. \tag{3.12}$$

Since

$$(1-q)^{\sigma+1/2} \;=\; \left(1-\frac{\beta\mu M^\beta}{2}\right)^{\sigma+1/2},$$

then from (3.10) and (3.12) follows the required estimate (3.11).

□

Corollary 3.4. *Let μ^* be an arbitrary number from the interval $0 < \mu^* < \mu_\beta$, where μ_β is defined by equality* (3.10)*. Then, under the conditions of Lemma 2, for any $s > 0$, the estimate*

$$\sum_{k=1}^{\infty} k^s f_k^2 e^{\mu k^\beta} \;\le\; const$$

is valid uniformly over the interval $0 \le \mu \le \mu^*$.

Proof. Indeed, set $\delta = (\mu_\beta - \mu^*)/2$. Then

$$\mu + \delta \;\le\; \mu^* + \delta \;=\; \mu_\beta - \delta.$$

In this case, using estimates (3.9) and (3.11), we obtain

$$\sum_{k=1}^{\infty} k^s f_k^2 e^{\mu k^\beta} \;\le\; C\sum_{k=1}^{\infty} f_k^2 e^{(\mu+\delta)k^\beta} \;\le\; C\sum_{k=1}^{\infty} f_k^2 e^{(\mu_\beta-\delta)k^\beta} \;\le$$

$$\le\; C\left(1-\frac{\mu_\beta-\delta}{\mu_\beta}\right)^{-\sigma-1/2} \;=\; \text{const}.$$

□

4. Solvability of the Cauchy problem

4.1. **Auxiliary identities.** For any vector $v \in R^2$, we put $|v| = \sqrt{v_1^2 + v_2^2}$ and introduce the norm of the $(2\times 2)-$matrix A by equality

$$\|A\| \;=\; \sup_{|v|=1} |Av|.$$

Let the function $F(z)$ be analytic in a disk of radius $R > 0$. For any matrix A with norm $\|A\| < R$, we define the matrix function

$$F(A) \;=\; \sum_{m=0}^{\infty} \frac{F^{(m)}(0)}{m!} A^m$$

(see. [4], volume. V, §4).

Note that for any two vectors $u \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$, the equality holds:

$$\frac{d}{dt}(F(tA)u, v) \;=\; (F'(tA)Au, v), \quad \|tA\| \le R.$$

Proposition 4.1. *If the function $F(z)$ is even and the function $G(z)$ is odd, then for any μ from the disk of convergence of these functions the following equalities hold:*

$$F(\mu J) = F(i\mu)I, \quad G(\mu J) = -i\,G(i\mu)J.$$

Proof. Since $J^2 = -I$, then

$$F(\mu J) \;=\; \sum_{k=0}^{\infty} \frac{F^{(2k)}(0)}{(2k)!}\mu^{2k}J^{2k} \;=\; \sum_{k=0}^{\infty} \frac{F^{(2k)}(0)}{(2k)!}\mu^{2k}(-I)^k \;=$$

$$= \sum_{k=0}^{\infty} \frac{F^{(2k)}(0)}{(2k)!}(i\mu)^{2k}I \;=\; F(i\mu)I.$$

Further,

$$G(\mu J) \;=\; \sum_{k=0}^{\infty} \frac{G^{(2k+1)}(0)}{(2k+1)!}\mu^{2k+1}J^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{G^{(2k+1)}(0)}{(2k+1)!}\mu^{2k+1}(-I)^kJ \;=$$

$$= \sum_{k=0}^{\infty} \frac{G^{(2k+1)}(0)}{(2k+1)!}(i\mu)^{2k}\mu J \;=\; -i\sum_{k=0}^{\infty} \frac{G^{(2k+1)}(0)}{(2k+1)!}(i\mu)^{2k+1}J \;=\; -iG(i\mu)J.$$

□

Corollary 4.2. *For any $a \in \mathbb{R}$ and $b \in \mathbb{R}$, the equalities hold*

$$\cos(aI + bJ) \;=\; (\cos a\cosh b)I \;-\; (\sin a\sinh b)J, \tag{4.1}$$

$$\sin(aI + bJ) \;=\; (\sin a\cosh b)I \;+\; (\cos a\sinh b)J. \tag{4.2}$$

Proof. Indeed, according to the above,

$$\cos(aI + bJ) \;=\; \cos aI\cos bJ \;-\; \sin aI\sin bJ \;=$$

$$= (\cos a\cos ib)I - (-i\sin a\sin ib)J = (\cos a\cosh b)I - (\sin a\sinh b)J.$$

The validity of equality (4.2) is checked in the same way.

$$\sin(aI + bJ) \;=\; \sin aI\cos bJ \;+\; \cos aI\sin bJ \;=$$

$$= (\sin a\cosh b)I + \cos a(-i\sin ib)J = (\sin a\cosh b)I + (\cos a\sinh b)J.$$

□

4.2. **Proof of Theorem 1.2.** The proof of Theorems 1.2 and 1.3 is based on the Fourier method, the application of which shows that if a solution to the Cauchy problem (1.3)-(1.4) exists, then its decomposition into a Fourier series should have the form

$$u(x,t) \;=\; \sum_{k=1}^{\infty} [\cos(tk^{\alpha/2}J^{\alpha/2})f_k]T_k(x). \tag{4.3}$$

In this section, we show that, if the conditions of Theorem 1.2 are fulfilled, function (4.3) is indeed a solution to the Cauchy problem (1.3)-(1.4) on a half-line $t > 0$.

According to (2.3), equality is valid

$$\cos(tk^{\alpha/2}J^{\alpha/2}) \;=\; \cos\left[tk^{\alpha/2}\left(\cos\frac{\pi\alpha}{4}I+\sin\frac{\pi\alpha}{4}J\right)\right].$$

In order to shorten the entries, we will use the following notation throughout to the end of this paragraph:

$$\beta=\frac{\alpha}{2}, \quad a \;=\; \cos\frac{\pi\alpha}{4}, \quad b \;=\; \sin\frac{\pi\alpha}{4}. \tag{4.4}$$

Then, according to (4.1),

$$\cos(tk^{\beta}J^{\beta})=\cos\left(tk^{\beta}a\right)\cosh\left(tk^{\beta}b\right)I-\sin\left(tk^{\beta}a\right)\sinh\left(tk^{\beta}b\right)J. \tag{4.5}$$

Set

$$u_1(x,t) \;=\sum_{k=1}^{\infty}\cos\left(tk^{\beta}a\right)\cosh\left(tk^{\beta}b\right)f_kT_k(x), \tag{4.6}$$

$$u_2(x,t) \;=\sum_{k=1}^{\infty}\sin\left(tk^{\beta}a\right)\sinh\left(tk^{\beta}b\right)(Jf_k)T_k(x). \tag{4.7}$$

Then the function (4.3) can be written as follows:

$$u(x,t) \;=\; u_1(x,t)-u_2(x,t). \tag{4.8}$$

Lemma 4.3. *Let $f\in G_2^{\sigma}[-\pi,\pi]$, where $\sigma<2/\alpha$. Then the function (4.6) for every $t>0$ belongs to $L_2[-\pi,\pi]$ and the equality holds*

$$\lim_{t\to 0}\int_{-\pi}^{\pi}|u_1(x,t)-f(x)|^2\,dx \;=\; 0. \tag{4.9}$$

Proof. According to (4.6),

$$f(x)-u_1(x,t) \;=\; \sum_{k=1}^{\infty}\left[1-\cos(k^{\beta}ta)\cosh(k^{\beta}tb)\right]f_kT_k(x)$$

Taking into account the Parseval equality (2.1), we get:

$$\int_{-\pi}^{\pi}|u_1(x,t)-f(x)|^2\,dx \;=\; \pi\sum_{k=1}^{\infty}\left[1-\cos(k^{\beta}ta)\cosh(k^{\beta}tb)\right]^2|f_k|^2. \tag{4.10}$$

We fix an arbitrary $T>0$. For any t from the interval $0\le t\le T$ and for all $k\in\mathbb{N}$, the estimate

$$\left[1-\cos(k^{\beta}ta)\cosh(k^{\beta}tb)\right]^2 \;\le\; 4\cosh^2(k^{\beta}tb) \;\le\; 4e^{2k^{\beta}Tb}$$

is valid.

According to Lemma 1, the number series

$$\sum_{k=1}^{\infty}e^{2k^{\beta}Tb}|f_k|^2$$

converges for any $T>0$. Therefore, according to Weierstrass's uniform convergence theorem (see [13], Chapter 7, Th. 7.10), the series on the right side of (4.10) converges uniformly on the interval $0\le t\le T$ and is a continuous function. Since at $t=0$ the sum of this series is zero, the required equality (4.9) holds.

□

Lemma 4.4. *Let $f \in G_2^\sigma[-\pi, \pi]$, where $\sigma < 2/\alpha$. Then the function* (4.7)*) for every $t > 0$ belongs to $L_2[-\pi, \pi]$ and the equality holds*

$$\lim_{t\to 0} \int_{-\pi}^{\pi} |u_2(x,t)|^2 \, dx \;=\; 0.$$

Proof. is quite similar to the proof of Lemma 2.3. To do this, we should apply Proposition 1 to decomposition (4.7)):

$$\int_{-\pi}^{\pi} |u_2(x,t)|^2 \, dx \;=\; \pi \sum_{k=1}^{\infty} \left[\sin(k^\beta ta)\sinh(k^\beta tb)\right]^2 |f_k|^2,$$

then use the estimate

$$\left[\sin(k^\beta ta)\sinh(k^\beta tb)\right]^2 \;\le\; \sinh^2(k^\beta tb) \;\le\; e^{2k^\beta Tb}.$$

$\square$

Corollary 4.5. *The function* (4.3) *satisfies the condition* (1.5)*.*

The validity of this statement follows from Lemmas 2.3 and 2.4 and equality (4.8).

Lemma 4.6. *Let $f \in G_2^\sigma[-\pi, \pi]$, where $\sigma < 2/\alpha$. Then the function* (4.3) *has a derivative with respect to t, which for every $t > 0$ belongs to $L_2[-\pi, \pi]$ and satisfies the condition* (1.6)*.*

Proof. First, we show that the formally differentiated series (4.3) converges in the norm $L_2[-\pi, \pi]$ uniformly over t on any interval $0 \le t \le T$.

We have,

$$u_t(x,t) \;=\; -\sum_{k=1}^{\infty} \; k^\beta [\sin(tk^\beta J^\beta) J^\beta f_k] T_k(x). \tag{4.11}$$

According to (2.3), (4.2) and (4.4),

$$\sin(tk^\beta J^\beta) \;=\; \sin\left[tk^\beta\,(aI + bJ)\right] \;=$$

$$=\; \sin(tk^\beta a)\cdot\cosh(tk^\beta b)I \;+\; \cos(tk^\beta a)\cdot\sinh(tk^\beta b)J.$$

Set

$$v_1(x,t) \;=\; -\sum_{k=1}^{\infty} k^\beta \sin(k^\beta ta)\cdot\cosh(k^\beta tb)(J^\beta f_k)T_k(x), \tag{4.12}$$

$$v_2(x,t) \;=\; -\sum_{k=1}^{\infty} k^\beta \cos(k^\beta ta)\cdot\sinh(k^\beta tb)(J^{\beta+1} f_k)T_k(x).$$

Then the derivative (4.11) of the function (4.3) can be written as follows:

$$u_t(x,t) \;=\; v_1(x,t) + v_2(x,t).$$

According to Parseval's equality (see Proposition 2.1),

$$\int_{-\pi}^{\pi} |v_1(x,t)|^2 \, dx \;=\; \pi \sum_{k=1}^{\infty} k^{2\beta} [\sin(k^\beta ta)\cdot\cosh(k^\beta tb)]^2 |f_k|^2. \tag{4.13}$$

Next, let's use the estimate

$$[\sin(k^\beta ta)\cdot\cosh(k^\beta tb)]^2 \ \leq\ e^{2tk^\beta}.$$

From this estimate and the corollary of Lemma 2.1, it follows that the convergence of the series (4.12) is uniform over t on any interval $0 \leq t \leq T$. Consequently, the integral (4.13) tends to zero at $t \to 0$.

Let's move on to estimate the integral

$$\int_{-\pi}^{\pi} |v_2(x,t)|^2\,dx \ =\ \pi\sum_{k=1}^{\infty} k^{2\beta}[\cos(k^\beta ta)\cdot\sinh(k^\beta tb)]^2|f_k|^2.$$

To do this, we note that the following inequality holds:

$$[\cos(k^\beta ta)\cdot\sinh(k^\beta tb)]^2 \ \leq\ \frac{1}{4}\,e^{2k^\beta tb},$$

the application of which completes the proof of the lemma.

□

Lemma 4.7. *Let $f \in G_2^\sigma[-\pi,\pi]$, where $\sigma < 2/\alpha$. Then the function* (4.3) *is the solution of the Cauchy problem* (1.3)-(1.4).

Proof. 1) Formally differentiating the series (4.3) twice by t, we obtain

$$u_{tt}(x,t) \ =\ -\sum_{k=1}^{\infty} k^{2\beta}\left[\cos(tk^\beta J^\beta)J^{2\beta}f_k\right]T_k(x). \tag{4.14}$$

Let's apply the operator ∂^α to the series (4.3), considering that $\alpha = 2\beta$. As a result, we get

$$\partial^{2\beta}u(x,t) \ =\ \sum_{k=1}^{\infty} k^{2\beta}J^{2\beta}\left[\cos(tk^\beta J^\beta)f_k\right]T_k(x). \tag{4.15}$$

According to the Corollary 3.2 of Lemma 2.1, both series (4.14) and (4.15) converge in the metric $L_2[-\pi,\pi]$ uniformly over t on any interval of the positive half-line. In this case, it follows from the equality of these two series that function (4.3) satisfies equation (1.3).

2) According to the Corollary of Lemma 2.4, function (4.3) satisfies condition (1.5).

3) According to Lemma 2.5, function (4.3) satisfies condition (1.6).

Thus, it is proved that function (4.3) is a solution to the Cauchy problem (1.3)-(1.4).

□

Lemma 2.6 completes the proof of Theorem 1.2.

It follows from the Corollary of Lemma 2.3 that, if the conditions of Theorem 1.2 are fulfilled, the solution (4.3) of the Cauchy problem is an infinitely differentiable function with respect to x and t in the domain

$$[-\pi,\pi]\times\overline{\mathbb{R}_+} \ =\ \{(x,t)\in\mathbb{R}^2 \ :\ -\pi\leq x\leq\pi,\ t\geq 0\}.$$

4.3. **Proof of the Theorem 1.3.** Assume that $\sigma\alpha = 2$. We show that in this case, function (4.3) is a solution to the Cauchy problem (1.3)-(1.4) in the interval $0 \le t < T(\alpha)$, where $T(\alpha)$ is determined by equality (1.8). To do this, it suffices to prove that series (4.3) and all the series obtained from it by formal differentiation converge in $L_2[-\pi, \pi]$ uniformly over t in any interval $0 \le t \le T$, where $T < T(\alpha)$.

The proof follows the same pattern as the proof of Lemmas 2.3-2.5. Let us show, for example, how the convergence of the series (4.15) can be estimated.

Using equality (4.5), we represent this series as

$$u_{tt}(x,t) \ = \ w_1(x,t) \ - \ w_2(x,t),$$

where

$$w_1(x,t) \ = \sum_{k=1}^{\infty} k^{2\beta} \cos\left(tk^{\beta}a\right) \cosh\left(tk^{\beta}b\right) (J^{2\beta} f_k) T_k(x),$$

$$w_2(x,t) \ = \sum_{k=1}^{\infty} k^{2\beta} \sin\left(tk^{\beta}a\right) \sinh\left(tk^{\beta}b\right) (J^{2\beta+1} f_k) T_k(x).$$

Next, we apply Proposition 2.1, according to which the following equalities are fulfilled

$$\|w_1(\cdot,t)\|^2 \ = \pi \sum_{k=1}^{\infty} k^{4\beta} \left[\cos\left(tk^{\beta}a\right) \cosh\left(tk^{\beta}b\right)\right]^2 |f_k|^2, \tag{4.16}$$

$$\|w_2(\cdot,t)\|^2 \ = \pi \sum_{k=1}^{\infty} k^{4\beta} \left[\sin\left(tk^{\beta}a\right) \sinh\left(tk^{\beta}b\right)\right]^2 |f_k|^2. \tag{4.17}$$

To prove the uniform convergence of the series (4.16), we use the estimate

$$\left[\cos\left(tk^{\beta}a\right) \cosh\left(tk^{\beta}b\right)\right]^2 \ \le \ e^{2btk^{\beta}}.$$

It follows from this estimate that for $0 \le t \le T$, the series (4.16) is majorized by a number series

$$\sum_{k=1}^{\infty} k^{4\beta} e^{2bTk^{\beta}} |f_k|^2. \tag{4.18}$$

As a consequence of Lemma 2.2, the series (4.18) converges at $2bT < \frac{2}{\beta^{\beta}}$, i.e. at $T < T(\alpha)$. In this case, due to the Weierstrass theorem, the series (4.16) converges uniformly along t from the interval $0 \le t \le T$.

Similarly, the uniform convergence of the series (4.17) is proved, and the following estimate should be used:

$$\left[\sin\left(tk^{\beta}a\right) \sinh\left(tk^{\beta}b\right)\right]^2 \ \le \ \frac{1}{4}\, e^{2btk^{\beta}}.$$

From the uniform convergence on the interval $0 \le t \le T$ of the series (4.3) and all the series obtained from this series by direct differentiation, it follows that the function (4.3) satisfies equation (1.3), as well as the initial conditions (1.4).

Thus, Theorem 1.3 is proved.

References

[1] Alimov S. A., Ilhan O. A.; An inequality associated with some entire functions. Journal of Inequalities in Pure and Applied Mathematics,–2004.–5.–No3.

[2] Alimov S. A., Qudaybergenov A. K.; Determination of temperature at the outer boundary of a body. Journal of Mathematical Sciences,–2023.–274.– No2.–P.159–171.

[3] Ashurov R.R., Shakarova M., Umarov S.; An inverse problem for the subdiffusion equation with a non-local in time condition. Fractal Fract.,–2024.– 8.–378.

[4] Gantmacher F. R.;The Theory Of Matrices. AMS Chelsea Publishing, New York,– 1960.–P.576.

[5] Hadamard J.; Lectures on Cauchy's problem in linear partial differential equations.Yale University Press, New Haven,–1923.

[6] Hormander L. V.; The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis,Springer-Verlag,–1983.

[7] Kabanikhin S. I.; Inverse and Ill-posed Problems: Theory and Applications. Inverse Ill-posed Probl. Ser.,Walter de Gruyter,Berlin/Boston,–2012.–55.

[8] Kakhan Zh. P.; Absolyutno skhodyaschiesya ryady Fure. Mir,– 1976.

[9] Kilbas A.A., Srivastava H.M., Trujillo J.J.; Theory and applications of fractional differential equations.Elsevier,–2006.

[10] Kirane M., Samet B., Torebek B.T.; Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data.Journal of Differential Equations,–2017.– 217.–P.1–13.

[11] Landau L.D., Lifshitz E.M.; Course of Theoretical Physics III. Fizmatlit,Nauka,Moskva,– 2004.

[12] Lizama C.; Abstract linear fractional evolution equations. Handbook of fractional calculus with applications,– 2019.

[13] Rudin W.; Principles of Mathematical Analysis.McGraw-Hill,–1964.

[14] Samko St.G., Kilbas A.A., Marichev O.I.; Fractional integrals and derivatives: theory and applications. Gordon and Breach, New York,–1993.

[15] Tikhonov A. N., Goncharsky A. V., Stepanov V. V., Yagola A. G.; Numerical Methods for the Solution of Ill-Posed Problems. Kluwer Academic Publishers,–1995.

[16] Ul'yanov P. L.; Classes of infinitely differentiable functions. Dokl. Akad. Nauk SSSR ,–1989.–305.–No2.–P.287–290.

[17] Ul'yanov P. L.; Properties of functions in Gevrey classes. Dokl. Akad. Nauk SSSR,–1990.–314.–No4.–P.793–797.

[18] Ul'yanov P. L.; On classes of infinitely differentiable functions. Mat. Sb.,–1990.– 181.–No5.–P.589–609.

[19] Weyl H.; Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung. Vierteljahrcsschrift der Naturforschenden Gesellschaft in Zurich,– 1917.–62.–No12.–P.296–302.

[20] Whittaker E. T., Watson G. N.; A Course in Modern Analysis.Cambridge University Press, New York,–1966.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.23-28
DOI: 10.29229/uzmj.2025-2-2

Applying of the surface theory of non-Euclidean spaces to the solution of the Monge-Ampere equation of elliptic type

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. Solutions of many geometry problems in the whole are related to solutions of differential equations. A.D. Alexandrov's problem of recovering a surface from a given extrinsic curvature function leads to the solution of the Dirichlet problem for the Monge-Ampere equation in a convex domain. The paper presents a method for generalizing the problem of recovering a convex surface from extrinsic curvature in non-Euclidean spaces. For this purpose, a cylindrical mapping is defined, which is a generalization of the spherical mapping, and a method is given for determining the extrinsic curvature of a convex surface in non-Euclidean spaces. Formulas for the extrinsic curvature of convex surfaces in some specific non-Euclidean spaces are calculated. It has been proven that in non-Euclidean spaces it is possible to generalize the problem of A.D. Alexandrov, which makes it possible to prove the existence of a solution to the Monge-Ampere equation in non-convex and non-simply connected domains with different boundary conditions. The problem of A.D. Alexandrov will lead to the solution of the Monge-Ampere equation, which is a special case of solutions to the equation proved by I.Ya. Bakelman.

Keywords: Convex polyhedron, convex surface, extrinsic curvature, non-Euclidean space, spherical mapping, additive set function, hyperbolic space.
MSC (2020): 53A35, 35J96, 35J15.

1. Introduction

The study of intrinsic geometry was begun at the beginning of the 19th century by the great mathematician of this century, Gauss. He defined the concept of the extrinsic curvature of a convex surface as the area of its spherical mapping. One of Gauss's remarkable theorems is the equality of the intrinsic and extrinsic curvature of regular surfaces. In the 20th century, A.D. Alexandrov generalized the concept of spherical mapping for convex polyhedron and introduced the concept of extrinsic curvature of convex polyhedron as the area of its spherical mapping [1]. A.D. Alexandrov was the first to pose and solve the problem of the existence of a convex polyhedron with given values of extrinsic curvature at the vertices. Let us formulate this problem:

Let a convex polygon Γ bounding domain D be given on the plane. Points $A_1, A_2, ..., A_n$ are marked inside the domain D. A closed polyline L is given in space, which uniquely projects onto the polygon Γ. Consider the class W of convex polyhedra with a boundary L that uniquely projects into the domain D and vertices $\overline{A}_i$ that project onto points A_i and have no other vertices.

The problem, that is, A.D. Alexandrov's problem on the existence of a convex polyhedron with given values of extrinsic curvature, is as follows:
If numbers $\omega_1, \omega_2, ..., \omega_n$ are given, does there exist a convex polyhedron $F \in W$ with extrinsic curvature at the vertices of this polyhedron equal to the given numbers.

This problem was solved by A.D. Alexandrov himself and a positive answer was given [2]. A.V. Pogorelov also solved this problem in Euclidean space with another method, which was called Pogorelov's extreme method [13]. In Lobachevsky space this problem was solved by A.L. Verner [16]. The work of A. Artykbaev was devoted to solving this problem in other three-dimensional non-Euclidean spaces [3]. There are 27 three-dimensional spaces, according to the Cayley-Klein theory, but this problem is mainly solved in eight of these spaces. Also in [4] a general method is given for constructing an analogue of a spherical mapping in all spaces with projective metrics, that is, a method for determining the extrinsic curvature of convex surfaces.In addition, the possibility of formulating and solving the generalized problem of A.D. Alexandrov for non-convex and non-simply connected domains is indicated.

2. Preliminaries

For regular surfaces, A.D. Alexandrov's problem is formulated as follows [8, 9]:
Let a convex domain D with boundary ∂D be given on the plane. A closed curve L is given in space and uniquely projects onto the boundary ∂D of the domain D. Consider the Borel set $M \subset D$ and $\mu(M)$ is a positively completely additive function defined on the Borel sets $M \subset D$.

Problem: Under what conditions to the function $\mu(M)$, there is a convex surface, the extrinsic curvature of the domain M' projected onto M is equal to the value of $\mu(M)$, that is, $\omega_F(M') = \mu(M)$.

2.1. **How is A.D. Alexandrov's problem associated with the Monge-Ampere equation.** Let F be a regular convex surface with boundary L, which is uniquely projected into the domain D. The concept of extrinsic curvature of a convex surface transferred to a plane is introduced.If the surface is given by the equation:

$$z = z(x, y),\ (x, y) \in D \tag{2.1}$$

then each tangent plane given by the equation:

$$z - z_0 = z_x(x_0, y_0)(x - x_0) + z_y(x_0, y_0)(y - y_0) \tag{2.2}$$

is associated with a point on the unit sphere S_2 with coordinates:

$$\{\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}, \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}, -\frac{1}{\sqrt{1 + z_x^2 + z_y^2}}\} \tag{2.3}$$

This point on the sphere S_2 is called the spherical image of the point $(x_0, y_0) \in D$ relative to the surface F.If we consider the set $M \subset D$, then the spherical image of the points $(x, y) \in M$ forms a certain set $M^* \subset S_2$. The area of the set $M^* \subset S_2$ is called the extrinsic curvature of the set M relative to the surface F and is denoted by $\omega_F(M) = S(M^*)$ [6].

If the surface F is given by equation (2.1), then the area of its spherical image is calculated by the following formula:

$$S(M^*) = \iint\limits_M \frac{z_{xx} z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^{\frac{3}{2}}} dxdy \tag{2.4}$$

But the extrinsic curvature of the surface F transferred to the plane is defined as a function of the set [8, 9]:

$$\omega_F(M) = \iint\limits_M \varphi(x, y) dxdy \tag{2.5}$$

Equating expressions (2.4) and (2.5) we obtain:

$$z_{xx} z_{yy} - z_{xy}^2 = (1 + z_x^2 + z_y^2)^{\frac{3}{2}} \varphi(x, y) \tag{2.6}$$

(2.6) is called the Monge-Ampere equation. If $\varphi(x, y) > 0$, then (2.6) is elliptic [14].Equality (2.4), (2.5) and (2.6) show that finding a convex surface F with a given extrinsic curvature is equivalent to solve the Dirichlet problem for the elliptic Monge-Ampere equation [5, 12].

2.2. **Extrinsic curvature of convex surfaces of non-Euclidean space.** In [3], A. Artykbaev generalized the concept of a spherical mapping of convex surfaces for all non-Euclidean spaces with projective metrics and called it a cylindrical mapping. The area of the cylindrical mapping is analogous to Euclidean space and is called the extrinsic curvature of the convex surface of this non-Euclidean space. In a special case, the matching principle constructed in the work [3] for Galilean space can be given in the following diagram (Fig.1): A scheme of map-

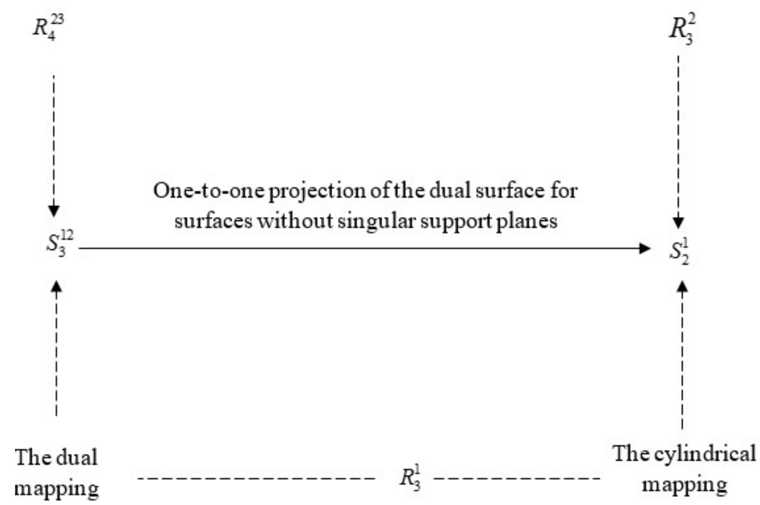

FIGURE 1.

pings leading to the concept of a cylindrical mapping of a convex surface Φ in a Galilean space R_3^1. The dotted lines denote inclusion maps of the spheres S_3^{12} (co-Galilean space) and S_2^1(the co-Euclidean plane) respectively into R_4^{23} (semi-Euclidean space) and R_3^2 (isotropic space). The arrows denote maps of Φ . But it should be noted that this scheme for a specific non-Euclidean space has an individual character. Therefore, in each case it is necessary to construct a corresponding mapping. In this regard, for specific non-Euclidean spaces, extrinsic curvature has been defined by various authors. But the principle of calculation differs little from each other. The extrinsic curvature of convex surfaces transferred to a plane in Galilean space is given in [3]. The form of the formula for extrinsic curvature in Galilean space significantly depends on the geometry of the plane onto which the convex surface is projected. If a surface Φ of a Galilean space $R_3^1\{x, y, z\}$ is projected onto the plane Oxy, the extrinsic curvature of the surface $\Phi : z = z(x, y)$ transferred to the plane is calculated by the formula:

$$\omega_F(M) = \iint\limits_M \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_y^2)^{\frac{3}{2}}} dxdy \tag{2.7}$$

If a surface $\Psi : x = x(y, z)$ is projected onto a singular plane Oyz, the extrinsic curvature is calculated by the formula:

$$\omega_F(M) = \iint\limits_M \frac{x_{yy}x_{zz} - x_{yz}^2}{(x_y^2 + x_z^2)^{\frac{3}{2}}} dydz \tag{2.8}$$

In the work [11], for an isotropic space, when the surface is given by the equation (2.1), the extrinsic curvature of the surface is calculated by the formula:

$$\omega_F(M) = \iint\limits_M \frac{z_{xx}z_{yy} - z_{xy}^2}{(z_x^2 + z_y^2)^{\frac{3}{2}}} dxdy \tag{2.9}$$

It is easy to notice that formulas (2.8) - (2.9) do not differ fundamentally.
It was also proven in [11] that the problem of the existence of a surface by extrinsic curvature in an isotropic space is equivalent to the problem of recovering a surface by the total curvature. We present a method for calculating extrinsic curvature in the example of a semi-hyperbolic space ${}^{10}S_3^2$, which is implemented on the sphere of a semi-pseudo-Euclidean space ${}^{10}R_4^3$. If $Ox_1x_2x_3$ is the coordinate system of the space ${}^{10}R_4^3$, then the sphere of this space is [10]:

$${}^{10}S_3^2 = \{(x_0, x_1, x_2, x_3) \in {}^{10}R_4^3 : -x_0^2 + x_1^2 + x_2^2 = -1\} \tag{2.10}$$

If we use an analogue of A.V. Pogorelov's mapping,

$$TX = \frac{X + e_0(e_0, X)}{|(e_0, X)|} \tag{2.11}$$

then the space ${}^{10}S_3^2$ is interpreted inside the sphere of the isotropic space R_3^2 [7]. The sphere of isotropic space $R_3^2\{x, y, z\}$ has the equation $x^2 + y^2 = 1$, that is, it is affinely a cylinder whose guide is parallel to the Oz axis. Then the points of space are:

$${}^{10}S_3^2\{(x, y, z) \in R_3^2 : x^2 + y^2 \leq 1\} \tag{2.12}$$

Moreover, on the plane $z = 0$, inside the circle $x^2+y^2 = 1$, the Lobachevsky plane is interpreted, which is the plane of space ${}^{10}S_3^2$. If the surface $F \subset {}^{10}S_3^2$ is given by the equation $z = z(x, y),\ x^2 + y^2 \leq 1$, then the extrinsic curvature of the surface transferred to the Oxy plane is calculated by the formula:

$$\omega_F(M) = \iint\limits_M \frac{z_{xx}z_{yy} - z_{xy}^2}{(z_x^2 + z_y^2)^{\frac{3}{2}}} (1 - z_x^2 - z_y^2)^{\frac{1}{2}} dxdy \tag{2.13}$$

3. Main part

The study of works associated with the extrinsic curvature of a surface, where the extrinsic curvature of a surface transferred to a plane was calculated, showed the following pattern. From this, for non-Euclidean space, the formula for the extrinsic curvature of a convex surface transferred to a plane includes the Monge-Ampere operator and some function depending on through the derivatives z_x and z_y [3, 9, 11]. Based on this reasoning, we formulate the following theorem:

Theorem 3.1. *The solution to A.D. Alexandrov's problem using the surface theory of non-Euclidean spaces is a particular solution to the Dirichlet problem of the Monge-Ampere equation for*

$$z_{xx}z_{yy} - z_{xy}^2 = R(z_x, z_y)\varphi(x, y) \tag{3.1}$$

elliptic type, where, $\varphi(x, y) > 0$.

Proof. We consider formulas for calculating extrinsic curvature in three-dimensional non-Euclidean spaces. In this case, we present the formula expression $R(p,q) = R(z_x, z_y)$ in the spaces under consideration. In particular for Galilean and isotropic space:

$$R(z_x, z_y) = (1+z_y^2)^{\frac{3}{2}}, \quad R(z_x, z_y) = (z_x^2 + z_y^2)^{\frac{3}{2}}$$

For Minkowski space:

$$R(z_x, z_y) = (1 - z_x^2 - z_y^2)^{\frac{3}{2}}$$

For Lobachevsky space:

$$R(z_x, z_y) = \frac{(1-z_x^2-z_y^2)^{\frac{3}{2}}}{(1-x^2-y^2)^{\frac{1}{2}}}$$

For semihyperbolic space ${}^{10}S_3^2$:

$$R(z_x, z_y) = \frac{(z_x^2+z_y^2)^{\frac{3}{2}}}{(1-z_x^2-z_y^2)^{\frac{1}{2}}}$$

It should be noted that in elliptic and semi-elliptic spaces the extrinsic curvature does not have the property of monotonicity; for this reason, A.D. Alexandrov's problem has no solution [15]. □

3.1. **Applying of the surface theory of non-Euclidean spaces to the solution of A.D. Alexandrov's problem.** Solving the problem of recovering convex surfaces from a given extrinsic curvature in non-Euclidean spaces, from the point of view of solving the Monge-Ampere equation of elliptic type, did not lead to new results. Since from the point of view of equations it is a special case of the known results [8, 9]. Let us recall that A.D. Alexandrov's problem is considered only on a convex domain of the plane or in the whole plane. When considering a problem in the whole plane, a limit cone is specified, to which the surfaces under consideration tend. I.Ya. Bakelman, using the solution to the problem of recovering a convex surface from a given extrinsic curvature, proved the existence and uniqueness of the Dirichlet problem only for convex domain on the plane.

But thanks to the singularities of the geometry of non-Euclidean spaces, it is possible to generalize A.D. Alexandrov's problem to non-Euclidean spaces. Firstly, it is possible to generalize A.D. Alexandrov's problem for non-convex and non-simply connected domains. This possibility was proven in [3], when A.D. Alexandrov's problem was solved in Galilean space. In addition, changing the geometry of the domain of definition of the problem will lead to a change in the boundary condition required for the Dirichlet problem.For example, in Galilean space this problem was posed and solved in the following formulation:

Let a closed convex curve L_2 and a convex curve L_1 lying strictly inside L_2 be given on the plane. We denote by K the domain enclosed between L_1 and L_2. Next, inside the domain K, a function of Borel sets $\mu(M)$ $M \subset K$ is given and L a spatial closed curve uniquely projected on the curve L_2 is given in the half-space $z > 0$, by the equation $f(t),\ t \in L_2$.

Theorem 3.2. *If $\mu(M)-$ is a non-negative completely additive function of the set $M \subset K$ is bounded for all $M \subset K$ for which $\overline{M} \subset K$, then there exists a solution to the boundary value problem with boundary conditions:*

$$z|_{t\in L_1} = 0, \quad z|_{t \in L_2} = f(t)$$

in the ring-shaped domain K.

The proof of the theorem is based on methods developed by A.D. Alexandrov. First, the problem is formulated so that it makes sense in the class of polyhedra. After this, in a generalized sense, it is solved in the class of convex polyhedra.The solution to the problem for convex surfaces is obtained by passing to the limit from polyhedra [1, 2]. The entire procedure for proving the theorem is based on the method of I.Ya. Bakelman developed for Euclidean space [9]. In addition to the fact that the convexity condition is removed from the domain of the Monge-Ampere solution under consideration. There are also no requirements for the function $R(z_x, z_y)$ to be a summable function. In many cases, solving this problem in non-Euclidean spaces does not necessarily fulfill this condition.
Finally, it can be stated that the applying of the theory of surfaces of non-Euclidean spaces to the solution of the problem of the existence of a convex surface with a given extrinsic curvature will lead to new solutions of the Monge-Ampere equation with different boundary conditions not only for a convex but also for a non-convex, non-simply connected domains.

References

[1] Alexandrov A.D.; Convex polyhedra. Springer Monographs in Mathematics.–2005.

[2] Alexandrov A.D.; Intrinsic geometry of convex surfaces. Classics of Soviet Mathematics. –2006.

[3] Artykbayev A.; Recovering convex surfaces from the extrinsic curvature in Galilean space.Math. USSR Sb.–1984.–V.47.–No.1. –pp. 195–214.

[4] Artykbayev A., Sokolov D.D.; Geometry as a whole in space-time. Tashkent, Fan (in Russian). –1991.

[5] Artykbayev A., Ibodullayeva N.M.; The problem of recovering a surface from extrinsic curvature and solving the Monge-Ampere equation. Itogi nauki i texniki Ser. Sovrem. Math i ee pri. –2021. –201. –pp. 123–131.

[6] Artykbayev A., Ibodullayeva N.M.; Generalized extrinsic curvature of a surface. AIP. Conf. Proc. –2024.–V.3004.–No.030007. –pp. 1–5.

[7] Artykbaev A., Saitova S.S.; Interpretation of geometry on manifolds as a geometry in a space with projective metric. J. Math. Sci.–2022.–V.265–No.1.–pp.1–10.

[8] Bakelman I.Ya., Verner A.L., Kantor B.E.; Introduction to the differential geometry as a whole.Moscow, Nauka (in Russian). –1973.

[9] Bakelman I.Ya.; Convex analysis and nonlinear geometric elliptic equations. Springer,Berlin-Heidelberg.–2011.

[10] Chen B.Y.; Pseudo-riemannian geometry, $\delta-$invariants and applications. World Scientific Book.–2011.

[11] Ismoilov Sh.Sh.; Geometry of the Monge-Ampere equation in an isotropic space. Uzbek Math.J.–2022.–V.65.–No.2.–pp.66–77.

[12] Nam Q.L.; Analysis of Monge-Ampere equations. American Math. Soc.–2024.

[13] Pogorelov A.V.; Extrinsic geometry of convex surfaces. Moscow, Science.–1973.

[14] Polyanin A.D.; Lectures on nonlinear equations of mathematical physics. Moskva Ipmex. RAN.–2023.

[15] Vereshagen B.M.; Recovering of a closed convex surface from a given Gaussian curvature function. Voprosi Glob. Geom (in Russian).–1979.–pp.7–12.

[16] Verner A.L.; Recovering of a total convex surface in Lobachevsky space by the extrinsic curvature. Uch. zap. Len. ped. in-ta. im. A.I.Gersena.–1960.–V.218.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.29-37
DOI: 10.29229/uzmj.2025-2-3

An investigation of non-local and nonlinear boundary value problems for the fractional subdiffusion equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov

Abstract. The paper focuses on the analysis of boundary value problems that are both non-local and nonlinear for the fractional subdiffusion equation within the domain $[0,1] \times (0,T]$. The equation is defined using the Caputo fractional derivative of order $0 < \alpha < 1$ with respect to the time variable. An a priori estimate is established for the solution, ensuring its stability. The existence of the solution is analyzed through an auxiliary problem and the Greens function method. The solution to the integral equation is proven in the space $C[0,1]$ using the method of successive approximations. The main result demonstrates that, under certain conditions, the problem has a unique solution.

Keywords: Fractional subdiffusion equation, Caputo fractional derivative, non-local boundary condition, a priori estimate, Greens function, successive approximations, existence of solution.

MSC (2020): 35R11;35B45.

1. Introduction

Consider the fractional subdiffusion equation in the domain $(0,1) \times (0,T]$,

$$\partial_{0t}^{\alpha} u(x,t) - u_{xx}(x,t) = f(x,t), \tag{1.1}$$

with the non-local condition

$$u(x,0) = g(x, u(x,T)), \quad x \in [0,1], \tag{1.2}$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T], \tag{1.3}$$

where ∂_{0t}^{α} is the Caputo fractional derivative of order $0 < \alpha < 1$ in the time variable, $f(x,t)$, $g(x,w)$ are given functions.

Definition 1.1. A function $u(t) \in C([0,1] \times [0,T])$ with the properties $\partial_{0t}^{\alpha} u(x,t), u_{xx}(x,t) \in C((0,1) \times (0,T])$ and satisfying conditions (1.1)–(1.3) is called a regular solution of the non-local problem (1.1)–(1.3).

We first state the main results related to the problem (1.1)–(1.3), with detailed proofs provided in the subsequent sections.

Theorem 1.2. *Let $f(x,t) \in C([0,1] \times [0,T])$, $g(x,w)$ be continuous functions of their arguments $(x,w) \in [0,1] \times \mathbb{R}$, satisfying the following Lipschitz condition*

$$|g(x,w)| \le L|w|, \quad 0 < L < \frac{1}{\sqrt{E_\alpha(T^\alpha)}}. \tag{1.4}$$

Then the solution of (1.1)–(1.3) satisfies the following a priori estimate

$$\|u(\cdot,t)\|^2 \le \left[\frac{L^2 E_{\alpha,\alpha}(T^\alpha)}{1 - L^2 E_\alpha(T^\alpha)} E_\alpha(t^\alpha) + E_{\alpha,\alpha}(t^\alpha)\right] \frac{T}{\alpha} \max_{[0,T]} \|f(\cdot,t)\|,$$

where $\|u(\cdot,t)\|^2 = \int_0^1 u^2(x,t)dx$ and $E_\alpha(\cdot), E_{\alpha,\tau}(\cdot)$ are the Mittag–Leffler functions.

Theorem 1.3. *Let the following conditions hold:*

1. $f(x,t) \in C([0,1] \times [0,T])$;
2. $L < \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)T^{1-\frac{\alpha}{2}}}$;
3. *condition* (1.4) *holds for the function* $g(x,w)$.

Then the problem (1.1)–(1.3) *has a unique solution.*

Many scientists have studied various non-local conditional problems for subdiffusion equations.

If $g = \psi(x)$, then such initial-boundary value problems have been studied in papers such as [2] and [9].

It should be noted that various problems have been considered even when the function g is linear, as can be seen in [3], [4], and [10].

The subdiffusion equation with the condition

$$u(\xi) = \delta u(0) + \varphi, \quad 0 < \xi \leq T, \quad \delta = \text{const}$$

(instead of condition (1.2)) is studied in detail in [3]. In that work, the values of the parameter that ensure the existence and uniqueness of the solution are determined. In other cases, the authors establish certain orthogonality conditions for $f(t)$ and φ that guarantee the existence of a solution; however, uniqueness is not ensured in those cases.

In [4] and [10], the authors employed linear non-local conditions depending on three parameters to solve equation (1.1).

The work [4] involves a pointwise non-local condition of the form:

$$\alpha u(0) + \beta u(T) = \gamma u(\xi) + \varphi,$$

while the paper [10] includes an integral non-local condition:

$$\alpha u(0) + \beta u(T) + \gamma \int_0^T u(\eta)\, d\eta = \varphi.$$

Both problems are formulated in terms of the Caputo fractional derivative of order $0 < \alpha < 1$, and the elliptic part is represented by a self-adjoint positive operator in a separable Hilbert space. The authors establish existence and uniqueness theorems for the solutions of both problems. Furthermore, they identify sufficient conditions that guarantee the uniqueness of the solution. The influence of the parameters α, β, and γ on the existence and uniqueness of the solutions is thoroughly analyzed.

In all of the aforementioned works, the Fourier method was employed. In contrast, the present study differs by incorporating a nonlinear non-local condition, and an unconventional approach is used to solve the problem.

2. Preliminaries

In this section, we introduce the definitions of the fractional integral and derivative, present an auxiliary problem essential for proving the main theorems, and state necessary lemmas.

Let $h(t)$ be a function defined on the interval $[a,b]$, and let $\sigma > 0$. The Riemann–Liouville fractional integral of order σ is defined by (see [6]):

$$D_{0t}^{-\sigma} h(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} h(\tau)\, d\tau, \tag{2.1}$$

provided that the right-hand side exists point-wise. As usual, $\Gamma(\sigma)$ is Euler's gamma function.

The Caputo fractional derivative of order $0<\rho<1$ for the function $h(t)$ is defined as (see, e.g., [6], p. 92):

$$\partial_{0t}^{\alpha} h(t) = \frac{1}{\Gamma(1-\rho)} \int\limits_0^t \frac{h'(\xi)}{(t-\xi)^{\rho}} d\xi, \quad t>0,$$

provided that the right-hand side exists point-wise.

For $0<\rho\le 1$ and an arbitrary complex number μ, by $E_{\rho,\mu}(z)$ we denote the Mittag–Leffler function with two parameters (see, e.g., [6], p. 56):

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n+\mu)}.$$

If the parameter $\mu=1$, then we have the classical Mittag-Leffler function: $E_{\rho}(z)=E_{\rho,1}(z)$. Note also $E_{1,1}(z)=E_1(z)=e^z$.

The Beta function $B(a,b)$ is defined for real numbers $a>0$, $b>0$ by the improper integral

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt. \tag{2.2}$$

The Beta function is related to the Gamma function by the identity

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.3}$$

Consider the following auxiliriary problem to find the function $u(x,t)$

$$\partial_{0t}^{\alpha} u(x,t) - u_{xx}(x,t) = f(x,t), \tag{2.4}$$

subject to the initial and boundary conditions

$$u(x,0) = \varphi(x), \quad 0\le x\le 1, \tag{2.5}$$

$$u(0,t)=0, \quad u(1,t)=0, \quad 0\le t\le T. \tag{2.6}$$

where ∂_{0t}^{α} is the Caputo fractional derivative of order $0<\alpha<1$ in the time variable, $f(x,t), \varphi(x)$ are given functions.

The solution of the problem (2.4)–(2.6) is defined in the same manner as in Definition 1.1.

A solution to problem (2.4)–(2.6) can be represented by the formula

$$u(x,t) = \int_0^1 \varphi(\xi) D_{0t}^{\alpha-1} G(x,t,\xi,0)\, d\xi + \int_0^t \int_0^1 G(x,t,\xi,\tau) f(\xi,\tau)\, d\xi d\tau, \tag{2.7}$$

where $G(x,t,\xi,\tau)$ is the Green function corresponding to the problem. This function is constructed in Remark 6.1 of [7] and has the form

$$G(x,t,\xi,\tau) = \sum_{m=-\infty}^{+\infty} \left[P(2m+x-\xi,t-\tau) - P(2m+x+\xi,t-\tau)\right], \quad 0<\xi<1, \quad 0<\tau<t\le T,$$

where

$$P(x,t) = \frac{t^{\frac{\alpha}{2}-1}}{2} e^{1,\frac{\alpha}{2}}_{1,\frac{\alpha}{2}} \left(-|x| t^{-\frac{\alpha}{2}}\right),$$

and

$$e^{\mu,\delta}_{\gamma,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n+\mu)\Gamma(\delta-\beta n)}, \quad \gamma>0, \quad \gamma>\beta,$$

is the Wright-type function (see [11], p. 23).

Lemma 2.1. *[see,[11], p. 46]*

- *If* $\delta \geq 0$, $\beta \in (0,1)$, *then* $e^{1,\delta}_{1,\beta}(-x) > 0$ *for any positive* x.
- *If* $\delta \geq \beta$, *then when* $x > 0$ *the function* $e^{1,\delta}_{1,\beta}(-x)$ *is strictly decreasing.*

Lemma 2.2. *[see [11], p. 47] If* $\delta \geq 1$, $\beta \in (0,1)$ *then for any positive* x, *the inequalities*

$$0 < e^{1,\delta}_{1,\beta}(-x) \leq \frac{1}{\Gamma(\delta)} e^{-x^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}(1-\beta)}.$$

Lemma 2.3 (see [11], p. 49)**.** *Let* $\delta < 1$, $\beta \in (0,1)$, *then for any positive* x *and* t, $a \in (0,x)$, $\xi \in [\beta, 1]$, $\omega \in (\frac{1}{2}, \min\{1, \frac{1}{2\beta}\})$ *is the inequality true*

$$\left| t^{\delta-1} e^{1,\delta}_{1,\beta}(-\frac{x}{t^\beta}) \right| \leq \frac{1}{\beta\pi} \big(C_\beta(\xi,\omega)\big)^{\frac{\delta-1}{\beta}} \Gamma(\frac{1-\delta}{\beta})(x-a)^{\frac{\delta-1}{\beta}} e^{1,1}_{1,\beta}(-\frac{a}{t^\beta}),$$

where

$$C_\beta(\xi,\omega) = (1-\beta)\left(\frac{-cos\omega\pi}{\xi-\beta}\right)^{\frac{1-\beta}{1-\beta}} \left(\frac{cos\beta\omega\pi}{1-\xi}\right)^{\frac{1-\beta}{1-\beta}}.$$

Using Lemma 2.2, we obtain the following estimate:

$$\left| t^{\delta-1} e^{1,\delta}_{1,\beta}(-\frac{x}{t^\beta}) \right| \leq \frac{1}{\beta\pi} \big(C_\beta(\xi,\omega)\big)^{\frac{\delta-1}{\beta}} \Gamma(\frac{1-\delta}{\beta})(x-a)^{\frac{\delta-1}{\beta}} e^{-x^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}(1-\beta)} = C(x-a)^{-\gamma_0} e^{-kx^\gamma}, \tag{2.8}$$

where

$$C = \frac{1}{\beta\pi}\big(C_\beta(\xi,\omega)\big)^{\frac{\delta-1}{\beta}} \Gamma(\frac{1-\delta}{\beta}), \quad \gamma_0, \gamma, k > 0.$$

To estimate the Green function $G(x,t,\xi,\tau)$, we decompose it into three parts:

$$G(x,t,\xi,\tau) = G_{-1}(x,t,\xi,\tau) + G_0(x,t,\xi,\tau) + G_1(x,t,\xi,\tau),$$

where

$$G_{-1}(x,t,\xi,\tau) = \sum_{m=-\infty}^{-1} \left[P(2m+x-\xi,t-\tau) - P(2m+x+\xi,t-\tau)\right],$$

$$G_0(x,t,\xi,\tau) = P(x-\xi,t-\tau) - P(x+\xi,t-\tau),$$

$$G_1(x,t,\xi,\tau) = \sum_{m=1}^{\infty} \left[P(2m+x-\xi,t-\tau) - P(2m+x+\xi,t-\tau)\right].$$

By substituting $m = -n$ in $G_{-1}(x,t,\xi,\tau)$, we rewrite

$$G_{-1}(x,t,\xi,\tau) = \sum_{n=1}^{\infty} \left[P(-2n+x-\xi,t-\tau) - P(-2n+x+\xi,t-\tau)\right].$$

Applying the estimate (2.8) to $G_1(x,t,\xi,\tau)$, we have

$$|G_{-1}(x,t,\xi,\tau)| \leq \sum_{n=1}^{\infty} |P(-2n+x-\xi,t-\tau)| + |P(-2n+x+\xi,t-\tau)|$$

$$\leq \sum_{n=1}^{\infty} C|-2n+x-\xi|^{-\gamma_0} e^{-k|x-\xi-2n|^{\gamma}} + C|-2n+x+\xi|^{-\gamma_0} e^{-k|x+\xi-2n|^{\gamma}} \leq C.$$

Similarly, for $G_1(x,t,\xi,\tau)$, the same bound holds:

$$|G_1(x,t,\xi,\tau)| \leq C.$$

Next, we consider an estimate for the function $G_0(x,t,\xi,\tau)$.

$$|G_0(x,t,\xi,\tau)| \leq |P(x-\xi,t-\tau)| + |P(x+\xi,t-\tau)|.$$

According to Lemma 2.1, we obtain the following estimate:

$$|P(x-\xi,t-\tau)| + |P(x+\xi,t-\tau)| \leq \frac{(t-\tau)^{\frac{\alpha}{2}-1}}{2}\left| e^{1,\frac{\alpha}{2}}_{1,\frac{\alpha}{2}}\left(-|x-\xi|(t-\tau)^{-\frac{\alpha}{2}}\right)\right|$$

$$+\frac{(t-\tau)^{\frac{\alpha}{2}-1}}{2}\left| e^{1,\frac{\alpha}{2}}_{1,\frac{\alpha}{2}}\left(-|x+\xi|(t-\tau)^{-\frac{\alpha}{2}}\right)\right| \leq C(t-\tau)^{\frac{\alpha}{2}-1}.$$

Hence, the Green function satisfies

$$|G(x,t,\xi,\tau)| \leq C|t-\tau|^{\frac{\alpha}{2}-1}, \quad 0<\tau<t\leq T. \tag{2.9}$$

Lemma 2.4 (see [1])**.** *Let $y(t)\geq 0$ be an absolutely continuous function satisfying the inequality*

$$\partial_{0t}^{\alpha} y(t) \leq c_1 y(t) + c_2(t), \quad 0<\alpha<1,$$

for almost all $t\in[0,T]$, where $c_1>0$ and $c_2(t)$ is a nonnegative integrable function on $[0,T]$. Then

$$y(t) \leq y(0)E_{\alpha}(c_1 t^{\alpha}) + \Gamma(\alpha)E_{\alpha,\alpha}(c_1 t^{\alpha}) D_{0t}^{-\alpha} c_2(t).$$

3. Proof of the Theorem 1.2

We multiply both sides of equation (1.1) by $u(x,t)$ and integrate with respect to x over the interval $[0,1]$:

$$\int_0^1 u(x,t)\partial_{0t}^{\alpha} u(x,t)\,dx = \int_0^1 u(x,t)u_{xx}(x,t)\,dx + \int_0^1 u(x,t)f(x,t)\,dx =$$

$$u(x,t)u_x(x,t)\big|_0^1 - \int_0^1 u_x^2(x,t)\,dx + \int_0^1 u(x,t)f(x,t)\,dx,$$

or equivalently,

$$\int_0^1 u(x,t)\partial_{0t}^{\alpha} u(x,t)\,dx = -\int_0^1 u_x^2(x,t)\,dx + \int_0^1 u(x,t)f(x,t)\,dx. \tag{3.1}$$

According to Lemma 1 in Alikhanov's work [1], for $0<\alpha<1$ the following inequality holds:

$$\int_0^1 u(x,t)\partial_{0t}^{\alpha} u(x,t)\,dx \geq \frac{1}{2}\partial_{0t}^{\alpha}\int_0^1 u^2(x,t)\,dx. \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\frac{1}{2}\partial_{0t}^{\alpha}\int_0^1 u^2(x,t)\,dx \leq -\int_0^1 u_x^2(x,t)\,dx + \int_0^1 u(x,t)f(x,t)\,dx \leq \int_0^1 u(x,t)f(x,t)\,dx.$$

Using the Cauchy inequality, we have

$$\int_0^1 u(x,t)f(x,t)\,dx \le \frac{1}{2}\left(\int_0^1 u^2(x,t)\,dx + \int_0^1 f^2(x,t)\,dx\right).$$

Thus, the following inequality holds:

$$\partial_{0t}^{\alpha}||u(\cdot,t)||^2 \le ||u(\cdot,t)||^2 + ||f(\cdot,t)||^2.$$

By Lemma 2.4, we get

$$||u(\cdot,t)||^2 \le ||u(\cdot,0)||^2 E_\alpha(t^\alpha) + \Gamma(\alpha)E_{\alpha,\alpha}(t^\alpha)D_{0t}^{-\alpha}||f(\cdot,t)||^2. \tag{3.3}$$

Evaluating (3.3) at $t = T$ and using the non-local condition (1.2), it follows that

$$||u(\cdot,T)||^2 \le ||g(\cdot,u(\cdot,T))||^2 E_\alpha(T^\alpha) + \Gamma(\alpha)E_{\alpha,\alpha}(T^\alpha)D_{0T}^{-\alpha}||f(\cdot,T)||^2.$$

According to (1.4), we have

$$||u(\cdot,T)||^2(1 - L^2E_\alpha(T^\alpha)) \le \Gamma(\alpha)E_{\alpha,\alpha}(T^\alpha)D_{0T}^{-\alpha}||f(\cdot,T)||^2.$$

If

$$L < \frac{1}{\sqrt{E_\alpha(T^\alpha)}},$$

then the following estimate holds:

$$||u(\cdot,T)||^2 \le \frac{\Gamma(\alpha)}{1 - L^2E_\alpha(T^\alpha)}E_{\alpha,\alpha}(T^\alpha)D_{0T}^{-\alpha}||f(\cdot,T)||^2. \tag{3.4}$$

Rewriting (3.3) by taking (1.2) into account yields

$$||u(\cdot,t)||^2 \le L^2||u(\cdot,T)||^2 E_\alpha(t^\alpha) + \Gamma(\alpha)E_{\alpha,\alpha}(t^\alpha)D_{0t}^{-\alpha}||f(\cdot,t)||^2.$$

Using (3.4), we obtain

$$||u(\cdot,t)||^2 \le \frac{L^2\Gamma(\alpha)E_{\alpha,\alpha}(T^\alpha)}{1 - L^2E_\alpha(T^\alpha)}D_{0T}^{-\alpha}||f(\cdot,T)||^2E_\alpha(t^\alpha) + \Gamma(\alpha)E_{\alpha,\alpha}(t^\alpha)D_{0t}^{-\alpha}||f(\cdot,t)||^2. \tag{3.5}$$

From (2.1), we get the following estimate:

$$D_{0t}^{-\alpha}||f(\cdot,t)||^2 \le \frac{T}{\alpha\Gamma(\alpha)}\max_{t\in[0,T]}||f(\cdot,t)||^2. \tag{3.6}$$

Apply (3.6) to (3.5) to get

$$||u(\cdot,t)||^2 \le \left[\frac{L^2\Gamma(\alpha)E_{\alpha,\alpha}(T^\alpha)}{1 - L^2E_\alpha(T^\alpha)}E_\alpha(t^\alpha) + E_{\alpha,\alpha}(t^\alpha)\right]\frac{T}{\alpha}\max_{t\in[0,T]}||f(\cdot,t)||^2.$$

4. Proof of Theorem 1.3

First, we prove the existence of the solution to problem (1.1)–(1.3). From (2.7) we have

$$u(x,t)=\int_0^1 \varphi(\xi)D_{0t}^{\alpha-1}G(x,t,\xi,0)d\xi+\int_0^t\int_0^1 G(x,t,\xi,\tau)f(\xi,\tau)d\xi d\tau.$$

Setting $u(x,T)=v(x)$, we obtain:

$$v(x)=\int_0^1 D_{0T}^{\alpha-1}G(x,T,\xi,0)g(\xi,v(\xi))d\xi+F(x), \tag{4.1}$$

where

$$F(x)=\int_0^T\int_0^1 G(x,T,\xi,\tau)f(\xi,\tau)d\xi d\tau.$$

We prove the existence of the solution to integral equation (4.1) in the space $C[0,1]$ by the method of successive approximations. For this, we consider the sequence of functions

$$v_n(x)=\int_0^1 D_{0T}^{\alpha-1}G(x,T,\xi,0)g(\xi,v_{n-1}(\xi))d\xi, \quad n=1,2,\dots,$$

with the initial approximation $v_0=F(x)$.

Next, using the estimate (2.9), we get

$$|v_0(x)|\le\int_0^T\int_0^1|G(x,T,\xi,\tau)f(\xi,\tau)|d\xi d\tau\le C\int_0^T\int_0^1(T-\tau)^{\frac{\alpha}{2}-1}|f(\xi,\tau)|d\xi d\tau,$$

$$\le\frac{2C}{\alpha}T^{\frac{\alpha}{2}}\max_{[0,1]\times[0,T]}|f(x,t)|.$$

For the next steps,

$$v_1(x)=\int_0^1 D_{0T}^{\alpha-1}G(x,T,\xi,0)g(v_0(\xi))d\xi=\frac{1}{\Gamma(1-\alpha)}\int_0^1\int_0^T(T-\eta)^{-\alpha}G(x,\eta,\xi,0)g(v_0(\xi))d\eta d\xi,$$

hence,

$$|v_1(x)|=\left|\frac{1}{\Gamma(1-\alpha)}\int_0^T(T-\eta)^{-\alpha}\left(\int_0^1 G(x,\eta,\xi,0)g(v_0(\xi))d\xi\right)d\eta\right|.$$

Using the Lipschitz condition for g and the previous estimate, we have

$$|v_1(x)|\le\frac{L}{\Gamma(1-\alpha)}\frac{2C}{\alpha}T^{\frac{\alpha}{2}}\max_{[0,1]\times[0,T]}|f(x,t)|\int_0^T(T-\eta)^{-\alpha}\eta^{\frac{\alpha}{2}-1}d\eta.$$

By the Beta function integral formula (2.2) and relation with Gamma function (2.3)

$$|v_1(x)|\le\frac{L}{\Gamma(1-\alpha)}\frac{2C}{\alpha}T^{\frac{\alpha}{2}}\max|f|\cdot T^{1-\frac{\alpha}{2}}\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)},$$

and thus

$$|v_1(x)|\le\frac{\Gamma\left(\frac{\alpha}{2}\right)L}{\Gamma\left(1-\frac{\alpha}{2}\right)}\frac{2C}{\alpha}T\max_{[0,1]\times[0,T]}|f(x,t)|.$$

Proceeding similarly for $v_2(x), v_3(x)$ and, in general, for $v_n(x)$, we obtain the estimate

$$|v_n(x)| \leq \left(\frac{\Gamma\left(\frac{\alpha}{2}\right) L}{\Gamma\left(1-\frac{\alpha}{2}\right)} \right)^n \frac{2C}{\alpha} T^{n\left(1-\frac{\alpha}{2}\right)+\frac{\alpha}{2}} \max_{[0,1]\times[0,T]} |f(x,t)|.$$

Now consider the functional series

$$v_0(x) + \sum_{n=1}^{\infty} (v_n(x) - v_{n-1}(x)). \tag{4.2}$$

and prove that series (4.2) converges uniformly on the interval $[0,1]$. Comparing it with the number series

$$\frac{2C}{\alpha} T^{\frac{\alpha}{2}} \max|f| + M \sum_{n=1}^{\infty} \left(\frac{\Gamma\left(\frac{\alpha}{2}\right) L T^{1-\frac{\alpha}{2}}}{\Gamma\left(1-\frac{\alpha}{2}\right)} \right)^{n-1},$$

where

$$M = T^{1-\frac{\alpha}{2}} \left(\frac{\Gamma\left(\frac{\alpha}{2}\right) L}{\Gamma\left(1-\frac{\alpha}{2}\right)} T^{1-\frac{\alpha}{2}} + 1 \right) \frac{2C}{\alpha} \max|f|,$$

by the DAlembert ratio test, the series converges provided that

$$L < \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) T^{1-\frac{\alpha}{2}}}.$$

In the previous steps, we constructed a sequence of functions $\{v_n(x)\}$using the method of successive approximations and established uniform bounds for each term. These bounds allowed us to apply the DAlembert ratio test and conclude that the sequence converges uniformly on the interval $[0,1]$. Consequently, the limiting function $v(x)$ exists and is continuous, and it serves as the solution to the integral equation (4.1).

Given the non-local initial condition $u(x,0) = g(x, u(x,T))$, and recognizing that $u(x,T) = v(x)$, we can now fully construct the solution $u(x,t)$ of the original problem. This is done using the representation (2.7), which expresses the solution via the Green function as:

$$u(x,t) = \int_0^1 g(\xi, v(\xi)) D_{0t}^{\alpha-1} G(x,t,\xi,0)\, d\xi + \int_0^t \int_0^1 G(x,t,\xi,\tau) f(\xi,\tau)\, d\xi d\tau. \tag{4.3}$$

Therefore, under the assumptions stated in conditions (1.2) and (2.5), we have constructed a well-defined, continuous, and explicit solution to the non-local and nonlinear boundary value problem involving a fractional Caputo derivative.

It is proven in [7] that the function of the form (4.3) satisfies the conditions of Definition 1.1. Since the conditions of Theorem 1.3 include those of Theorem 1.2, the uniqueness of the solution to problem (2.4)–(2.6) is also ensured under these conditions.

5. Acknowledgement.

The author acknowledges financial support from the Ministry of Innovative Development of the Republic of Uzbekistan, Grant No F-FA-2021-424.

References

[1] Alikhanov A.A.; A Priori Estimates for Solutions of Boundary Value Problems for Fractional-Order Equations. Differential Equations,–2010.–46.–5.–P.658–664.

[2] Ashurov R.R., Muhiddinova O.T.; Initial-Boundary Value Problem for a Time-Fractional Subdiffusion Equation with an Arbitrary Elliptic Differential Operator. Lobachevskii Journal of Mathematics,–2021.–42.–3.–P.517–525.

[3] Ashurov R.R., Fayziyev Yu.E.; On the non-local Problems in Time for Time-Fractional Subdiffusion Equations. Fractal Fract.,–2022.–6.–41.–P.1–21.

[4] Ashurov R., Nuraliyeva N.; A Three-Parameter Problem for Fractional Differential Equation with an Abstract Operator. Lobachevskii Journal of Mathematics,–2024.–45.–P.5788–5801.

[5] Boudabsa L., Simon T.; Some Properties of the Kilbas-Saigo Function. Mathematics,–2021.–9.–P.217.

[6] Kilbas A. A., Srivastava H. M., Trujillo J. J., Theory and Applications of Fractional Differential Equations, Elsevier, North-Holland, Mathematics Studies, 2006.

[7] Mamchuev M.O.; Solutions of the Main Boundary Value Problems for the Time-Fractional Telegraph Equation by the Green Function Method. Fract. Calc. Appl. Anal.,–2017.–20.–P.190–211. https://doi.org/10.1515/fca-2017-0010.

[8] Mamchuev M.O.; Fundamental Solution of a System of Fractional Partial Differential Equations. Differ. Uravn.,–2010.–46.–8.–P.1113–1124.

[9] Muhiddinova O.T.; Initial-Boundary Value Problem for a Subdiffusion Equation with Caputo derivative. International Journal of Applied Mathematics,–2022.–35.–1.–P.135–146.

[10] Nuraliyeva N.; A Three-Parameter non-local Problem Involving the Integral Condition for the Subdiffusion Equation. BIM,–2024.–7.–P.55–70.

[11] Pskhu A.V.; Fractional Partial Differential Equations. Nauka, Moscow,–2005.–199 p. (in Russian).

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.38-53
DOI: 10.29229/uzmj.2025-2-4

A note on the NBVP with Samarskii-Ionkin condition I for elliptic equations

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In the present paper, the nonlocal boundary value problem with Samarskii-Ionkin condition I for elliptic equations in a Banach space with the positive operator is investigated. The main theorems on well-posedness of this problem are established. In practice, the coercive stability estimates for solution of four types of nonlocal boundary value problems with Samarskii-Ionkin condition I for elliptic differential equations are proved.

Keywords: Samarskii-Ionkin condition; elliptic equations; coercive stability; well-posedness; positive operator.

MSC (2020): 35L10; 35L90; 35B35.

1. Introduction

Elliptic partial differential equations have applications in almost all areas of mathematics, from harmonic analysis to geometry and to Lie theory, as well as numerous applications in physics and engineering. The well-posedness of the local boundary value problem for the elliptic equation

$$-v''(t)+Av(t)=f(t)\ (0\leq t\leq T),\ v(0)=v_0,\ v(T)=v_T \tag{1.1}$$

in an arbitrary Banach space E with the positive operator A and its related applications have been investigated by many researchers (see, for example, [7],[26], [24], and the references given therein).

In mathematical modeling, elliptic equations are used together with local boundary conditions specifying the solution on the boundary of the domain. In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological or environmental processes often involve nonclassical conditions. Such conditions usually are identified as nonlocal boundary conditions and reflect situations when the data on the domain boundary cannot be measured directly, or when the data on the boundary depend on the data inside the domain. The well-posedness of various nonlocal boundary value problems for partial differential and difference equations has been studied extensively by many researchers (see, e.g. [3],[6],[8],[23], [31],[34], and the references given therein). The survey paper [9] contains the recent results on the local and nonlocal well-posed problems for second order differential and difference equations. Results on the stability of differential problems for second order equations and of difference schemes for approximate solution of the second order problems were presented.

Recently, various nonlocal boundary value problems with Samarskii-Ionkin condition for partial differential have been investigated by many researchers (see, e.g. [27],[16], and the references given therein).

In the present paper, the nonlocal boundary value problem with Samarskii–Ionkin condition I for the elliptic equation

$$\begin{cases} -\frac{d^2u(t)}{dt^2}+Au(t)=f(t), 0<t<T, \\ u(0)=u(T)+\varphi, u'(T)+\mu u(T)=\psi \end{cases} \tag{1.2}$$

in a Banach space E with the positive operator A and $\mu \geq 0$ is investigated.

A function $u(t)$ is called a solution of the problem (1.2) if the following conditions are satisfied:

- i) $u(t)$ is twice continuously differentiable on the interval $(0.T)$ and continuously differentiable on the segment $[0,T]$. The derivatives at the endpoints of the segment are understood as the corresponding unilateral derivatives.
- ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,T]$, and the function $Au(t)$ is continuous on the segment $[0,T]$.
- iii) $u(t)$ satisfies the equation and the nonlocal boundary condition (1.2).

In the present paper, we study the well-posedness of the nonlocal boundary value problem (1.2). Throughout the paper, the main theorems on the well-posedness of the nonlocal boundary value problem are established. In applications, the new coercive stability estimates for solution of four types of problems for elliptic equations are obtained.

2. Auxiliary Results for the Solution of Problem (1.1)

In this section, we give some auxiliary statements from [7] which will be useful in the sequel. The operator $B = A^{\frac{1}{2}}$ has better spectral properties than the positive operator A. Indeed, the operator $(-B)$ is a generator of an analytic semigroup $\exp\{-tB\}$ $(t \geq 0)$ with exponentially decreasing norm, when $t \longrightarrow +\infty$, i.e. the following estimates

$$\|\exp\,(-tB)\|_{E\to E}\,, \|tB\exp(-tB)\|_{E\to E} \leq M(B)e^{-\delta(B)t}\ (t>0) \tag{2.1}$$

hold for some number $M(B) \in [1,+\infty)$, $\delta(B) \in (0,+\infty)$. From that it follows that the operator $I - e^{-2TB}$ has the bounded inverse and the following estimate holds:

$$\left\|(I-e^{-2TB})^{-1}\right\|_{E\to E} \leq M(B)(1-e^{-2T\delta(B)})^{-1}. \tag{2.2}$$

The following formula

$$v(t) = (I-e^{-2TB})^{-1}\{(e^{-tB}-e^{-(2T-t)B})v_0 + (e^{-(T-t)B}-e^{-(T+t)B})v_T \tag{2.3}$$

$$-(e^{-(T-t)B}-e^{-(T+t)B})(2B)^{-1}\int_0^T (e^{-(T-s)B}-e^{-(T+s)B})f(s)ds\}$$

$$+(2B)^{-1}\int_0^T (e^{-|t-s|B}-e^{-(t+s)B})f(s)ds$$

holds for the exact solution of the problem (1.1) under sufficiently smooth data v_0, v_T and $f(t)$.

First, we denote by $C^{\alpha}(E), (0<\alpha<1)$, the Banach space obtained by completion of the set of all smooth E-valued functions $\varphi(t)$ on $[0,T]$ in the norm

$$\|\,\varphi\,\|_{C^{\alpha}(E)} = \max_{0\leq t\leq T} ||\varphi(t)||_E + \sup_{0\leq t<t+\tau\leq T} \frac{\|\,\varphi(t+\tau)-\varphi(t)\,\|_E}{\tau^{\alpha}}.$$

Theorem 2.1. *Suppose v''_0, $v''_T \in E_\alpha$, $f(t) \in C^\alpha(E)(0 < \alpha < 1)$. Then the boundary value problem (1.1) is well-posed in Hölder space $C^\alpha(E)$, if A is the positive operator in Banach space E. For the solution $v(t)$ in $C^\alpha(E)$ of the boundary value problem the coercive inequality*

$$\|v''\|_{C^\alpha(E)} + \|Av\|_{C^\alpha(E)} + \|v''\|_{C(E_\alpha)} \tag{2.4}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \parallel f \parallel_{C^\alpha(E)} + \frac{M}{\alpha}[\| v''_0||_{E_\alpha} + ||v''_T||_{E_\alpha}]$$

holds, where M does not depend on α, v_0, v_T and $f(t)$.

Here, the Banach space $E_\alpha = E_\alpha(B, E)$ $(0 < \alpha < 1)$ consists of those $v \in E$ for which the norm

$$\parallel v \parallel_{E_\alpha} = \sup_{z>0} z^{1-\alpha} \parallel B exp\{-zB\}v \parallel_E + \parallel v \parallel_E$$

is finite. Moreover, the positivity of A is a necessary condition for well-posedness of the problem (1.1) in $C(E)$. However, the problem (1.1) is not well posed in $C(E)$ for all positive operators. It turns out that a Banach space E can be restricted to a Banach space E' $(D(A) \subset E' \subset E)$ in such a manner that the restricted problem (1.1) in E' will be well posed in $C(E')$. The role of E' will be played here by the fractional spaces $E_\alpha = E_\alpha(B, E)(0 < \alpha < 1)$.

Theorem 2.2. *Let A be the positive operator in a Banach space E and $f(t) \in C(E_\alpha)$ $(0 < \alpha < 1)$. Then for the solution $v(t)$ in $C(E_\alpha)$ of the boundary value problem (1.1) the coercive inequality*

$$\parallel v'' \parallel_{C(E_\alpha)} + \parallel Av \parallel_{C(E_\alpha)} \tag{2.5}$$

$$\leq M[\parallel Av_0 \parallel_{E_\alpha} + \parallel Av_T \parallel_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1} \parallel f \parallel_{C(E_\alpha)}]$$

holds, where M does not depend on α, v_0, v_T and $f(t)$.

3. Main theorems on the well-posedness of the problem (1.2)

Let us give lemma that will be needed below.

Lemma 3.1[10]. Let A be positive operator in a Banach space E. then the operator $\left(I - (B-\mu)(B+\mu)^{-1}e^{-TB}\right)$ has an inverse $\left(I - (B-\mu)(B+\mu)^{-1}e^{-TB}\right)^{-1}$ and the following estimates hold:

$$\|B(B+\mu)^{-1}\|_{E\to E}, \|(B-\mu)(B+\mu)^{-1}\|_{E\to E} \leq M_1(B), \tag{3.1}$$

$$\parallel \left(I - (B-\mu)(B+\mu)^{-1}e^{-TB}\right)^{-1} \parallel_{E\to E} \leq \frac{M_2(B)}{1 - M_1(B)e^{-\delta(B)T}}. \tag{3.2}$$

Here $\delta(B) > \frac{\ln M_1(B)}{T}$.

We consider the problem (1.2). Using the formula (2.3) and $u(0) = u(T) + \varphi$, we get

$$u(t) = R\{\left(e^{-tB} - e^{-(2T-t)B}\right)(u(T) + \varphi) \tag{3.3}$$

$$+ \left(e^{-(T-t)B} - e^{-(T+t)B}\right)\left(u(T) - (2B)^{-1}\int_0^T \left(e^{-(T-s)B} - e^{-(T+s)B}\right) f(s)ds\right)\Big\}$$

$$+(2B)^{-1}\int_0^T \left(e^{-|t-s|B} - e^{-(t+s)B}\right) f(s)ds.$$

Here and in this paper we wil put $R=\left(I-e^{-2TB}\right)^{-1}$. Taking the derivative, we get

$$u'(t)=R\left\{-B\left(e^{-tB}+e^{-(2T-t)B}\right)(u(T)+\varphi)\right. \tag{3.4}$$

$$\left.+\left(e^{-(T-t)B}+e^{-(T+t)B}\right)\left(Bu(T)-2^{-1}\int_0^T\left(e^{-(T-s)B}-e^{-(T+s)B}\right)f(s)ds\right)\right\}$$

$$+2^{-1}\left(-\int_0^t e^{-(t-s)B}f(s)ds+\int_t^T e^{-(s-t)B}f(s)ds+\int_0^T e^{-(t+s)B}f(s)ds\right).$$

Applying the formula (3.4) and the nonlocal condition $u'(T)+\mu u(T)=\psi$, we get

$$\left(B+\mu-(B-\mu)\,e^{-TB}\right)u(T)$$

$$=(I-e^{-TB})^{-1}\left[2Be^{-TB}\varphi+\int_0^T(e^{-(T-s)B}-e^{-(T+s)B})f(s)ds\right]+(I+e^{-TB})\psi.$$

From that it follows

$$u(T)=\left(B+\mu-(B-\mu)\,e^{-TB}\right)^{-1}\left\{2B\left(I-e^{-TB}\right)^{-1}e^{-TB}\varphi\right. \tag{3.5}$$

$$\left.+(I-e^{-TB})^{-1}\int_0^T(e^{-(T-s)B}-e^{-(T+s)B})f(s)ds+(I+e^{-TB})\psi\right\}.$$

It is easy to show that $u(t)$ defined on $[0,T]$ by formulas (3.3), (3.4), and (3.5) is a unique solution in $C(E)$ of the problem (1.2) if, for example, $\varphi\in D(A^2),\psi\in D(A^{\frac{3}{2}})$ and $Af(t)\in C(E)$ or $f'(t)\in C(E)$. Sufficient conditions for the well-posedness of the nonlocal boundary value problem (1.2) can be established if one considers this problem in certain spaces of smooth $E-$valued functions defined on $[0,T]$.
Note that for the solution of problem (1.2) the coercivity inequality

$$\|u''\|_{C^\alpha(E)}+\|Au\|_{C^\alpha(E)}\le M_C[\|f\|_{C^\alpha(E)}+\ ||A\varphi||_E+||A\psi||_E]$$

fails. Nevertheless, we have the following result.

Theorem 3.1. *Suppose A is the positive operator in Banach space E and $B\psi\in E_\alpha$, $A\varphi-f(0)+f(T)\in E_\alpha$, $f(t)\in C^\alpha(E)$ $(0<\alpha<1)$. Then the nonlocal boundary value problem (1.2) is well-posed in Hölder space $C^\alpha(E)$. For the solution $u(t)$ in $C^\alpha(E)$ of the nonlocal boundary value problem (1.2) the coercive inequality*

$$\|u''\|_{C^\alpha(E)}+\|Au\|_{C^\alpha(E)}+\|u''\|_{C(E_\alpha)} \tag{3.6}$$

$$\le\frac{M}{\alpha(1-\alpha)}\parallel f\parallel_{C^\alpha(E)}+\frac{M}{\alpha}\left[\|A\varphi-f(0)+f(T)\|_{E_\alpha}+\|A\psi\|_{E\alpha}\right]$$

holds, where M does not depend on α, φ,ψ and $f(t)$.

Proof. By Theorem 2.1 we have the following estimate

$$\|u''\|_{C^\alpha(E)}+\|Au\|_{C^\alpha(E)}+\|u''\|_{C(E_\alpha)} \tag{3.7}$$

$$\le\frac{M}{\alpha(1-\alpha)}\parallel f\parallel_{C^\alpha(E)}+\frac{M}{\alpha}\left[\|Au(0)-f(0)\|_{E_\alpha}+\|Au(T)-f(T)\|_{E_\alpha}\right]$$

for the solution of the problem (1.2). We have that

$$Au(0)-f(0)=A\varphi-f\left(0\right)+f(T)+Au(T)-f(T).$$

Therefore, to prove the theorem it suffices to establish the estimate for $\|Au(T)-f(T)\|_{E_\alpha}$. Applying the formula (3.5), we get

$$Au(T)-f(T)=\left(B+\mu-\left(B-\mu\right)e^{-TB}\right)^{-1}B\Big\{2\left(I-e^{-TB}\right)^{-1}e^{-TB}A\varphi$$

$$+(I-e^{-TB})^{-1}\int_0^T B(e^{-(T-s)B}-e^{-(T+s)B})f(s)ds+(I+e^{-TB})B\psi$$

$$-\left(B+\mu-\left(B-\mu\right)e^{-TB}\right)B^{-1}f(T)\}=\Big(I-\left(B-\mu\right)\left(B+\mu\right)^{-1}e^{-TB}\Big)^{-1}B\left(B+\mu\right)^{-1}$$

$$\times\Big\{\left(I-e^{-TB}\right)^{-1}e^{-TB}\left[2\left(A\varphi-f(0)+f(T)\right)+\left(I+e^{-TB}\right)\left(f(T)-f(0)\right)\right]$$

$$+(I+e^{-TB})\left(B\psi+\mu(I-e^{-TB})B^{-1}f(T)\right)$$

$$+\left(I-e^{-TB}\right)^{-1}\left[\int_0^T Be^{-(T-s)B}\left(f(s)-f\left(T\right)\right)ds-\int_0^T Be^{-(T+s)B}\left(f(s)-f\left(0\right)\right)ds\right]\Big\}.$$

Then using the triangle inequality, the estimates (2.1), (3.1), (3.2), and the definition of the spaces $C^\alpha\left(E\right)$ and E_α, we get

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}\left(Au(T)-f(T)\right)\right\|_E$$

$$\leq\|(B-\mu)(B+\mu)^{-1}\|_{E\to E}\|\left(I-(B-\mu)(B-\mu)^{-1}e^{-TB}\right)^{-1}\|_{E\to E}$$

$$\times\Big\{2\left\|\left(I-e^{-TB}\right)^{-1}\right\|_{E\to E}\lambda^{1-\alpha}\left\|Be^{-(\lambda+T)B}\left(A\varphi-f\left(0\right)+f(T)\right)\right\|_E$$

$$+\left\|I+e^{-TB}\right\|_{E\to E}\lambda^{1-\alpha}\left\|Be^{-(\lambda+T)B}\left(f\left(0\right)-f(T)\right)\right\|_E$$

$$+\left\|I+e^{-TB}\right\|_{E\to E}\lambda^{1-\alpha}\left\|Be^{-\lambda B}B\psi\right\|_E+\left\|I-e^{-TB}\right\|_{E\to E}\lambda^{1-\alpha}\left\|e^{-\lambda B}f(T)\right\|_E$$

$$+\left\|\left(I-e^{-TB}\right)^{-1}\right\|_{E\to E}\lambda^{1-\alpha}\int_0^T\left\|B^2e^{-(\lambda+(T-s))B}\right\|_{E\to E}\left\|f\left(s\right)-f\left(T\right)\right\|_E ds$$

$$+\left\|\left(I-e^{-TB}\right)^{-1}\right\|_{E\to E}\lambda^{1-\alpha}\int_0^T\left\|B^2e^{-(\lambda+T+s)B}\right\|_{E\to E}\left\|f\left(s\right)-f\left(0\right)\right\|_E ds\Big\}$$

$$\leq M_3\left[\left\|A\varphi-f\left(0\right)+f(T)\right\|_E+\left\|B\psi\right\|_{E\alpha}\right]$$

$$+M_4\lambda^{1-\alpha}\left(\int_0^T\frac{(T-s)^\alpha}{\left(\lambda+(T-s)\right)^2}ds+\int_0^T\frac{s^\alpha}{\left(\lambda+T+s\right)^2}ds\right)\|f\|_{C^\alpha(E)}$$

for any $\lambda>0$. Since

$$\int_0^T\frac{\lambda^{1-\alpha}s^\alpha}{\left(\lambda+s\right)^2}ds\leq\int_0^\infty\frac{p^\alpha}{\left(1+p\right)^2}dp\leq\frac{2}{\left(1+\alpha\right)\left(1-\alpha\right)},$$

we have that

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}\left(Au(T)-f(T)\right)\right\|_E$$

$$\leq M_3\left[\|A\varphi - f(0) + f(T)\|_{E_\alpha} + \|B\psi\|_{E_\alpha}\right] + M_4\frac{1}{1-\alpha}\|f\|_{C^\alpha(E)}$$

for any $\lambda > 0$. Therefore

$$\|Au(T) - f(T)\|_{E_\alpha} \leq M_3\left[\|A\varphi - f(0) + f(T)\|_{E_\alpha} + \|B\psi\|_{E_\alpha}\right] + M_4\frac{1}{1-\alpha}\|f\|_{C^\alpha(E)}. \quad (3.8)$$

Theorem 3.1 is proved.

Theorem 3.2. *Suppose A is the positive operator in a Banach space E and $B\psi \in E_\alpha$, $A\varphi \in E_\alpha$, $f(t) \in C(E_\alpha)$ $(0 < \alpha < 1)$. Then for the solution $u(t)$ in $C(E_\alpha)$ of the boundary value problem (1.2) the coercive inequality*

$$\|u''\|_{C(E_\alpha)} + \|Au\|_{C(E_\alpha)} \quad (3.9)$$

$$\leq M(\mu)[\|A\varphi\|_{E_\alpha} + \|B\psi\|_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1}\|f\|_{C(E_\alpha)}]$$

holds, where $M(\mu)$ does not depend on α, φ, ψ and $f(t)$.

Proof. By Theorem 2.2 we have the following estimate

$$\|u''\|_{C(E_\alpha)} + \|Au\|_{C(E_\alpha)}$$

$$\leq M[\|Au(0)\|_{E_\alpha} + \|Au(T)\|_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1}\| f \|_{C(E_\alpha)}]$$

for the solution of problem (1.2). We have that

$$Au(0) = A\varphi + Au(T).$$

Therefore, to prove the theorem it suffices to establish the estimate for $\|Au(T)\|_{E_\alpha}$. Applying formula (3.5), we get

$$Au(T) = \left(I - (B-\mu)(B+\mu)^{-1}e^{-TB}\right)^{-1} B(B+\mu)^{-1}$$

$$\times\Big\{2\left(I - e^{-TB}\right)^{-1}e^{-TB}A\varphi$$

$$+(I - e^{-TB})^{-1}\int_0^T B(e^{-(T-s)B} - e^{-(T+s)B})f(s)ds + (I + e^{-TB})B\psi\Big\}.$$

Using the triangle inequality, the estimates (2.1), (3.1), (3.2), and the definition of the spaces E_α, we get

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}Au(T)\right\|_E$$

$$\leq \|(B-\mu)(B+\mu)^{-1}\|_{E\to E}\|\left(I - (B-\mu)(B-\mu)^{-1}e^{-TB}\right)^{-1}\|_{E\to E}$$

$$\times\Big\{2\left\|\left(I - e^{-TB}\right)^{-1}\right\|_{E\to E}\lambda^{1-\alpha}\left\|Be^{-(\lambda+T)B}A\varphi\right\|_E$$

$$+\left\|I + e^{-TB}\right\|_{E\to E}\lambda^{1-\alpha}\left\|Be^{-\lambda B}B\psi\right\|_E$$

$$+\left\|\left(I - e^{-TB}\right)^{-1}\right\|_{E\to E}\lambda^{1-\alpha}\int_0^T\left\|B^2e^{-(\lambda+(T-s))B}f(s)\right\|_E ds$$

$$+\left\|\left(I - e^{-TB}\right)^{-1}\right\|_{E\to E}\lambda^{1-\alpha}\int_0^T\left\|B^2e^{-(\lambda+s)B}f(s)\right\|_E ds\Big\}$$

$$\leq M_1\left[\|A\varphi\|_{E_\alpha}+\|B\psi\|_{E_\alpha}\right]+M_2\lambda^{1-\alpha}\left(\int_0^T\frac{ds}{(\lambda+T-s)\,(T-s)^{1-\alpha}}\right.$$

$$\left.+\int_0^T\frac{ds}{(\lambda+T+s)\,s^{1-\alpha}}\right)\|f\|_{C(E_\alpha)}$$

for any $\lambda>0$. Since

$$\int_0^T\frac{\lambda^{1-\alpha}ds}{(\lambda+s)\,s^{1-\alpha}}\leq\int_0^\infty\frac{p^{\alpha-1}}{p+1}dp\leq\frac{1}{\alpha\,(1-\alpha)},$$

we have that

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}Au(T)\right\|_E\leq M_1\left[\|A\varphi\|_{E_\alpha}+\|B\psi\|_{E_\alpha}\right]+M_3\frac{1}{\alpha\,(1-\alpha)}\|f\|_{C(E_\alpha)}$$

for any $\lambda>0$. Therefore

$$\|Au(T)\|_{E_\alpha}\leq M_1(\mu)\left[\|A\varphi\|_{E_\alpha}+\|B\psi\|_{E_\alpha}\right]+M_3(\mu)\frac{1}{\alpha\,(1-\alpha)}\|f\|_{C(E_\alpha)}. \qquad (3.10)$$

Theorem 3.2 is proved.

Note that the nonlocal boundary value problem (1.2) can be rewritted as the system of the sequential Dirichlet and Robin boundary value problems for elliptic equations in a Banach space E with the positive operator A. Actually, the solution $u(t)$ of the problem (1.2) can be presented in the form

$$u(t)=E(t)+O(t), \qquad (3.11)$$

where $E(t)$ and $O(t)$ are abstract even and odd functions defined on the segment $[0,T]$, respectively. We have that

$$E(t)=\frac{u(t)+u(T-t)}{2},O(t)=\frac{u(t)-u(T-t)}{2}.$$

By the definition of functions

$$E(0)=E(T),E'(0)=-E'(T),$$

$$O(0)=-O(T),O'(0)=O'(T),$$

Using the nonlocal condition

$$u(0)=u(T)+\varphi,$$

we can write

$$E(0)+O(0)=E(T)+O(T)+\varphi.$$

Then, using $E(0)=E(T),O(0)=-O(T)$, we get $O(0)=\frac{\varphi}{2},O(T)=-\frac{\varphi}{2}$. Using the local condition

$$u'(T)+\mu u(T)=\psi,$$

we can write

$$E'(T)+O'(T)+\mu\left(E(T)+O(T)\right)=\psi.$$

Therefore,

$$E'(T)+\mu E(T)=-O'(T)-\mu O(T)+\psi$$

and using conditions $E(0)=E(T),E'(0)=-E'(T)$, we get

$$-E'(0)+\mu E(0)=-O'(T)-\mu O(T)+\psi.$$

So, it is easy to see that $O(t)$ is the solution of the Dirichlet boundary value problem for the elliptic equation

$$\begin{cases} -\frac{d^2O(t)}{dt^2} + AO(t) = \frac{1}{2}(f(t) - f(T-t)), 0 < t < T, \\ O(0) = \frac{\varphi}{2}, O(T) = -\frac{\varphi}{2} \end{cases} \tag{3.12}$$

in an arbitrary Banach space E with the positive operator A and the function $E(t)$ is the solution of the Robin boundary value problem for the elliptic equation

$$\begin{cases} -\frac{d^2E(t)}{dt^2} + AE(t) = \frac{1}{2}(f(t) + f(T-t)), 0 < t < T, \\ -E'(0) + \mu E(0) = -O'(T) + \frac{1}{2}\mu\varphi + \psi, \\ E'(T) + \mu E(T) = -O'(T) + \frac{1}{2}\mu\varphi + \psi \end{cases} \tag{3.13}$$

in an arbitrary Banach space E with the positive operator A, respectively.

By Theorems 2.1 and 2.2, we have the following estimates

$$\|O''\|_{C^\alpha(E)} + \|AO\|_{C^\alpha(E)} + \|O''\|_{C(E_\alpha)} \tag{3.14}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \|f\|_{C^\alpha(E)} + \frac{M}{\alpha} \|A\varphi - f(0) + f(T)\|_{E_\alpha},$$

$$\|O''\|_{C(E_\alpha)} + \|AO\|_{C(E_\alpha)} \tag{3.15}$$

$$\leq M[\|A\varphi\|_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1} \|f\|_{C(E_\alpha)}]$$

for the solution of the problem (3.12) and

$$\|E''\|_{C^\alpha(E)} + \|AE\|_{C^\alpha(E)} + \|E''\|_{C(E_\alpha)} \tag{3.16}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \|f\|_{C^\alpha(E)} + \frac{M}{\alpha} \left\| AE(0) - \frac{1}{2}(f(0) + f(T)) \right\|_{E_\alpha},$$

$$\|E''\|_{C(E_\alpha)} + \|AE\|_{C(E_\alpha)} \tag{3.17}$$

$$\leq M[\|AE(0)\|_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1} \| f \|_{C(E_\alpha)}]$$

for the solution of the problem (3.13), respectively. Moreover, we have that the following formulas

$$O(t) = (I - e^{-2TB})^{-1} \left\{ \left[e^{-tB} - e^{-(2T-t)B} - e^{-(T-t)B} + e^{-(T+t)B} \right] \frac{\varphi}{2} \right. \tag{3.18}$$

$$\left. -(e^{-(T-t)B} - e^{-(T+t)B})(2B)^{-1} \int_0^T (e^{-(T-s)B} - e^{-(T+s)B}) \frac{1}{2}(f(s) - f(T-s))ds \right\}$$

$$+(2B)^{-1} \int_0^T (e^{-|t-s|B} - e^{-(t+s)B}) \frac{1}{2}(f(s) - f(T-s))ds,$$

$$E(t) = (I - e^{-2TB})^{-1} \left\{ \left[e^{-tB} - e^{-(2T-t)B} + e^{-(T-t)B} - e^{-(T+t)B} \right] E(0) \right. \tag{3.19}$$

$$\left. -(e^{-(T-t)B} - e^{-(T+t)B})(2B)^{-1} \int_0^T (e^{-(T-s)B} - e^{-(T+s)B}) \frac{1}{2}(f(s) + f(T-s))ds \right\}$$

$$+(2B)^{-1} \int_0^T (e^{-|t-s|B} - e^{-(t+s)B}) \frac{1}{2}(f(s) + f(T-s))ds$$

give solutions of problems (3.13) and (3.12), respectively. Applying formula (3.18), we get

$$O'(T) = (I - e^{-2TB})^{-1} \tag{3.20}$$

$$\times\left[-B\left(I+e^{-TB}\right)^{2}\frac{\varphi}{2}-\int_{0}^{T}(e^{-(T-s)B}-e^{-(T+s)B})\frac{1}{2}\left(f(s)-f(T-s)\right)ds\right]$$

$$=(I-e^{-2TB})^{-1}\left\{-B\left(I+e^{-TB}\right)^{2}\frac{\varphi}{2}-\left(I-e^{-TB}\right)^{2}B^{-1}\frac{1}{2}\left(f(T)-f(0)\right)\right.$$

$$\left.-\int_{0}^{T}(e^{-(T-s)B}-e^{-(T+s)B})\frac{1}{2}\left(f(s)+f(T-s)-f(T)+f(0)\right)ds\right\}.$$

Using the triangle inequality, the estimates (2.1), (2.2), and the definition of the spaces E_α, we get

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}BO'(T)\right\|_{E} \tag{3.21}$$

$$\leq\|(I-e^{-2TB})^{-1}\|_{E\to E}\left\{\|\frac{1}{2}\left(I+e^{-TB}\right)^{2}\|_{E\to E}\lambda^{1-\alpha}\left\|Be^{-\lambda B}A\varphi\right\|_{E}\right.$$

$$+\|(I-e^{-2TB})^{-1}\|_{E\to E}\left\{\|\frac{1}{2}\left(I+e^{-TB}\right)^{2}\|_{E\to E}\right.$$

$$\times\lambda^{1-\alpha}\left\|Be^{-\lambda B}\left(A\varphi-f(0)+f(T)\right)\right\|_{E}+\lambda^{1-\alpha}$$

$$\times\int_{0}^{T}\|2\left(I-e^{-2sB}\right)\|_{E\to E}\left\|B^{2}e^{-(\lambda+2T-s)B}\frac{1}{2}\left(f(s)+f(T-s)-f(0)+f(T)\right)\right\|_{E}ds$$

$$\leq M_1\left\|A\varphi-f(0)+f(T)\right\|_{E_\alpha}+M_2\lambda^{1-\alpha}\int_{0}^{T}\frac{ds}{(\lambda+2T-s)\,(T-s)^{1-\alpha}}\|f\|_{C^{\alpha}(E)}$$

$$\leq M_1\left\|A\varphi-f(0)+f(T)\right\|_{E_\alpha}+M_3\frac{1}{\alpha\,(1-\alpha)}\|f\|_{C^{\alpha}(E)},$$

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}BO'(T)\right\|_{E} \tag{3.22}$$

$$\leq\|(I-e^{-2TB})^{-1}\|_{E\to E}\left\{\|\frac{1}{2}\left(I+e^{-TB}\right)^{2}\|_{E\to E}\lambda^{1-\alpha}\left\|Be^{-\lambda B}A\varphi\right\|_{E}\right.$$

$$+\lambda^{1-\alpha}\int_{0}^{T}\|\frac{1}{2}\left(I-e^{-2sTB}\right)\|_{E\to E}\left\|B^{2}e^{-(\lambda+(T-s))B}\frac{1}{2}\left(f(s)+f(T-s)\right)\right\|_{E}ds$$

$$\leq M_1\left\|A\varphi\right\|_{E_\alpha}+M_2\lambda^{1-\alpha}\int_{0}^{T}\frac{ds}{(\lambda+T-s)\,(T-s)^{1-\alpha}}\|f\|_{C(E_\alpha)}$$

$$\leq M_1\left\|A\varphi\right\|_{E_\alpha}+M_3\frac{1}{\alpha\,(1-\alpha)}\|f\|_{C(E_\alpha)}.$$

Applying formula (3.19) and condition $-E'(0)+\mu E(0)=-O'(T)+\frac{1}{2}\mu\varphi+\psi$, we get

$$E(0)=\left(B+\mu-(B-\mu)\,e^{-TB}\right)^{-1}$$

$$\times\left\{-(I-e^{-TB})^{-1}\int_{0}^{T}e^{-(2T-s)B}\frac{1}{2}\left(f(s)+f(T-s)\right)ds\right.$$

$$+(I-e^{-TB})^{-1}\int_{0}^{T}e^{-B}\frac{1}{2}\left(f(s)+f(T-s)-f(0)-f(T)\right)ds$$

$$+(I+e^{-TB})\left(-O'(T)+\frac{1}{2}\mu\varphi+\psi+B^{-1}\frac{1}{2}\left(f(0)+f(T)\right)\right)\bigg\}.$$

Then,

$$AE(0)-\frac{1}{2}\left(f(0)+f(T)\right)=B\left(B+\mu-\left(B-\mu\right)e^{-TB}\right)^{-1}$$

$$\times\bigg\{-(I-e^{-TB})^{-1}B\int_0^T e^{-(2T-s)B}\frac{1}{2}\left(f(s)+f(T-s)\right)ds$$

$$+(I-e^{-TB})^{-1}B\int_0^T e^{-B}\frac{1}{2}\left(f(s)+f(T-s)-f(0)-f(T)\right)ds$$

$$+B(I+e^{-TB})\left(-O'(T)+\frac{1}{2}\mu\varphi+\psi+\frac{1}{2}\left(f(0)+f(T)\right)\right)$$

$$-B^{-1}\left(B+\mu-\left(B-\mu\right)e^{-TB}\right)\frac{1}{2}\left(f(0)+f(T)\right)\bigg\}$$

$$=B\left(B+\mu-\left(B-\mu\right)e^{-TB}\right)^{-1}$$

$$\times\bigg\{-(I-e^{-TB})^{-1}B\int_0^T e^{-(2T-s)B}\frac{1}{2}\left(f(s)+f(T-s)\right)ds$$

$$+(I-e^{-TB})^{-1}B\int_0^T e^{-B}\frac{1}{2}\left(f(s)+f(T-s)-f(0)-f(T)\right)ds$$

$$+(I+e^{-TB})-BO'(T)+\frac{1}{2}\mu\left(A\varphi-f(0)+f(T)\right)$$

$$+B\psi-\mu B^{-1}f(T)+B^{-1}\left(B-\mu\right)e^{-TB}\frac{1}{2}\left(f(0)+f(T)\right)\bigg\}.$$

Using the triangle inequality, the estimates (2.1), (2.2),(3.22), (3.21), and the definition of the spaces E_α, we get

$$\left\|AE(0)-\frac{1}{2}\left(f(0)+f(T)\right)\right\|_{E_\alpha}\leq M_3\frac{1}{\alpha\left(1-\alpha\right)}\left\|f\right\|_{C^\alpha(E)} \tag{3.23}$$

$$+M_4\left[\left\|A\varphi-f\left(0\right)+f(T)\right\|_{E_\alpha}+\left\|B\psi\right\|_{E\alpha}\right],$$

$$\left\|AE(0)\right\|_{E_\alpha}\leq M_1(\mu)\left[\left\|A\varphi\right\|_{E_\alpha}+\left\|B\psi\right\|_{E_\alpha}\right]+M_3(\mu)\frac{1}{\alpha\left(1-\alpha\right)}\left\|f\right\|_{C(E_\alpha)}. \tag{3.24}$$

Combining the estimates (3.14), (3.16), and (3.23), we get the coercive inequalities (3.6) and (3.15), (3.17), and (3.24), we get the coercive inequalities (3.9). Theorems 3.1 and 3.2 are established.

4. Applications of main Theorems 3.1 and 3.2

Finally, we consider the applications of Theorems 3.1 and 3.2 to the elliptic equations.
First, we consider the boundary value problem for the two dimensional elliptic equations

$$\begin{cases} -\frac{\partial^2 u}{\partial y^2}-\left(a(x)u_x\right)_x+\delta u=f(y,x),\ 0<y<T,\ 0<x<l,\\ u(0,x)=u(T,x)+\varphi(x),u_y(T,x)+\mu u(T,x)=\psi(x),\ 0\leq x\leq l,\\ u(y,0)=u(y,l),\ \ u_x(y,0)=u_x(y,l),\ \ 0\leq y\leq T, \end{cases} \tag{4.1}$$

where $\delta > 0$ is a sufficiently large number and $a(x), \varphi(x), \psi(x)$ and $f(t,x)$ are given smooth functions and they satisfy every compatibility conditions which guarantees the problem (4.1) has a smooth solution $u(t,x)$. We will assume that $a(x) \geq a > 0$ and $a(l) = a(0)$.Here $\delta > 0$ is a sufficiently large number.

We introduce the Banach spaces $C^\beta[0,l]$ $(0 < \beta < 1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite

$$\| \varphi \|_{C^\beta[0,l]} = \| \varphi \|_{C[0,l]} + \sup_{0 \leq x < x+\tau \leq l} \frac{|\varphi(x+\tau) - \varphi(x)|}{\tau^\beta},$$

where $C[0,1]$ is the space of the all continuous functions $\varphi(x)$ defined on $[0,1]$ with the usual norm

$$\| \varphi \|_{C[0,l]} = \max_{0 \leq x \leq l} |\varphi(x)|.$$

It is known that the differential expression

$$A^x v = -a(x) \left(v_x(x)\right)_x + \delta v(x)$$

define a positive operator A^x acting in $C^\beta[0,l]$ with domain $C^{\beta+2}[0,l]$ and satisfying the conditions $v(0) = v(l),\ v_x(0) = v_x(l)$. Therefore, we can replace boundary value problems (4.1) by the abstract boundary value problem (1.2). Using the results of Theorems 3.1 and 3.2, we can obtain the following result.

Theorem 4.1. *Assume that* $\psi(x) \in C^{1+2\alpha+\beta}[0,l]$, $-\left(a(x)\varphi_x(x)\right)_x + \delta\varphi(x) - f(0,x) + f(T,x) \in C^{2\alpha+\beta}[0,l]$, $f(t,x) \in C^\alpha\left(C^\beta[0,l]\right)$. *Then, for the solution of the boundary value problem (4.1) the following coercive inequalities are valid:*

$$\| u \|_{C^{2+\alpha}\left(C^\beta[0,l]\right)} + \| u \|_{C^\alpha\left(C^{2+\beta}[0,l]\right)} + \| u \|_{C\left(C^{2\alpha+\beta}[0,l]\right)} \leq M(\alpha,\delta,\mu) \Big[\| f \|_{C^\alpha\left(C^\beta[0,l]\right)}$$

$$+ ||\psi||_{C^{1+2\alpha+\beta}[0,l]} + || - \left(a(\cdot)\varphi_x(\cdot)\right)_x + \delta\varphi(\cdot) - f(0,\cdot) + f(T,\cdot)||_{C^{2\alpha+\beta}[0,l]} \Big],$$

$$\| u \|_{C^2\left(C^{2\alpha+\beta}[0,l]\right)} + \| u \|_{C\left(C^{2+2\alpha+\beta}[0,l]\right)}$$

$$\leq M(\alpha,\delta,\mu) \Big[\| f \|_{C\left(C^{2\alpha+\beta}[0,l]\right)} + ||\psi||_{C^{1+2\alpha+\beta}[0,l]} + ||\varphi||_{C^{2+2\alpha+\beta}[0,l]} \Big], \quad 0 < 2\alpha + \beta < 1.$$

Here $M(\alpha,\delta,\mu)$ is independent of $\varphi(x), \psi(x)$ and $f(y,x)$.

Second, we consider the nonlocal boundary value problems for the two dimensional elliptic equations with involution in x

$$\begin{cases} -\frac{\partial^2 u(y,x)}{\partial y^2} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(y,x)}{\partial x}\right) + \delta u(y,x) - \beta\left(\frac{\partial}{\partial x}\left(a(-x)\frac{\partial u(y,-x)}{\partial x}\right)\right) \\ = f(y,x), 0 < y < T, x \in (-l,l), \\ u(0,x) = u(T,x) + \varphi(x), u_y(T,x) + \mu u(T,x) = \psi(x),\ x \in [-l,l], \\ u(y,-l) = u(y,l) = 0, 0 \leq y \leq T, \end{cases} \quad (4.2)$$

where $\delta > 0$ is a sufficiently large number and $a(x), \varphi(x), \psi(x)$ and $f(t,x)$ are given smooth functions and they satisfy every compatibility conditions which guarantees the problem (4.2) has a smooth solution $u(y,x)$.We will assume that $a \geq a(x) = a(-x) \geq \sigma > 0$, $\sigma - a|\beta| \geq 0$.

Theorem 4.2. *Assume that* $\psi(x)=0, -\left(a(x)\varphi_x(x)\right)_x-\beta\left(a(-x)\varphi_x(-x)\right)_x+\delta\varphi(x)-f(0,x)+f(T,x)=0, x\in[-l,l],\ f(t,x)\in C^{\alpha}\left(L_2[-l,l]\right)$. *Then, for the solution of the boundary value problem (4.2) the following coercive inequalities are valid:*

$$\| u \|_{C^{2+\alpha}(L_2[-l,l])}+\| u \|_{C^{\alpha}\left(W_2^2[-l,l]\right)}\leq M(\alpha,\delta,\mu,\sigma)\| f \|_{C^{\alpha}(L_2[-l,l])},$$

where $M(\alpha,\delta,\mu,\sigma)$ *is independent of* $f(y,x)$.

The proof of Theorem 4.2 is based on the abstract Theorem 3.1, on the self-adjointness and positivity in $L_2\left[-l,l\right]$ of a differential operator A^x defined by the formula (see,[6])

$$A^x v(x)=-\left(a(x)v_x(x)\right)_x-\beta\left(a(-x)v_x\left(-x\right)\right)_x+\delta v\left(x\right)$$

with the domain $D(A^x)=\left\{u\in W_2^2\left[-l,l\right]: u\left(-l\right)=u\left(l\right)=0\right\}$.

Third, let Ω be the unit open cube in the $n-$dimensional Euclidean space $\mathbf{R}^n$ $(0<x_k<1, 1\leq k\leq n)$ with boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,T]\times\Omega$ we consider the mixed boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u(y,x)}{\partial y^2}-\sum\limits_{r=1}^{n}\alpha_r(x)\frac{\partial^2 u(y,x)}{\partial x_r^2}+\delta u(y,x)=f(y,x), \\ x=(x_1,\ldots,x_n)\in\Omega, 0<y<T, \\ u(0,x)=u(T,x)+\varphi(x), u_y(T,x)+\mu u(T,x)=\psi(x),\ x\in\overline{\Omega}, \\ u(y,x)=0,\ x\in S, \end{cases} \tag{4.3}$$

where $\delta>0$ is a sufficiently large number and $a_r(x),\varphi(x),\psi(x)$ and $f(y,x)$ are given smooth functions and they satisfy every compatibility conditions which guarantees the problem (4.3) has a smooth solution $u(t,x)$. We will assume that $a_r(x)\geq a_0>0$.

We introduce the Banach spaces $C_{01}^{\beta}(\overline{\Omega})$ $(\beta=(\beta_1,\ldots,\beta_n), 0<x_k<1, k=1,\ldots,n)$ of all continuous functions satisfying a Hölder condition with the indicator $\beta=(\beta_1,\ldots,\beta_n), \beta_k\in(0,1), 1\leq k\leq n$ and with weight $x_k^{\beta_k}(1-x_k-h_k)^{\beta_k}, 0\leq x_k<x_k+h_k\leq 1, 1\leq k\leq n$ which equipped with the norm (see, e.g., [34])

$$\begin{aligned} \| f \|_{C_{01}^{\beta}(\overline{\Omega})}=&\| f \|_{C(\overline{\Omega})} \\ &+\sup_{0\leq x_k<x_k+h_k\leq 1, 1\leq k\leq n}|f(x_1,\ldots,x_n)-f(x_1+h_1,\ldots,x_n+h_n)| \\ &\times\prod_{k=1}^{n}h_k^{-\beta_k}x_k^{\beta_k}(1-x_k-h_k)^{\beta_k}, \end{aligned}$$

where $C(\overline{\Omega})$-is the space of the all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{C(\overline{\Omega})}=\max_{x\in\overline{\Omega}}|f(x)|.$$

It is known that the differential expression

$$A^x v=-\sum_{r=1}^{n}\alpha_r(x)\frac{\partial^2 v(y,x)}{\partial x^2}+\delta v(y,x)$$

defines a positive operator A^x acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x)\subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $v=0$ on S. Therefore, we can replace boundary value problems (4.3) by the abstract boundary value problems (1.2). Using the results of Theorems 3.1, we can obtain that

Theorem 4.3. *Assume that*

$$\psi(x)=0,-\sum_{r=1}^{n}\alpha_r(x)\frac{\partial^2\varphi(x)}{\partial x^2}+\delta\varphi(x)-f(0,x)+f(T,x)=0,x\in\overline{\Omega},f(y,x)\in C^{\alpha}\left(C_{01}^{\beta}(\overline{\Omega})\right).$$

Then for the solution of the boundary value problem (4.3) the following coercive inequality is valid:

$$\|u\|_{C^{2+\alpha}\left(C_{01}^{\beta}(\overline{\Omega})\right)}+\sum_{r=1}^{n}\left\|\frac{\partial^2 u}{\partial x_r^2}\right\|_{C^{\alpha}\left(C_{01}^{\beta}(\overline{\Omega})\right)}\le M(\alpha,\delta,\mu)\|f\|_{C^{\alpha}\left(C_{01}^{\beta}(\overline{\Omega})\right)},$$

$$0<\alpha<1,\beta=\{\beta_1,\cdot\cdot\cdot,\beta_n\},0<\beta_k<1,1\le k\le n,$$

where $M(\alpha,\delta,\mu)$ is independent of $f(y,x)$.

Fourth, we consider the boundary value problem on the range

$$\{0\le y\le T,x\in\mathbf{R}^n\}$$

for $2m-$order multidimensional elliptic equations

$$\begin{cases} -\frac{\partial^2 u}{\partial y^2}+\sum\limits_{|r|=2m}a_r(x)\frac{\partial^{|\tau|}u}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}+\delta u(y,x)=f(y,x),\\ 0<y<T,\ x,r\in\mathbf{R}^n,|r|=r_1+\cdots+r_n,\\ u(0,x)=u(T,x)+\varphi(x),u_y(T,x)+\mu u(T,x)=\psi(x),x\in\mathbf{R}^n, \end{cases}\tag{4.4}$$

where $\delta>0$ is a sufficiently large number and $a_r(x),\varphi(x),\psi(x)$ and $f(y,x)$ are given smooth functions and they satisfy every compatibility conditions which guarantees the problem (4.4) has a smooth solution $u(t,x)$. We will assume that $a_r(x)\ge a_0>0$. We will assume that the symbol

$$B^x(\xi)=\sum_{|r|=2m}a_r(x)\left(i\xi_1\right)^{r_1}...\left(i\xi_n\right)^{r_n},\xi=(\xi_1,\cdot\cdot\cdot,\xi_n)\in R^n$$

of the differential operator of the form

$$B^x=\sum_{|r|=2m}a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}\tag{4.5}$$

acting on functions defined on the space $\mathbf{R}^n$, satisfies the inequalities

$$0<M_1|\xi|^{2m}\le(-1)^m B^x(\xi)\le M_2|\xi|^{2m}<\infty$$

for $\xi\neq 0$. The problem (4.4) has a unique smooth solution. This allows us to reduce the boundary value problem (4.4) to the boundary value problem (1.2) in a Banach space $E=C^{\beta}(R^n)$ of all continuous bounded functions defined on $\mathbf{R}^n$ satisfying a Hölder condition with the indicator $\beta\in(0,1)$ with a strongly positive operator $A^x=B^x+\delta I$ defined by (4.5).

Theorem 4.4. *Assume that*

$$\psi(x)\in C^{m+2m\alpha+\beta}(R^n),$$

$$\sum_{|r|=2m}a_r(x)\frac{\partial^{|\tau|}\varphi(x)}{\partial x_1^{r_1}...\partial x_n^{r_n}}+\delta\varphi(x)-f(0,x)+f(T,x)\in C^{2m\alpha+\beta}(R^n),$$

$f(y,x) \in C^{\alpha}\left(C^{\beta}(R^n)\right)$. Then, for the solution of the boundary value problem (4.4) the following coercivity inequalities are satisfied

$$\| u \|_{C^{2+\alpha}\left(C^{\beta}(R^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|} u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C^{\alpha}\left(C^{\beta}(R^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|} u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C\left(C^{2m\alpha+\beta}(R^n)\right)}$$

$$\leq M(\alpha,\delta,\mu) \left[\|f\|_{C^{\alpha}\left(C^{\beta}(R^n)\right)} + \left\| \sum_{|r|=2m} a_r(\cdot) \frac{\partial^{|\tau|}\psi(\cdot)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta\psi(\cdot) - \mu f(T,\cdot) \right\|_{C^{\beta}(R^n)} \right.$$

$$\left. + \left\| \sum_{|r|=2m} a_r(\cdot) \frac{\partial^{|\tau|}\varphi(\cdot)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta\varphi(\cdot) - f(0,\cdot) + f(T,\cdot) \right\|_{C^{2m\alpha+\beta}(R^n)} \right],$$

$$\| u \|_{C^{2}\left(C^{2m\alpha+\beta}(R^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|} u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C\left(C^{2m\alpha+\beta}(R^n)\right)}$$

$$\leq M(\alpha,\delta,\mu) \left[\|f\|_{C\left(C^{2m\alpha+\beta}(R^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|} \varphi}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C^{2m\alpha+\beta}(R^n)} \right.$$

$$\left. + \sum_{|\tau|=m} \left\| \frac{\partial^{|r|} \psi}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C^{2m\alpha+\beta}(R^n)} \right], 0 < 2m\alpha + \beta < 1,$$

where $M(\alpha,\delta,\mu)$ does not depend on $\varphi(x), \psi(x)$ and $f(y,x)$.

The proof of Theorem 4.4 is based on the abstract Theorems 3.1 and 3.2, the positivity of the operator A^x in $C^{\beta}(R^n)$, the structure of the fractional spaces $E_{\alpha}((A^x)^{\frac{1}{2}}, C(R^n))$ and the coercivity inequality for an elliptic operator A^x in $C^{\beta}(R^n)$.

5. Conclusion and future plans

1. In the present paper, the nonlocal boundary value problem with Samarskii–Ionkin condition I for elliptic equations in a Banach space with a positive operator is investigated. The main theorems on well-posedness of this problem are proved. In practice, the coercive stability estimates for solution of four types of nonlocal boundary value problems with Samarskii–Ionkin condition I for elliptic differential equations are proved.
2. Investigate the high order of accuracy for the numerical solution of the nonlocal boundary value problem for elliptic partial differential equations(see,[7]).

3. Investigate the uniform two-step difference schemes and asymptotic formulas for the solution of the nonlocal boundary value perturbation problem

$$\begin{cases} -\varepsilon^2 \frac{d^2 v(t)}{dt^2} + Av(t) = f(t), \ 0 < t < T, \\ u(0) = u(T) + \varphi, u'(T) + \mu u(T) = \psi \end{cases}$$

for a linear elliptic equation in a Banach space E with the positive operator A and with $\varepsilon \in (0,\infty)$ parameter multiplying the highest order derivative term (see, [7]).

6. Acknowledgements

The work was supported by the RUDN Program 5-100 and within the framework of the target program BR05236656 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan. The authors also express their gratitude to the Prof. A.L. Skubachevskii for his useful advice on improving the article.

References

[1] Al-Hammouri A.; The Source Identification Problem for Telegraph-Elliptic Differential Equations. PhD Thesis. Yakın Doğu Üniversitesi. Lefkoşa. Turkiye.–2020.

[2] Avalishvili G., Avalishvili M.; Nonclassical problem with integral boundary conditions for elliptic system. Complex Var. Elliptic Equ.,–2018.–63.–No6.–P.836-853.

[3] Ashyralyev A.; A note on the Bitsadze-Samarskii type nonlocal boundary value problem in a Banach space. Journal of Mathematical Analysis and Applications,–2008.–344.–No1.–P.557-573.

[4] Ashyralyev A., Hamad, A.; On the well-posedness of elliptic equations with nonlocal boundary conditions. TWMS J. Pure Appl. Math.,–2025.–16–No1.–P. 59-70.

[5] Ashyralyev A., Agirseven D., Agarwal R.P.; Stability estimates for delay parabolic differential and difference equations. Appl. Comput. Math.,–2020.–19.–P.175-204.

[6] Ashyralyev A., Sarsenbi A.M.; Well-posedness of an elliptic equation with involution. Electron. J. Differ. Equ.,–2015.–2015. No284.–P.1-8.

[7] Ashyralyev A., Sobolevskii P.E.; New Difference Schemes for Partial Differential Equations. Operator Theory: Advances and Applications .–148.- 443 pages. Birkhauser Verlag. Basel, Boston, Berlin.–2004.

[8] Ashyralyev A., Tetikoğlu Özesenli S. F.; A note on Bitsadze-Samarskii type nonlocal boundary value problems: well-posedness. Numer. Funct. Anal.Optim.,–2013.–34.–No9.–P.939-975.

[9] Ashyralyev A., Pastor J., Piskarev S., Yurtsever H.A.;Well-posed and ill-posed boundary value problems for PDE. Abstract and Applied Analysis.–2015.– 948321.–P.1-63.

[10] Ashyralyev A., Hamad A.; On the nonlocal boundary value problem for the elliptic differential equations with integral type Samarskii-Ionkin conditions. Contemporary Mathematics. Fundamental Directions,–2025.–71.–No1.–P.1-17.

[11] Ashyralyev C.; Stability estimates for solution of Neumann-type overdetermined elliptic problem. Numer. Funct. Anal. Optim.,–2017.–38.–No10.–P.1226-1243.

[12] Ashyralyev C.; Numerical solution to Bitsadze-Samarskii type elliptic overdetermined multipoint NBVP. Boundary Value Problems,–2017.–2017.–No74.–P.1-22.

[13] Ciupaila R., Sapagovas M., Štikonienė O.; Numerical solution of nonlinear elliptic equation with nonlocal condition. Nonlinear Anal. Modelling Control,–2013,–18,–No4.–P.–412–426.

[14] Dukenbayeva A.A.; On a generalised Samarskii–Ionkin type problem for the Poisson equation in the disk. Matematicheskiy Zhurnal,–2018.–18. No1.–P.78–87.

[15] Gasimova Kh. A.; On the Dirichlet problem for a class of non-uniformly elliptic equations with measure data. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics,–2023,–43. No4.–P.72–84.

[16] Il'in V.A., Kritskov L.V.; Properties of spectral expansions corresponding to non-self-adjoint differential operators. J. Math. Sci. (N.Y.),–2003.–116.–No5.–P.3489–3550.

[17] Hamad A.; Fractional Spaces Generated by Positive Operators with the Nonlocal Condition and Their Applications. PhD Thesis. Yakın Doğu Üniversitesi. Lefkoşa. Turkiye.–2019.

[18] Tetikoğlu Özesenli F. S.; High Order of Accuracy Difference Schemes for Bitsadze-Samarskii Problems. PhD Thesis. Fatih University. Istanbul. Turkiye.–2012.

[19] Tetikoğlu F. S.; Fractional Spaces Generated by Positive Differential and Difference Operators with Periodic Conditions and Their Applications. PhD Thesis.Istanbul University.Istanbul. Turkiye.–2015.

[20] Nalbant N.; The Structure of Fractional Spaces Generated by Positive Operators with the Nonlocal Condition and Their Applications. PhD Thesis. Istanbul University. Istanbul. Turkiye.–2016.

[21] Ozturk E.; Nonlocal Boundary Value Problems for Elliptic Differential and Difference Equations. PhD Thesis.Uludağ University. Bursa. Turkiye–2013.

[22] Karachik V.V., Turmetov B.Kh.; On solvability of some nonlocal boundary value problems for biharmonic equation. Mathematica Slovaca,–2020.–P.70.–No2.–P.329–342.

[23] Kirane M., Torebek B.T.; On a nonlocal problem for the Laplace equation in the unit ball with fractional boundary conditions. Math. Methods Appl. Sci.,–2016.–39.–P.1121-1128.

[24] Krein S.G.; Linear Differential Equations in Banach Space. American Mathematical Society.Providence. R.I.–1971. Translated from the Russian by J. M. Danskin, Translations of Mathematical Monographs– 29.

[25] Lunardi A.; Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhauser Verlag. Basel, Boston, Berlin.–1995.

[26] Skubachevskii A.L.; Elliptic Functional Differential Equations and Applications. Birkhauser Verlag. Basel, Boston, Berlin.–1997.

[27] Sadybekov M.A., Yessirkegenov N.A.; Spectral properties of a Laplace operator with Samarskii–Ionkin type boundary conditions in a disk. AIP Conf Proc. –2016.–1759:020139. https://doi.org/10.1063/1.

[28] Sadybekov M.A., Turmetov B.Kh., Torebek B.T.; Solvability of nonlocal boundary-value problems for the Laplace equation in the ball. Electron J Differ Equ.,–2014.–2014.–Article No157.

[29] Sadybekov M.A., Dukenbayeva A.A.; On boundary value problem of the Samarskii–Ionkin type for the Laplace operator in a ball. Kazakh Math. J.,–2020.-20.–N1.–P.84-94.

[30] Sadybekov M.A., Dukenbayeva A.A.; On boundary value problems of the Samarskii–Ionkin type for the Laplace operator in a ball. An International Journal Complex Variables and Elliptic Equations,–2022.–67.–No2.–P.369-383.

[31] Sapagovas M., Griskoniene V., Štikonienė O.; Application of M-matrices theory to numerical investigation of a nonlinear elliptic equation with an integral condition. Nonlinear Anal. Modelling Control,–2017.–22.–No4.–P.489–504.

[32] Sapagovas M., Stikoniene O., Ciupaila R., Joksiene Z.; Convergence of iterative methods for elliptic equations with integral boundary conditions. Electron. J. Differ. Equ.,–2016.–2016. P.1-16.

[33] Shakhmurov V., Musaev H.; Nonlocal separable elliptic equations and applications. TWMS J. Pure Appl. Math.,–2024.–15.–No2.–P.257-268.

[34] Sobolevskii P.E.; Elliptic equations in a Banach space. Differentsial'nye Uravneniya,–1968.–4.–No7.–P.1346–1348.

[35] Turmetov B.Kh., Karachik V.V.; On solvability of some boundary value problems for a biharmonic equation with periodic conditions. Filomat,–2018.–32.–No3.–P.947–953.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.54-67
DOI: 10.29229/uzmj.2025-2-5

Time-optimal control problem for a fourth order parabolic equation with involution in a two-dimensional domain

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. This paper considers the optimal time control problem for a fourth-order parabolic equation with involution in a square domain. The solution with the control function on the border of the considered domain is given. The constraints on the control are determined to ensure that the average value of the solution within the considered domain attains a given value. The initial-boundary problem is solved by the Fourier method, and the control problem under consideration is analyzed with the Volterra integral equation of the first kind. The existence of the control function was proved by the Laplace transform method, and an optimal estimate of the minimum time required for a thin film to reach a given average height is found.

Keywords: fourth order parabolic equation, admissible control, Volterra integral equation, Laplace transform, involution, minimal time, thin film.

MSC (2020): 35K15, 35K35, 58J35.

1. Introduction

Three kinds of optimal control problems: time optimal control problem, target optimal control problem and norm optimal control problem, are important and interesting problems of optimal control theory. The aim of this work is to study the time-optimal control problem for a fourth order parabolic equation. It is known that fourth-order parabolic equations were introduced to describe the epitaxial growth of nanoscale thin films [28]. Therefore, interest in materials science has been increasing in recent years.

A control problem associated with parabolic equations was studied by Friedman [21]. A great deal of developments in the controllability theory of the linear second order parabolic equation were initiated by Fattorini and Russell [22, 23]. Control problems for the infinite-dimensional case were studied by Egorov [20], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions. The time-varying bang-bang property of time optimal controls for heat equation and its applications is studied in [6].

Initially, the problem of time optimal control associated with a second-order parabolic-type equation in a bounded $n-$dimensional domain was studied in [1, 2] and the minimal time estimate for achieving a given average temperature was found. In [3], a mathematical model of thermocontrol processes was studied.

Control problem for a second-order parabolic equation in a bounded two-dimensional domain with a Dirichlet boundary condition is studied in [24]. In these article, the existence of the control function was proved. Boundary control problems related to fast heating of the rod when the conductivity of the rod is different were studied in [10, 11] and the existence of an admissible control function was proved. Similar control problems were studied in work [12] for equations of the second order parabolic type with different boundary conditions.

The monographs [25, 31] of Lions and Fursikov provide a lot of information about optimal control problems. General numerical optimization and control problems associated with second-order parabolic equations have been studied in many publications such as [4]. [32] studies some practical problems for control problems related to parabolic equations of the second order.

The control problems for the pseudoparabolic equation in the bounded domain were studied in works [14, 15, 16] and the existence of an admissible control function was proved using the Laplace transform method.

The optimal time control problem for a fourth-order parabolic equation in a multidimensional domain is studied in [13]. In this work, an estimate of the optimal time to reach an average thickness of a thin film is found. In [26], the null boundary control problem associated with a fourth-order parabolic equation in a one-dimensional bounded domain was considered by the method of reducing the control problem to well-posed problems proposed by Guo and Littman [27]. In [40], the null interior controllability for a fourth order parabolic equation was studied. The method they used is based on Lebeau-Rabbiano inequality. Further research results on the global dynamic behavior of solutions associated with fourth-order parabolic equations for the epitaxial thin film model were studied by Chen [7]. In [17], the boundary control problem for a fourth-order parabolic equation in a bounded one-dimensional domain is studied. Optimal time problems for the fourth-order parabolic equation in the two-dimensional domain are studied in works [18, 19].

In recent years, there has also been a growing interest in the study of mixed problems for parabolic-type equations involving involution. An inverse problems for equations of parabolic type with involution are studied in work [35, 36]. In [33], a boundary value problem for the heat equation associated with involution in a one-dimensional domain is studied. Many boundary value problems for parabolic type equations with involution were studied in works [5, 29]. Boundary problems for fourth-order parabolic equation with involution is studied in work [30].

The solution of some inverse problems for the nonlocal analogue of the fourth-order parabolic equation when the domain is a multidimensional parallelepiped was studied in [37]. The inverse problem for a fractional-order parabolic equation involving a nonlocal biharmonic operator in a two-dimensional domain is studied in detail in [34].

In the present paper, the time optimal control problem for a fourth-order parabolic equation with involution is considered. The difference of this work from the previous works is that in this problem, the time optimal control problem for the fourth-order parabolic type equation related to involution is studied. Section 2 presents the time-optimal problem and the main theorem. In Section 3, the control problem studied in this paper is reduced to the main integral equation by the Fourier method, which is the Volterra integral equation of the first type. In Section 4, the existence of a solution to the main integral equation is proved using the Laplace transform method. In Section 5, the minimum time for a thin film to reach a given average thickness was estimated.

2. Statement of problem

In the present paper, we consider the fourth order parabolic equation with involution in the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi, \quad 0 < y < \pi\}$

$$\frac{\partial}{\partial t}u(x,y,t) + \frac{\partial^4}{\partial x^4}u(x,y,t) + \frac{\partial^4}{\partial y^4}u(x,y,t) +$$

$$+ \varepsilon_1 \frac{\partial^4}{\partial x^4}u(\pi - x, y, t) + \varepsilon_2 \frac{\partial^4}{\partial y^4}u(x, \pi - y, t) = 0, \quad (x, y, t) \in \Omega_T := \Omega \times (0, \infty), \tag{2.1}$$

with boundary value conditions

$$u(0, y, t) \; = \; \varphi(y)\, \nu(t), \quad u(\pi, y, t) \; = \; 0, \quad u(x, 0, t) \; = \; 0, \quad u(x, \pi, t) \; = \; 0, \tag{2.2}$$

and

$$u_{xx}(0, y, t) \; = \; 0, \quad u_{xx}(\pi, y, t) \; = \; 0, \quad u_{yy}(x, 0, t) \; = \; 0, \quad u_{yy}(x, \pi, t) \; = \; 0, \tag{2.3}$$

and initial value condition

$$u(x, y, 0) \;=\; 0, \quad 0 \leq x, y \leq \pi, \tag{2.4}$$

where ε_i $(i = 1, 2)$ are nonzero real numbers such that $|\varepsilon_i| < 1$, $\varphi(y)$ is a given function and $\nu(t)$ is the control function.

In what follows, by $\overline{\mathbb{R}}_+$ we denote the nonnegative half-line, $\overline{\mathbb{R}}_+ = \{t \in \mathbb{R} : t \geq 0\}$. Assume $M > 0$ is given constant. It is called that the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ is *admissible*, if it fulfills the conditions $\nu(0) = 0$ and $|\nu(t)| \leq M$ on the half-line $\overline{\mathbb{R}}_+$.

Differential equations with modified arguments are equations in which the unknown function and its derivatives are evaluated with modifications of time or space variables; such equations are called, in general, functional differential equations. Among such equations, one can single out, equations with involutions [8].

Definition 2.1. ([9, 39]) A function $f(x) \not\equiv x$ maps bijectively a set of real numbers D, such that

$$f(f(x)) \;=\; x \quad \text{or} \quad f^{-1}(x) \;=\; f(x),$$

is called an involution on D.

It can be seen that equation (2.1) for $\varepsilon_i = 0$ $(i = 1, 2)$ is a classical fourth order parabolic equation. If $\varepsilon_i \neq 0$, equation (2.1) relates the values of the second derivatives at two different points and becomes a nonlocal equation. It is known that boundary control problems for the fourth order parabolic equation in the case $\varepsilon_i = 0$ were studied in details in work [19].

Now we present the following time optimal control problem.

Time Optimal Problem. *For a given constant $\theta > 0$ problem consist looking for the minimal value of $T \geq 0$, such that for $t \geq T$ the solution $u(x, y, t)$ of the initial-boundary value problem* (2.1)-(2.4) *with some admissible control $\nu(t)$ exists and for some $T_1 > T$ satisfies the equation*

$$\int\limits_0^\pi \int\limits_0^\pi u(x, y, t)\, dy\, dx = \theta, \quad T \leq t \leq T_1, \tag{2.5}$$

where function $u(x, y, t)$ describes the height of the growth interface at the spatial point $(x, y) \in \Omega$ at time $t \geq 0$, and θ is the average value of the growth interface height of the thin film.

Consider the following spectral problem

$$\frac{\partial^4}{\partial x^4}U(x,y) + \frac{\partial^4}{\partial y^4}U(x,y) + \varepsilon_1 \frac{\partial^4}{\partial x^4}U(\pi - x, y) + \varepsilon_2 \frac{\partial^4}{\partial y^4}U(x, \pi - y) = \lambda\, U(x,y), \quad (x,y) \in \Omega, \tag{2.6}$$

with boundary value conditions

$$U(0, y) = U(\pi, y) = 0, \quad U_{xx}(0, y) = U_{xx}(\pi, y) = 0,$$

and

$$U(x, 0) = U(x, \pi) = 0, \quad U_{yy}(x, 0) = U_{yy}(x, \pi) = 0, \quad (x, y) \in \partial\Omega,$$

where ε_i $(i = 1, 2)$ are nonzero real numbers such that $|\varepsilon_i| < 1$. If we say $U(x, y) = X(x)\, Y(y)$ in Eq. (2.6), we have the following spectral problems.

Spectral problem 1.

$$X''''(x) + \varepsilon_1 X''''(\pi - x) \;=\; \mu_1 X(x), \quad 0 < x < \pi,$$

$$X(0) = X(\pi) = 0, \quad X''(0) = X''(\pi) = 0, \quad 0 \leq x \leq \pi.$$

Spectral problem 2.

$$Y''''(y) + \varepsilon_2 Y''''(\pi - y) \;=\; \mu_2 Y(y), \quad 0 < y < \pi,$$

$$Y(0) = Y(\pi) = 0, \quad Y''(0) = Y''(\pi) = 0, \quad 0 \le y \le \pi.$$

We can see the solutions of the above spectral problems 1 and 2 in the works of [30, 34]. Then, from spectral problems 1 and 2 we have the following eigenvalues:

$$\lambda_{m,n,1} = (1 - \varepsilon_1)\, 16\, m^4 + (1 - \varepsilon_2)\, 16\, n^4, \quad m, n \in \mathbb{N}, \tag{2.7}$$

$$\lambda_{m,n,2} = (1 - \varepsilon_1)\, 16\, m^4 + (1 + \varepsilon_2)\, (2n + 1)^4, \quad m \in \mathbb{N}, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{2.8}$$

$$\lambda_{m,n,3} = (1 + \varepsilon_1)\, (2m + 1)^4 + (1 - \varepsilon_2)\, 16\, n^4, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N}, \tag{2.9}$$

$$\lambda_{m,n,4} = (1 + \varepsilon_1)\, (2m + 1)^4 + (1 + \varepsilon_2)\, (2n + 1)^4, \quad m, n \in \mathbb{N}_0, \tag{2.10}$$

and we have the following eigenfunctions

$$U_{m,n,1}(x, y) = \sin 2mx\, \sin 2ny, \quad m, n \in \mathbb{N},$$

$$U_{m,n,2}(x, y) = \sin 2mx\, \sin(2n + 1)y, \quad m \in \mathbb{N}, \quad n \in \mathbb{N}_0,$$

$$U_{m,n,3}(x, y) = \sin(2m + 1)x\, \sin 2ny, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N},$$

and

$$U_{m,n,4}(x, y) = \sin(2m + 1)x\, \sin(2n + 1)y, \quad m, n \in \mathbb{N}_0.$$

Assume, the given function $\varphi \in C^5([0, \pi])$ satisfies the conditions

$$\varphi(0) = \varphi(\pi) = \varphi^{(2)}(0) = \varphi^{(2)}(\pi) = 0, \quad \varphi_n \ge 0, \tag{2.11}$$

where

$$\varphi_n = \frac{2}{\pi} \int_0^{\pi} \varphi(y)\, \sin ny\, dy, \quad n = 1, 2, \ldots. \tag{2.12}$$

We set

$$\Phi_{mn} \;=\; \frac{8\,(1 + \varepsilon_1)}{\pi}\, \frac{(2m + 1)^2\, \varphi_{2n+1}}{2n + 1}, \quad m, n = 0, 1, \ldots, \tag{2.13}$$

and

$$\lambda_{0,0,4} = 2 + \varepsilon_1 + \varepsilon_2 = \lambda_0, \quad \Phi_{00} = \frac{8\,(1 + \varepsilon_1)\, \varphi_1}{\pi} = \Phi_0,$$

where φ_n is defined by (2.12).

Theorem 2.2. *Let*

$$0 < \theta < \frac{\Phi_0\, M}{\lambda_0}.$$

Set

$$T_0 \;=\; -\frac{1}{\lambda_0} \ln\!\left(1 - \frac{\theta\, \lambda_0}{\Phi_0\, M}\right).$$

Then a solution T_{min} of the Time optimal problem exists, and the estimate $T_{min} \le T_0$ is vaild.

3. Integral equation for control function

In this section, we derive the basic integral equation to find the control function. We first introduce the classical solution of the initial boundary value problem (2.1)-(2.4).

By the solution of mixed problem (2.1)–(2.4) we mean the function $u(x,y,t)$ expressed in the following form

$$u(x,y,t) \;=\; \frac{\pi-x}{\pi}\,\varphi(y)\,\nu(t) - w(x,y,t), \tag{3.1}$$

where the function $w(x,y,t)$ with the regularity $w(x,y,t)\in C^{4,4,1}_{x,y,t}(\Omega_T)\cap C(\bar{\Omega}_T)$ and $w_{xx}, w_{yy}\in C(\bar{\Omega})$ is the solution to the mixed problem

$$\frac{\partial}{\partial t}w(x,y,t)+\frac{\partial^4}{\partial x^4}w(x,y,t)+\frac{\partial^4}{\partial y^4}w(x,y,t)+\varepsilon_1\,\frac{\partial^4}{\partial x^4}w(\pi-x,y,t)+$$

$$+\varepsilon_2\,\frac{\partial^4}{\partial y^4}w(x,\pi-y,t) \;=\; \varphi(y)\,\frac{\pi-x}{\pi}\,\nu'(t)+\varphi^{(4)}(y)\,\frac{\pi-x}{\pi}\,\nu(t)+\varepsilon_2\,\varphi^{(4)}(\pi-y)\,\frac{\pi-x}{\pi}\,\nu(t),$$

with homogeneous boundary value conditions

$$w(0,y,t) \;=\; w(\pi,y,t) \;=\; 0,\quad w_{xx}(0,y,t) \;=\; w_{xx}(\pi,y,t) \;=\; 0,$$

$$w(x,0,t) \;=\; w(x,\pi,t) \;=\; 0,\quad w_{yy}(x,0,t) \;=\; w_{yy}(x,\pi,t) \;=\; 0,$$

and initial value condition

$$w(x,y,0)=0.$$

Using the condition (2.11) we can write

$$\frac{2}{\pi}\int\limits_0^\pi \varphi^{(4)}(y)\,\sin ny\,dy \;=\; n^4\,\varphi_n,\quad \frac{2}{\pi}\int\limits_0^\pi \varphi^{(4)}(\pi-y)\,\sin ny\,dy \;=\; (-1)^{n+1}\,n^4\,\varphi_n,\quad n\in\mathbb{N},$$

where φ_n is the Fourier coefficient of the function $\varphi(y)$.

We solve the above mixed problem by the Fourier method. Thus, we get (see [38])

$$
\begin{aligned}
w(x,y,t) &= \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\varphi_{2n}}{m}\Bigg(\int_0^t e^{-\lambda_{m,n,1}(t-s)}\,\nu'(s)\,ds\Bigg)\sin 2mx\,\sin 2ny+\\
&+\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\varphi_{2n+1}}{m}\Bigg(\int_0^t e^{-\lambda_{m,n,2}(t-s)}\,\nu'(s)\,ds\Bigg)\sin 2mx\,\sin(2n+1)y+\\
&+\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\varphi_{2n}}{2m+1}\Bigg(\int_0^t e^{-\lambda_{m,n,3}(t-s)}\,\nu'(s)\,ds\Bigg)\sin(2m+1)x\,\sin 2ny+\\
&+\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varphi_{2n+1}}{2m+1}\Bigg(\int_0^t e^{-\lambda_{m,n,4}(t-s)}\,\nu'(s)\,ds\Bigg)\sin(2m+1)x\,\sin(2n+1)y+\\
&+\frac{16}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n^4\,\varphi_{2n}\,(1-\varepsilon_2)}{m}\Bigg(\int_0^t e^{-\lambda_{m,n,1}(t-s)}\,\nu(s)\,ds\Bigg)\sin 2mx\,\sin 2ny+\\
&+\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^4\,\varphi_{2n+1}\,(1+\varepsilon_2)}{m}\Bigg(\int_0^t e^{-\lambda_{m,n,2}(t-s)}\,\nu(s)\,ds\Bigg)\sin 2mx\,\sin(2n+1)y+\\
&+\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^4\,\varphi_{2n+1}\,(1+\varepsilon_2)}{2m+1}\Bigg(\int_0^t e^{-\lambda_{m,n,4}(t-s)}\,\nu(s)\,ds\Bigg)\sin(2m+1)x\,\sin(2n+1)y+\\
&+\frac{32}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{n^4\,\varphi_{2n}\,(1-\varepsilon_2)}{2m+1}\Bigg(\int_0^t e^{-\lambda_{m,n,3}(t-s)}\,\nu(s)\,ds\Bigg)\sin(2m+1)x\,\sin 2ny,
\end{aligned}
\tag{3.2}
$$

where $\lambda_{m,n,i}$, $(i=\overline{1,4})$ are defined by (2.7)-(2.10), respectively.

By (3.1) and (3.2), we have the solution of the initial-boundary problem (2.1)–(2.4)

$$
\begin{aligned}
u(x,y,t) &= \frac{\pi-x}{\pi}\,\varphi(y)\,\nu(t)-\\
&-\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\varphi_{2n}}{m}\Bigg(\int_0^t e^{-\lambda_{m,n,1}(t-s)}\,\nu'(s)\,ds\Bigg)\sin 2mx\,\sin 2ny-\\
&-\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\varphi_{2n+1}}{m}\Bigg(\int_0^t e^{-\lambda_{m,n,2}(t-s)}\,\nu'(s)\,ds\Bigg)\sin 2mx\,\sin(2n+1)y-\\
&-\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\varphi_{2n}}{2m+1}\Bigg(\int_0^t e^{-\lambda_{m,n,3}(t-s)}\,\nu'(s)\,ds\Bigg)\sin(2m+1)x\,\sin 2ny-\\
&-\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varphi_{2n+1}}{2m+1}\Bigg(\int_0^t e^{-\lambda_{m,n,4}(t-s)}\,\nu'(s)\,ds\Bigg)\sin(2m+1)x\,\sin(2n+1)y-\\
&-\frac{16}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n^4\,\varphi_{2n}\,(1-\varepsilon_2)}{m}\Bigg(\int_0^t e^{-\lambda_{m,n,1}(t-s)}\,\nu(s)\,ds\Bigg)\sin 2mx\,\sin 2ny-
\end{aligned}
$$

$$-\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^4\,\varphi_{2n+1}\,(1+\varepsilon_2)}{m}\left(\int\limits_0^t e^{-\lambda_{m,n,2}(t-s)}\,\nu(s)\,ds\right)\,\sin 2mx\,\sin(2n+1)y-$$

$$-\frac{32}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{n^4\,\varphi_{2n}\,(1-\varepsilon_2)}{2m+1}\left(\int\limits_0^t e^{-\lambda_{m,n,3}(t-s)}\,\nu(s)\,ds\right)\,\sin(2m+1)x\,\sin 2ny-$$

$$-\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^4\,\varphi_{2n+1}\,(1+\varepsilon_2)}{2m+1}\left(\int\limits_0^t e^{-\lambda_{m,n,4}(t-s)}\,\nu(s)\,ds\right)\,\sin(2m+1)x\,\sin(2n+1)y.$$

Using the solution of the mixed problem (2.1)-(2.4) and the condition (2.5) we can write

$$\phi(t)=\nu(t)\int\limits_0^\pi\int\limits_0^\pi\frac{\pi-x}{\pi}\,\varphi(y)\,dy\,dx-$$
$$-\frac{8}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varphi_{2n+1}}{(2m+1)^2\,(2n+1)}\int\limits_0^t e^{-\lambda_{m,n,4}(t-s)}\,\nu'(s)\,ds-$$
$$-\frac{8}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^3\,\varphi_{2n+1}\,(1+\varepsilon_2)}{(2m+1)^2}\int\limits_0^t e^{-\lambda_{m,n,4}(t-s)}\,\nu(s)\,ds, \tag{3.3}$$

where $\phi(t)=\theta$ for $t\in[T,T_1]$.

According to Parseval equality, we get

$$\int\limits_0^\pi\int\limits_0^\pi\varphi(y)\,\frac{\pi-x}{\pi}\,dx\,dy\;=\;\frac{8}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varphi_{2n+1}}{(2m+1)^2\,(2n+1)}. \tag{3.4}$$

Using the definition of the admissible control $\nu(t)$ and (3.3), (3.4), we can write

$$\phi(t)\;=\;\frac{8\,(1+\varepsilon_1)}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+1)^2\,\varphi_{2n+1}}{2n+1}\int\limits_0^t e^{-\lambda_{m,n,4}(t-s)}\,\nu(s)\,ds, \tag{3.5}$$

where $\lambda_{m,n,4}$ is defined by (2.10).

Let us introduce the function

$$K(t)\;=\;\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi_{mn}\,e^{-\lambda_{m,n,4}t},\quad t>0, \tag{3.6}$$

where Φ_{mn} is defined by (2.13).

Then equality (3.5) takes the form

$$\int\limits_0^t K(t-s)\,\nu(s)\,ds\;=\;\phi(t),\quad t>0, \tag{3.7}$$

where function $\phi(t)=\theta$ for $t\in[T,T_1]$.

For any $M_0 > 0$, we denote $W(M_0)$ the set of functions $\phi \in W_2^2(-\infty,+\infty)$ which satisfy the conditions

$$\|\phi\|_{W_2^2(\mathbb{R}_+)} \le M_0, \quad \phi(t)=0 \quad \text{for} \quad t \le 0.$$

The resulting Volterra integral equation (3.7) is the main equation for admissible control $\nu(t)$. Therefore, we present the following theorem.

Theorem 3.1. *There exists $M_0 > 0$ such that for any function $\phi \in W(M_0)$ the solution $\nu(t)$ of the equation (3.7) exists and satisfies condition $|\nu(t)| \le M$.*

Proposition 3.2. *Let $|\varepsilon_i| < 1$ $(i = 1,2)$ and $\alpha \in (\frac{3}{4}, 1)$. Then the kernel $K(t)$ satisfies the estimate*

$$0 < K(t) \le \frac{C_\alpha}{t^\alpha}, \quad 0 < t \le 1.$$

Proof. According to condition (2.11), φ_n coefficients are non-negative. By (2.10) and (3.6), we have

$$0 < K(t) = \frac{8\,(1+\varepsilon_1)}{\pi}\sum_{m=0}^{\infty}(2m+1)^2\,e^{-(1+\varepsilon_1)(2m+1)^4\,t}\sum_{n=0}^{\infty}\frac{\varphi_{2n+1}}{2n+1}\,e^{-(1+\varepsilon_2)(2n+1)^4\,t} = A(t)\,W(t),$$

where

$$W(t) \;=\; \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\varphi_{2n+1}}{2n+1}\,e^{-(1+\varepsilon_2)(2n+1)^4\,t}, \quad t > 0,$$

and

$$A(t) = (1+\varepsilon_1)\sum_{m=0}^{\infty}(2m+1)^2\,e^{-(1+\varepsilon_1)(2m+1)^4\,t}.$$

For any $t \in (0,T]$, the function $W(t)$ satisfies the inequality

$$0 \;<\; W(T) \;\le\; W(t) \;<\; W(0). \tag{3.8}$$

Let the function $f(t) = t^\alpha e^{-\beta\,t}$ be given, where $\alpha, \beta > 0$. It is clear that the maximum value of this function is reached at the point $t_0 = \frac{\alpha}{\beta}$ and this value is equal to $\frac{\alpha^\alpha}{\beta^\alpha}e^{-\alpha}$. We will use this information in the following evaluation.

For any $\alpha \in (\frac{3}{4}, 1)$, the function $A(t)$ satisfies the estimate

$$\begin{aligned} A(t) &= (1+\varepsilon_1)\sum_{m=0}^{\infty}(2m+1)^2\,e^{-(1+\varepsilon_1)\,(2m+1)^4\,t} = \\ &= \frac{1+\varepsilon_1}{t^\alpha}\sum_{m=0}^{\infty}(2m+1)^2\,t^\alpha\,e^{-(1+\varepsilon_1)\,(2m+1)^4\,t} \le \\ &\le \frac{(1+\varepsilon_1)^{1-\alpha}\,\alpha^\alpha\,e^{-\alpha}}{t^\alpha}\sum_{m=0}^{\infty}\frac{(2m+1)^2}{(2m+1)^{4\,\alpha}} \le C_\alpha\,t^{-\alpha}, \end{aligned} \tag{3.9}$$

where

$$\sum_{m=0}^{\infty}\frac{(2m+1)^2}{(2m+1)^{4\,\alpha}} = \sum_{m=0}^{\infty}\frac{1}{(2m+1)^{4\alpha-2}} < +\infty.$$

The proof of proposition follows from (3.8) and (3.9).

□

4. Proof of Theorem 3.1

It is known that the Laplace transform of the function $\nu(t)$ is defined as follows

$$\widetilde{\nu}(p)=\int_0^\infty e^{-pt}\,\nu(t)\,dt, \quad \text{where } \; p=\sigma+i\,\tau, \quad \sigma>0, \quad \tau\in\mathbb{R}.$$

Using the Laplace transform and the integral equation (3.7), we get

$$\widetilde{\phi}(p)=\int_0^\infty e^{-pt}\,dt\int_0^t K(t-s)\,\nu(s)ds=\widetilde{K}(p)\,\widetilde{\nu}(p).$$

Then,

$$\widetilde{\nu}(p)=\frac{\widetilde{\phi}(p)}{\widetilde{K}(p)},$$

and we may write

$$\nu(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\widetilde{\phi}(p)}{\widetilde{K}(p)}\,e^{pt}dp=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\phi}(\sigma+i\,\tau)}{\widetilde{K}(\sigma+i\,\tau)}\,e^{(\sigma+i\,\tau)t}\,d\tau. \tag{4.1}$$

We present the following lemma, which is necessary to prove the existence of admissible control.

Lemma 4.1. *The following estimate is valid:*

$$|\widetilde{K}(\sigma+i\,\tau)|\geq\frac{C_\sigma}{\sqrt{1+\tau^2}}, \quad \sigma>0, \quad \tau\in\mathbb{R},$$

where $C_\sigma>0$ is a constant only depending on σ.

Proof. Using (3.6), we can write

$$\widetilde{K}(p)=\int_0^\infty K(t)\,e^{-pt}\,dt=\sum_{m,n=0}^\infty\Phi_{mn}\int_0^\infty e^{-(p+\lambda_{m,n,4})t}\,dt=\sum_{m,n=0}^\infty\frac{\Phi_{mn}}{p+\lambda_{m,n,4}},$$

and for $p=\sigma+i\,\tau$

$$\widetilde{K}(\sigma+i\,\tau)=\sum_{m,n=0}^\infty\frac{\Phi_{mn}}{\sigma+\lambda_{m,n,4}+i\,\tau}=\sum_{m,n=0}^\infty\frac{\Phi_{mn}\,(\sigma+\lambda_{m,n,4})}{(\sigma+\lambda_{m,n,4})^2+\tau^2}-i\,\tau\sum_{m,n=0}^\infty\frac{\Phi_{mn}}{(\sigma+\lambda_{m,n,4})^2+\tau^2}$$

$$=\mathrm{Re}\widetilde{K}(\sigma+i\,\tau)+i\,\mathrm{Im}\widetilde{K}(\sigma+i\,\tau),$$

where $\lambda_{m,n,4}$ is defined by (2.10), and

$$\mathrm{Re}\widetilde{K}(\sigma+i\,\tau)=\sum_{m,n=0}^\infty\frac{\Phi_{mn}\,(\sigma+\lambda_{m,n,4})}{(\sigma+\lambda_{m,n,4})^2+\tau^2},$$

$$\mathrm{Im}\widetilde{K}(\sigma+i\,\tau)=-\tau\sum_{m,n=0}^\infty\frac{\Phi_{mn}}{(\sigma+\lambda_{m,n,4})^2+\tau^2}.$$

We can see that the following inequalities hold

$$(\sigma+\lambda_{m,n,4})^2+\tau^2 \le [(\sigma+\lambda_{m,n,4})^2+1](1+\tau^2),$$

and

$$\frac{1}{(\sigma+\lambda_{m,n,4})^2+\tau^2} \ge \frac{1}{1+\tau^2}\,\frac{1}{(\sigma+\lambda_{m,n,4})^2+1}. \tag{4.2}$$

By (**??**), we get the estimates

$$|\mathrm{Re}\widetilde{K}(\sigma+i\,\tau)| = \sum_{m,n=0}^{\infty}\frac{\Phi_{mn}\,(\sigma+\lambda_{m,n,4})}{(\sigma+\lambda_{m,n,4})^2+\tau^2} \ge \frac{1}{1+\tau^2}\sum_{m,n=0}^{\infty}\frac{\Phi_{mn}\,(\sigma+\lambda_{m,n,4})}{(\sigma+\lambda_{m,n,4})^2+1} = \frac{C_{1,\sigma}}{1+\tau^2}, \tag{4.3}$$

and

$$|\mathrm{Im}\widetilde{K}(\sigma+i\,\tau)| = |\tau|\sum_{m,n=0}^{\infty}\frac{\Phi_{mn}}{(\sigma+\lambda_{m,n,4})^2+\tau^2} \ge \frac{|\tau|}{1+\tau^2}\sum_{m,n=0}^{\infty}\frac{\Phi_{mn}}{(\sigma+\lambda_{m,n,4})^2+1} = \frac{C_{2,\sigma}\,|\tau|}{1+\tau^2}, \tag{4.4}$$

where $C_{1,\sigma}$, $C_{2,\sigma}$ as follows

$$C_{1,\sigma} = \sum_{m,n=0}^{\infty}\frac{\Phi_{mn}\,(\sigma+\lambda_{m,n,4})}{(\sigma+\lambda_{m,n,4})^2+1},\quad C_{2,\sigma} = \sum_{m,n=0}^{\infty}\frac{\Phi_{mn}}{(\sigma+\lambda_{m,n,4})^2+1}.$$

By (**??**) and (4.4), we get the required estimate

$$|\widetilde{K}(\sigma+i\,\tau)| \ge \frac{C_\sigma}{\sqrt{1+\tau^2}},\quad \text{where}\quad C_\sigma = \min(C_{1,\sigma}, C_{2,\sigma}). \tag{4.5}$$

□

Consequently, proceeding to the limit as $\sigma \to 0$ from (4.1), we have the equality

$$\nu(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\phi}(i\,\tau)}{\widetilde{K}(i\,\tau)}\,e^{i\,\tau t}\,d\tau. \tag{4.6}$$

Proposition 4.2. [10] *Let $\phi(t) \in W(M_0)$. Then for the imaginary part of the Laplace transform of function $\phi(t)$ the inequality*

$$\int_{-\infty}^{+\infty}|\widetilde{\phi}(i\,\tau)|\sqrt{1+\tau^2}\,d\tau \le C_1\,\|\phi\|_{W_2^2(R_+)},$$

is valid, where $C_1 > 0$ is a constant.

Proposition 4.3. *Let $\phi(t) \in W(M_0)$. Then $\nu \in W_2^1(\mathbb{R}_+)$.*

Proof. By (4.5) and (4.6), we have the estimate

$$\int_{-\infty}^{+\infty}|\widetilde{\nu}(\tau)|^2(1+|\tau|^2)\,d\tau \;=\; \int_{-\infty}^{+\infty}\left|\frac{\widetilde{\phi}(i\,\tau)}{\widetilde{K}(i\,\tau)}\right|^2(1+|\tau|^2)\,d\tau \;\le$$

$$\le C_0\int_{-\infty}^{+\infty}|\widetilde{\phi}(i\,\tau)|^2(1+|\tau|^2)^2\,d\tau \;=\; C_0\|\phi\|^2_{W_2^2(\mathbb{R})},$$

where $C_0 = \min(C_{1,0}, C_{2,0})$ which is defined by (4.5). Next, we can write

$$|\nu(t) - \nu(s)| \ = \ \left| \int_s^t \nu'(\xi)\, d\xi \right| \ \leq \ \|\nu'\|_{L_2} (t-s)^{1/2}.$$

Hence, $\nu \in \operatorname{Lip} \alpha$, where $\alpha = 1/2$.

□

Proof of Theorem 3.1. Using (4.5), (4.6) and Proposition **??**, we have the estimate

$$|\nu(t)| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\widetilde{\phi}(i\,\tau)|}{|\widetilde{K}(i\,\tau)|} d\tau \leq \frac{1}{2\pi C_0} \int_{-\infty}^{+\infty} |\widetilde{\phi}(i\,\tau)| \sqrt{1+\tau^2} d\tau \leq$$

$$\leq \frac{C_1}{2\pi C_0} \|\phi\|_{W_2^2(R_+)} \leq \frac{C_1\, M_0}{2\pi C_0} = M,$$

where M_0 as follows

$$M_0 = \frac{2\pi C_0}{C_1}\, M.$$

□

5. Proof of Theorem 2.2

We consider the integral equation (3.7) for the case $t \in [T, T_1]$

$$\int_0^t K(t-s)\, \nu(s)\, ds \ = \ \theta,$$

where the kernel $K(t)$ is defined by (3.6).

Proposition 5.1. *For the kernel $K(t)$ the following estimate is valid:*

$$K(t) \geq \Phi_0\, e^{-\lambda_0 t}, \quad t > 0.$$

Proof. The proof of this proposition follows from the fact that the kernel $K(t)$ is positive on the half-line $t \geq 0$

□

We introduce the function

$$H(t) = \int_0^t K(t-s)\, ds = \int_0^t K(s)\, ds.$$

It is known $H(0) = 0$ and $H'(t) = K(t) > 0$. The physical meaning of this function is that $H(t)$ is the average thickness of the thin film in the Ω.

Define

$$H^* = \lim_{t \to \infty} H(t) = \int_0^\infty K(s)\, ds.$$

It is known that H^* is finite. Indeed,

$$H^* = \int_0^\infty K(s)\, ds = \sum_{m,n=0}^{\infty} \frac{\Phi_{mn}}{\lambda_{m,n,4}} < +\infty,$$

where Φ_{mn} and $\lambda_{m,n,4}$ are defined by (2.13), (2.10), respectively. In conclusion, we can say that the average thickness of a thin film in the domain Ω cannot be greater than H^*.

Proposition 5.2. *(see [19]) Let $0 < \theta < M\,H^*$. Then there exist $T > 0$ and an admissible control $\nu(t)$ and the following equality is valid:*

$$\int_0^T K(T-s)\,\nu(s)ds = \theta. \tag{5.1}$$

Remark 5.3. It can be seen that the value of T is the solution to the time optimal problem and it is the root of the following equation

$$H(T) = \frac{\theta}{M}. \tag{5.2}$$

Lemma 5.4. *Let be*

$$0 < \theta < \frac{\Phi_0\,M}{\lambda_0}.$$

Then there exists $T > 0$ so that

$$T < -\frac{1}{\lambda_0}\ln\left(1 - \frac{\theta\,\lambda_0}{\Phi_0\,M}\right),$$

and the Eq. (5.2) is fulfilled.

Proof. Using Proposition 5.1, we can write the following inequality

$$H(t) = \int_0^t K(s)\,ds \geq \Phi_0 \int_0^t e^{-\lambda_0 s}ds = \frac{\Phi_0}{\lambda_0}\left(1 - e^{-\lambda_0 t}\right). \tag{5.3}$$

At $t = T_0$ we consider the equation (5.3), then we have the equation

$$\frac{\Phi_0}{\lambda_0}\left(1 - e^{-\lambda_0\,T_0}\right) = \frac{\theta}{M}. \tag{5.4}$$

Thus,

$$T_0 = -\frac{1}{\lambda_0}\ln\left(1 - \frac{\theta\,\lambda_0}{\Phi_0\,M}\right).$$

According to (5.3) and (5.4) we get

$$0 < \frac{\theta}{M} \leq H(T_0).$$

Then it is clear that T $(0 < T < T_0)$ exists, which is a solution to the equation (5.2).

□

The proof of main Theorem 2.2 follows easily from Lemma 5.4.

Acknowledgments. The authors are grateful to Academician Sh.A. Alimov for his valuable comments.

References

[1] Albeverio S., Alimov Sh.A.; On one time-optimal control problem associated with the heat exchange process. Appl. Math. Opt., –2008. –57. –P.58–68.

[2] Alimov Sh.A.; On the null-controllability of the heat exchange process. Eurasian. Math. J., –2011. –2. –P.5-19.

[3] Alimov Sh.A., Ibragimov G.; Time optimal control problem with integral constraint for the heat transfer process. Eurasian Mathematical Journal, –2024. –15. –P.8–22.

[4] Altmüller A., Grüne L.; Distributed and boundary model predictive control for the heat equation. Technical report, University of Bayreuth, Department of Mathematics, –2012. https://doi.org/10.1002/gamm.201210010

[5] Ahmad B., Alsaedi A., Kirane M., Tapdigoglu R.; An inverse problem for space and time fractional evolution equations with an involution perturbation. Quaest. Math, –2017. –40. –P.151-160.

[6] Chen N., Wang Y., Yang D.; Time-varying bang-bang property of time optimal controls for heat equation and its applications. Systems and Control Letters, –2018. –112. –P.18–23. https://doi.org/10.1016/j.sysconle.2017.12.008

[7] Chen Y.; Global dynamical behavior of solutions for finite degenerate fourth-order parabolic equations with mean curvature nonlinearity. Communications in Analysis and Mechanics, –2023. –15. –P.658–694. https://doi.org/10.3934/cam.2023033

[8] Cabada A., Tojo F. A. F.; General Results for Differential Equations with Involutions, In: Differential Equations with Involutions, Atlantis Press, Paris, –2015. –P.17–23. http://dx.doi.org/10.2991/978-94-6239-121-5-2

[9] Carleman T.; Sur la theorie des equations integrales et ses applications, Verhandl. des internat. Mathem. Kongr. I. Zurich., –1932. –P.138–151.

[10] Dekhkonov F.N., Kuchkorov E.I.; On the time-optimal control problem associated with the heating process of a thin rod. Lobachevskii Journal of Mathematics, –2023. –44. –No3. –P.1134–1144. https://doi.org/10.1134/S1995080223030101

[11] Dekhkonov F.N.; Boundary control associated with a parabolic equation. Journal of Mathematics and Computer Science, –2024. –33. –P.146–154. https://doi.org/10.22436/jmcs.033.02.03

[12] Dekhkonov F.N.; Boundary control problem for the heat transfer equation associated with heating process of a rod. Bulletin of the Karaganda University. Mathematics Series, –2023. –110. –P.63–71. https://doi.org/10.31489/2023M2/63-71

[13] Dekhkonov F.N.; On the time-optimal control problem associated with the epitaxial growth of thin film. Uzbek Mathematical Journal, –2024. –68. –No4. P.37–47. ttp://dx.doi.org/10.29229/uzmj.2024-4-4

[14] Dekhkonov F.N.; On one boundary control problem for a pseudo-parabolic equation in a two-dimensional domain. Communications in Analysis and Mechanics, –2025, –17. –P.1–14. https://doi.org/10.3934/cam.2025001

[15] Dekhkonov F.N.; Boundary control problem associated with a pseudo-parabolic equation. Stochastic Modelling and Computational Sciences, –2023. –3. –P.117–128. https://doi.org/10.61485/SMCS.27523829/v3n1P9

[16] Dekhkonov F.N.; On the control problem associated with a pseudo-parabolic type equation in an one-dimensional domain. International Journal of Applied Mathematics, –2024. –37. –P.109–118. https://dx.doi.org/10.12732/ijam.v37i1.9

[17] Dekhkonov F.N.; Time-optimal control problem associated with a fourth-order parabolic equation. Mathematical notes of NEFU, –2024. –31. –P.70–80.

[18] Dekhkonov F. N., Li W.; On the boundary control problem associated with a fourth order parabolic equation in a two-dimensional domain. Discrete and Continuous Dynamical Systems - S, –2024. –17. –No8. –P.2478–2488.

[19] Dekhkonov F. N.; On the time-optimal control problem for a fourth order parabolic equation in a two-dimensional domain. Bulletin of the Karaganda University. Mathematics Series, –2024. –114. –P.57–70.

[20] Egorov Yu.V.; Optimal control in Banach spaces. Dokl. Akad. Nauk SSSR, –1963. –150. –No2. –P.241–244 (in Russian).

[21] Friedman A.; Differential equations of parabolic type. XVI, Englewood Cliffs, New Jersey,–1964. https://doi.org/10.1016/0022-247X(67)90040-6

[22] Fattorini H.O., Russell D. L.; Exact controllability theorems for linear parabolic equations in one space dimension. Arch. Rational Mech. Anal., –1971. –43. –P.272–292.

[23] Fattorini H.O.; Time-optimal control of solutions of operational differential equations. SIAM J. Control, –1964.–2.–P.49–65.

[24] Fayazova Z.K.; Boundary control of the heat transfer process in the space. Russian Mathematics, –2019. –63. –No12. –P.71–79. https://doi.org/10.26907/0021-3446-2019-12-82-90

[25] Fursikov A. V.; Optimal Control of Distributed Systems, Theory and Applications, 187, Amer. Math. Soc., Providence, RI, Transl. Math. Monogr., –2000.

[26] Guo Y. J. L.; Null boundary controllability for a fourth order parabolic equation. Taiwanese J. Math., –2002. –6. –P.421–431. https://doi.org/10.11650/twjm/1500558308

[27] Guo Y. J. L., Littman W.; Null boundary controllability for semilinear heat equations. Appl. Math. Opt., –1995. –32. –P.281–316. https://doi.org/10.1007/BF01187903

[28] King B.B., Stein O., Winkler M.; A fourth-order parabolic equation modeling epitaxial thin film growth. J. Math. Anal. Appl., –2003. –286. –P.459–490.

[29] Kopzhassarova A., Sarsenbi A.; Basis Properties of Eigenfunctions of Second- Order Differential Operators with Involution. Abstr. Appl. Anal., –2012. Article ID 576843, 6 pages. https://doi.org/10.1155/2012/576843

[30] Kirane M., Sarsenbi A. A.; Solvability of Mixed Problems for a Fourth-Order Equation with Involution and Fractional Derivative. Fractal Fract., –2023. –7. –P.131.

[31] Lions J.L.; Contróle optimal de systèmes gouvernés par deséquations aux dérivées partielles, Paris, Dunod Gauthier-Villars, –1968.

[32] Laroche B., Martin P., Rouchon P.; Motion planning for the heat equation. International Journal of Robust and Nonlinear Control, –2000. –10. –P.629-643. http://dx.doi.org/10.1002/1099-1239(20000715)10:8

[33] Mussirepova E., Sarsenbi A., Sarsenbi A.; The inverse problem for the heat equation with reflection of the argument and with a complex coefficient. Bound Value Probl, –2022. –1. artc. no. 99. https://doi.org/10.1186/s13661-022-01675-1

[34] Muratbekova M., Kadirkulov B., Koshanova M., Turmetov B.; On solvability of some inverse problems for a fractional parabolic equation with a nonlocal biharmonic operator. Fractal and Fractional, –2023. –7. 404. https://doi.org/10.3390/fractalfract7050404

[35] Turmetov B.K., Kadirkulov B.J.; An inverse problem for a parabolic equation with involution. Lobachevskii J. of Math., –2021. –42. –P.3006–3015.

[36] Turmetov B.K., Kadirkulov B.J.; On the solvability of an initial-boundary value problem for a fractional heat equation with involution. Lobachevskii J. Math., –2022. –43. –P.249–262. https://doi.org/10.1134/S1995080222040217

[37] Turmetov B., Karachik V.; On solvability of some inverse problems for a nonlocal fourth-order parabolic equation with multiple involution. AIMS Mathematics, –2024. –9. –P.6832–6849. https://doi.org/10.3934/math.2024333

[38] Tikhonov A. N., Samarsky A. A.; Equations of Mathematical Physics Nauka, Moscow, –1966.

[39] Wiener J.; Generalized Solutions of Functional-Dierential Equations, World Scientic Publishing, New Jersey, –1993.

[40] Yu H.; Null controllability for a fourth order parabolic equation. Sci. China Ser. F-Inf. Sci., –2009. –52. –P.2127–2132. https://doi.org/10.1007/s11432-009-0203-9

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.68-76
DOI: 10.29229/uzmj.2025-2-6

On the determination of the coefficient of a nonlinear term in a parabolic equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. A second-order parabolic equation with an inhomogeneity of the form $q(x',t)u^{1+\alpha}(x,t)$, where $x=(x',x_n)=(x_1,\cdots,x_{n-1},x_n)$ and $\alpha>0$, is considered in both direct and inverse problems. The inverse problem focuses on determining the coefficient $q(x',t)$ under heterogeneity. The trace of the direct problem solution is provided at the hyperplane $x_n=0$ as supplementary information. Conditions for the unique solvability of the direct problem are established. For the inverse problem, local existence and uniqueness theorems are proven.

Keywords: non-linear parabolic equation, inverse problem, overdetermination condition, integral equation, existence, uniqueness.
MSC (2020): 35A01, 35A02, 35L02, 35L03, 35R03.

1. Introduction. Statement of the problem

In recent years, there has been a growing body of scientific literature focused on solving both direct and inverse problems associated with nonlinear parabolic equations. Mathematical models that describe a series of physical and chemical processes, involving changes in the internal properties of materials, form a nonlinear system governed by a parabolic equation with an unknown coefficient. The complexity of such systems, along with their significant differences from conventional boundary value problems with fixed coefficients in parabolic equations, generates substantial theoretical interest in formulating their structure.

The determination of unknown coefficients in nonlinear parabolic partial differential equations from additional boundary conditions - such as measured data obtained at the boundary or at a finite time - is a well-established problem in the literature, commonly referred to as inverse coefficient problems [3]-[10]. The papers [3], [4] investigate the coefficient-to-data mappings associated with inverse problems involving unknown coefficients in nonlinear parabolic partial differential equations. The authors present a method that utilizes adjoint versions of the direct problem to derive equations that explicitly relate changes in the inputs (coefficients) to changes in the outputs (measured data). Using these equations, the authors demonstrate that the coefficient-to-data mappings are continuous, strictly monotone, and injective. Integral identities are derived, showing that the coefficient-to-data mapping is monotone and invertible when there is a single unknown coefficient, and the mapping is invertible when two unknown coefficients are determined simultaneously. The results of several numerical experiments are also presented. In [5]-[10], the problem of identifying potentially discontinuous diffusion coefficients in nonlinear parabolic equations is explored. A necessary condition for solving the parameter estimation problem is provided. Additionally, the study investigates a regularization method for solutions to ill-posed problems involving hemivariational inequalities in Banach spaces. Under the assumption that the hemivariational inequality is solvable, a strongly convergent approximation procedure is developed using the Browder-Tikhonov regularization method.

The work [14] deals with the determination of a pair (p,u) in the nonlinear parabolic equation $u_t-\triangle u+p(x)f(u)=0$, with initial and boundary conditions $u(x,0)=0$, $u\big|_{\partial\Omega\times[0,T]}=g$, from the overdetermination data $u(.,T) = q$. The problem is reduced to a nonlinear operator equation and solved based on a fixed point theory in ordered Banach spaces. In the paper [12], a nonlinear problem is considered, where the unknown coefficient depends on the derivative of the solution. A uniqueness result is established using the method of Carleman estimates. The applicability of this result is demonstrated through parameter identification problems in population dynamics and magnetism. For the magnetism application, we present numerical results based on a reconstruction method that employs a multiharmonic formulation of the problem. In [13], the authors prove a uniqueness result for the inverse problem of determining several non-constant coefficients in a system of two parabolic equations that corresponds to a Lotka-Volterra competition model. The result provides a sufficient condition for the uniqueness of determining four coefficients in the system.

In this study, we investigate the inverse problem of determining the multidimensional non-constant coefficients of an inhomogeneity term of the form $u^{1+\alpha}(x,t)$ in a parabolic equation. Consider the Cauchy problem for the nonlinear parabolic equation

$$\frac{\partial u}{\partial t} - \Delta u + q(x',t)u^{1+\alpha}(x,t) = 0, \quad (x,t) \in \mathbb{R}^n_T, \tag{1.1}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where Δ denotes the Laplacian in the spatial variables $x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$, and $\mathbb{R}^n_T := \{(x,t) \mid x \in \mathbb{R}^n, 0 < t \leq T\}$, $\alpha > 0$ is a real constant, and $q(x',t)$, $\varphi(x)$ are given functions.

The problem of finding a function $u(x,t)$ that satisfies (1.1) and (1.2) is referred to as the **direct problem**.

In this paper, for the direct problem we study the **inverse problem:** determine the function $q(x',t)$, $(x',t) \in \mathbb{R}^{n-1}_T$, given a solution $u(x,t)$ on the hyperplane $x_n = 0$:

$$u(x',0,t) = h(x',t), \quad h(x',0) = \varphi(x',0), \quad (x',t) \in \mathbb{R}^{n-1}_T. \tag{1.3}$$

The inverse problem for the linear case of the corresponding equation (1.1), along with the conditions (1.2) and (1.3), is explored in works [1], [2].

We will assume that $\varphi(x) \in H^{l+4}(\mathbb{R}^n)$, $h(x',t) \geq h_0 > 0$, $h_0 = const$, $h(x',t) \in H^{l+4,\frac{l+4}{2}}\left(\overline{\mathbb{R}}^{n-1}_T\right)$. The Holder spaces $H^{\gamma}(\Omega)$, $H^{\gamma,\gamma/2}(\Omega_T)$ and the norms in them are well known (see, for example, [6, pp. 10-20]). In what follows we will omit the indices Ω and Ω_T in the notations of the norms of these spaces, $|u|^{\gamma}_T \equiv |u|^{\gamma}_{\Omega_T}$, in our case $\Omega = \mathbb{R}^n$, or $\mathbb{R}^{n-1}$.

2. Study of the direct problem

The following statement is true:

Lemma 2.1. *The solution to the direct problem satisfies the integral equation*

$$u(x,t) = \Gamma(x,t) * \varphi(x) - \int_0^t \Gamma(x,t-\tau) * \left[q(x',\tau)u^{1+\alpha}(x,\tau)\right] d\tau, \tag{2.1}$$

where $\Gamma(x,t)$ *is the fundamental solution of the operator* $\frac{\partial}{\partial t} - \Delta$ *[6, p. 305]),* $f(x) * g(x) = \int_{\mathbb{R}^n} f(x-\xi)g(\xi)d\xi$.

Proof. For the right-hand side of (2.1) the following equalities hold:

$$\Gamma(x,t)*\varphi(x)-\int_0^t \Gamma(x,t-\tau)q(x',\tau)u^{1+\alpha}(x,\tau)d\tau=\Gamma(x,t)*\varphi(x)$$

$$+\int_0^t \Gamma(x,t-\tau)*\Big[u_\tau(x,\tau)-\triangle u(x,\tau)\Big]d\tau=\Gamma(x,t)*\varphi(x)$$

$$+\lim_{\tau\to t}\Big[\Gamma(x,t-\tau)*u(x,\tau)\Big]-\lim_{\tau\to 0+}\Big[\Gamma(x,t-\tau)*u(x,\tau)\Big]$$

$$-\int_0^t\int_{\mathbb{R}^n}\Big[\frac{\partial\Gamma}{\partial\tau}(x-\xi,t-\tau)+\triangle_\xi\Gamma(x-\xi,t-\tau)\Big]u(\xi,\tau)d\xi d\tau.$$

Using the relations $\lim\limits_{t\to 0+}\Gamma(x,t)=\delta(x)$, where $\delta(x)$ is the Dirac delta function, $\delta(x)*g(x)=g(x)$ and $\frac{\partial\Gamma}{\partial\tau}(x-\xi,t-\tau)+\triangle_\xi\Gamma(x-\xi,t-\tau)=0$, we conclude that the right-hand side of these equalities is equal to $u(x,t)$. Passing to the limit as $t\to 0+$ in (2.1), we obtain (1.2). ☐

Lemma 2.2. *Let $\varphi(x)\in H^{l+2}(\mathbb{R}^n)$, $q(x',t)\in H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^{n-1}_T\right)$, $0<l<1$, and $1\le\varphi(x)\le\varphi_0=const$. Then there exists $T^*\in(0,T]$ such that the direct problem has a non-negative solution $u(x,t)$ from the class $H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_T\right)$.*

Proof. According to Lemma 2.1, the solution to the direct problem satisfies the integral equation (2.1). We write (2.1) in operator form

$$u(x,t)=(Au)(x,t),\quad (x,t)\in\mathbb{R}^n_T, \tag{2.2}$$

where

$$(Au)(x,t)=\Phi(x,t)-\int_0^t\Gamma(x,t-\tau)*\left[q(x',\tau)u^{1+\alpha}(x,\tau)\right]d\tau, \tag{2.3}$$

and

$$\Phi(x,t):=\Gamma(x,t)*\varphi(x).$$

We will show that the operator A is contractive on a suitable set of functions if T is small.

Consider a closed ball

$$\mathbf{B}^1_T:=\Big\{\psi(x,t)\in H^{l,\frac{l}{2}}\left(\overline{\mathbb{R}}^n_T\right)\Big|\,|\psi-\Phi|^l_T\le 1\Big\}. \tag{2.4}$$

Since a closed subset of a complete metric space is complete, the ball $\mathbf{B}^1_T$ is a complete metric space with respect to the metric defined by the norm $|\cdot|^l_T$.

Under the conditions of the lemma $|\varphi|^l\le\varphi_0$. Let $u\in\mathbf{B}^1_T$, then according to (2.4) we have

$$|u|^l_T=|u-\Phi+\Phi|^l_T\le|u-\Phi|^l_T+|\Phi|^l_T\le 1+\varphi_0. \tag{2.5}$$

We prove that for sufficiently small T the operator A maps $\mathbf{B}^1_T$ to itself and is a contraction operator. Using (2.3), (2.5) and the estimate of the thermal volume potential in Holder norms [6, p. 318], we obtain

$$|Au-\Phi|^l_T=\left|\int_0^t\Gamma(x,t-\tau)*\left[q(x',\tau)u^{1+\alpha}(x,\tau)\right]d\tau\right|^l_T\le m(T)q_0\left(1+\varphi_0\right)^{1+\alpha}, \tag{2.6}$$

$q_0 = |q|_T^l \quad m(T) \to 0$ at $T \to 0$. By choosing $T^0 = \min\{T,\ T_1\}$, where T_1 is defined as the smallest positive root of the equation $m(T)q_0\left(1+\varphi_0\right)^{1+\alpha} = 1$ (this is possible due to the property of $m(T)$), we conclude that $Au \in \mathbf{B}^1_{T^0}$.

We will now show that the operator A for some T on the set $\mathbf{B}^1_T$ is contractive. First, note that if $u \in \mathbf{B}^1_T$, then

$$u(x,t) \geq \Phi(x,t) - |u(x,t) - \Phi(x,t)| \geq 0, \qquad (x,t) \in \mathbb{R}^n_T.$$

Let $u_i(x,t) \in \mathbf{B}^1_T$, $i = 1,2$ and write the difference $u_1^{1+\alpha}(x,t) - u_2^{1+\alpha}(x,t)$ in the form

$$u_1^{1+\alpha}(x,t) - u_2^{1+\alpha}(x,t) = (1+\alpha)\int\limits_{u_2(x,t)}^{u_1(x,t)} \eta^\alpha d\eta.$$

Next we change the variable

$$\eta = u_1(x,t)\eta' + u_2(x,t)\left(1-\eta'\right), \quad d\eta = \left[u_1(x,t) - u_2(x,t)\right]d\eta',$$
$$\eta = u_2(x,t) \Rightarrow \eta' = 0, \quad \eta = u_1(x,t) \Rightarrow \eta' = 1,$$

and in the end we get

$$u_1^{1+\alpha}(x,t) - u_2^{1+\alpha}(x,t) = \left[u_1(x,t) - u_2(x,t)\right] W_\alpha\Big(u_1(x,t), u_2(x,t)\Big), \tag{2.7}$$

where

$$W_\alpha\Big(u_1(x,t),\ u_2(x,t)\Big) := (1+\alpha)\int\limits_0^1 \left[u_1(x,t)\eta' + u_2(x,t)(1-\eta')\right]^\alpha d\eta'. \tag{2.8}$$

Using (2.5) and (2.7), we find the following estimates:

$$\Big|W_\alpha\Big(u_1(x,t),\ u_2(x,t)\Big)\Big|_T^l \leq (1+\alpha)\left(1+\varphi_0\right)^\alpha, \tag{2.9}$$

$$\Big|u_1^{1+\alpha}(x,t) - u_2^{1+\alpha}(x,t)\Big|_T^l \leq (1+\alpha)\left(1+\varphi_0\right)^\alpha \Big|u_1(x,t) - u_2(x,t)\Big|_T^l. \tag{2.10}$$

Taking into account (2.9) and (2.10), we estimate the difference

$$|Au_1 - Au_2|_T^l = \left|\int\limits_0^t \Gamma(x,t-\tau) * \Big[q(x',\tau)\left(u_1^{1+\alpha}(x,\tau) - u_2^{1+\alpha}(x,\tau)\right)\Big]d\tau\right|_T^l$$
$$\leq m(T)q_0(1+\alpha)\left(1+\varphi_0\right)^\alpha \Big|u_1(x,t) - u_2(x,t)\Big|_T^l. \tag{2.11}$$

It follows from (2.11) that if we choose $T_0 = \min\{T^0,\ T_2\}$, where T_2 is the smallest positive root of the equation $m(T)q_0(1+\alpha)\left(1+\varphi_0\right)^\alpha = 1$, then for $T \in (0,\ T)$ the operator A performs a contraction mapping of the set $\mathbf{B}^1_T$ into itself. Then, by the Banach fixed point principle, there is a unique solution of the operator equation (2.2) in $\mathbf{B}^1_T$. Therefore, solving (2.2), for example by the method of successive approximations (the Banach principle guarantees convergence to the solution), we uniquely determine the function $u(x,t)$. Moreover, under the conditions of the lemma $\Phi(x,t) \in H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_T\right)$, and since $q(x',t) \in H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^{n-1}_T\right)$, then each iteration leads to a function from the class $H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_T\right)$. Due to the uniform convergence of successive approximations to $u(x,t)$, we conclude that this function belongs to the class $H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_T\right)$ for $T \in (0, T_0)$ and is a solution of the direct problem. $\square$

3. Study of the inverse problem

Lemma 3.1. *The inverse problem (1.1)-(1.3) is equivalent to the problem of determining the functions $v(x,t) \in H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_T\right)$ and $q(x',t) \in H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^{n-1}_T\right)$ from the equations*

$$\frac{\partial v}{\partial t} - \triangle v + (1+\alpha) q(x',t) u^{\alpha}(x,t) v(x,t) = 0, \quad (x,t) \in \mathbb{R}^n_T, \tag{3.1}$$

$$v(x,0) = \varphi_{x_n}(x), \quad x \in \mathbb{R}^n, \tag{3.2}$$

$$v_{x_n}(x',0,t) = h_t - \triangle h + q(x',t) h^{1+\alpha}(x',t), \quad (x',t) \in \mathbb{R}^{n-1}_T, \tag{3.3}$$

where $u(x,t) \in H^{l+3,\frac{l+3}{2}}\left(\overline{\mathbb{R}}^n_T\right)$ is the solution to problem (1.1), (1.2).

Proof. Let (u,q) be a solution to problem (1.1)-(1.3). Then, setting $v(x,t) = u_{x_n}$, by direct differentiation of (1.1), (1.2), we obtain (3.1), (3.2). Equalities (3.3) are easily derived from equation (1.1) and condition (1.3).

Conversely, let (v,q) satisfy equations (3.1)-(3.3). Then, we show that the problem (1.1)-(1.3) has a unique solution (u,q) with the same function q. Uniqueness follows from the uniqueness of the solution to the problem (1.1), (1.2), proved in Section 2. To prove existence, note that since $v(x,t) = u_{x_n}(x,t)$, then

$$u(x,t) = \psi(x',t) + \int_0^{x_n} v(x',y,t)\,dy, \tag{3.4}$$

where $\psi(x',t) = v(x',0,t)$. Let us choose $\psi(x',t)$ so that (1.1)-(1.3) hold. From (1.3) and (3.4) it follows that $\psi(x',t) \equiv h(x',t)$. Further, from here and from (3.4) if the consistency condition $h(x',0) = \varphi(x',0)$ is satisfied, we obtain

$$u(x,0) = h(x',0) + \int_0^{x_n} v(x',y,0)\,dy = \varphi(x',0) + \int_0^{x_n} \varphi_y(x',y)\,dy = \varphi(x),$$

i.e. condition (1.2 is satisfied). It remains to verify that equality (1.1) holds. Integrating (3.1) from 0 to x_n, and using (3.3) we find

$$\int_0^{x_n} \Big[\frac{\partial v}{\partial t} - \triangle v + (1+\alpha) q(x',t) u^{\alpha}(x',y,t) v(x',y,t)\Big] dy = \int_0^{x_n} \Big[\frac{\partial u_t}{\partial y} - \left(\triangle_{x'} u(x',y,t)\right)_y\Big] dy$$

$$-v_{x_n}(x,t) + v_{x_n}(x',0,t) + q(x',t)\Big[u^{1+\alpha}(x,t) - u^{1+\alpha}(x',0,t)\Big]$$

$$= u_t(x,t) - u_t(x',0,t) - \triangle_{x'} u(x,t) + \triangle_{x'} u(x',0,t) - u_{x_n x_n}(x,t) + h_t - \triangle h$$

$$+ q(x',t) h^{1+\alpha}(x',t) + q(x',t)\Big[u^{1+\alpha}(x,t) - u^{1+\alpha}(x',0,t)\Big]$$

$$= u_t - \triangle u(x,t) + q(x',t) u^{1+\alpha}(x,t) = 0.$$

$\square$

The solution to problem (3.1), (3.2) is reduced to solving the integral equation

$$v(x,t) = \Gamma(x,t) * \varphi_{x_n}(x) - (1+\alpha) \int\limits_0^t \Gamma(x,t-\tau) * [q(x',t)u^\alpha(x,t)v(x,t)]\, d\tau, \quad (x,t) \in \mathbb{R}^n_T, \tag{3.5}$$

where $u(x,t) \in H^{l+3,\frac{l+3}{2}}\left(\overline{\mathbb{R}}^n_T\right)$ is the solution of equation (2.1).

Using (3.5) we calculate the derivative $v_{x_n}(x,t)$. In this case, taking into account the formula for differentiation of the convolution $f(x) * g(x) = f_{x_n}(x) * g(x) = f(x) * g_{x_n}(x)$, we have

$$v_{x_n}(x,t) = \Gamma(x,t) * \varphi_{x_n x_n}(x) - (1+\alpha) \int\limits_0^t \Gamma_{x_n}(x,t-\tau) * [q(x',t)u^\alpha(x,t)v(x,t)]\, d\tau, \tag{3.6}$$

where $(x,t) \in \mathbb{R}^n_T$. The kernel $\Gamma_{x_n}(x,t)$ has a weak singularity at $x=0$, $t=0$.

From (3.3) and (3.6, we obtain

$$q(x',t) = \frac{1}{h^{1+\alpha}(x',t)} \Big[\Gamma(x,t) * \varphi_{x_n x_n}(x)\big|_{x_n=0} - h_t + \triangle_{x'} h \Big]$$

$$- \frac{1+\alpha}{h^{1+\alpha}(x',t)} \int\limits_0^t \Gamma_{x_n}(x,t-\tau) * \Big[q(x',t)u^\alpha(x,t)v(x,t) \Big]_{x_n=0} d\tau, \quad (x',t) \in \mathbb{R}^{n-1}_T, \tag{3.7}$$

The system of equations (2.1), (3.5), (3.6) is a system of nonlinear integral equations in the domain $\mathbb{R}^n_T$ for unknown functions $u(x,t)$, $v(x,t)$ and $q(x',t)$. We write it in vector-operator form

$$U = L(U), \tag{3.8}$$

in which

$$U(x,t) = \Big(U_1,\, U_2,\, U_3 \Big) = \Big(u(x,t),\, v(x,t),\, q(x',t) \Big), \quad L(U) = \Big(L_1(U),\, L_2(U),\, L_3(U) \Big)$$

and the operators $L_1(U)$, $L_2(U)$, $L_3(U)$ are defined in accordance with the right-hand sides of equations (2.1), (3.5), (3.6). We show that for sufficiently small T the operator $L(U)$ is contractive on a suitable set of functions.

Let's denote

$$U^0(x,t) = \Big(U_1^0,\, U_2^0,\, U_3^0 \Big) := \Big(u_0(x,t),\, v_0(x,t),\, q_0(x',t) \Big)$$

$$:= \Big(\Phi(x,t),\ \Gamma(x,t) * \varphi_{x_n}(x),\ \frac{1}{h^{1+\alpha}(x',t)} \Big[\Gamma(x,t) * \varphi_{x_n x_n}(x)\big|_{x_n=0} - h_t + \triangle_{x'} h \Big] \Big). \tag{3.9}$$

From (3.9) the estimate follows

$$\Big| U^0 \Big|_T^l = \max\Big\{ \big|u_0\big|_T^l,\, \big|v_0\big|_T^l,\, \big|q_0\big|_T^l \Big\} \le \max\Big\{ \varphi_{00},\, \frac{1}{h_0}\big[\varphi_{00} + \big| h_t + \triangle_{x'} h \big|_T^l \big] \Big\} := K, \tag{3.10}$$

where $\varphi_{00} := \max\Big\{ \big|\varphi\big|^l,\, \big|\varphi_{x_n}\big|^l,\, \big|\varphi_{x_n x_n}\big|^l \Big\}$.

Definition 3.2. Let $\mathcal{K}(T,K) = \Big\{ u(x,t),\, v(x,t),\, q(x',t) \Big\}$ be the set of functions belonging to the class $H^{l,l/2}\left(\mathbb{R}^n_T\right)$, and satisfying the conditions

$$\Big| U - U^0 \Big|_T^l \le K, \quad u(x,t) > 0, \quad (x,t) \in \mathbb{R}^n_T. \tag{3.11}$$

Theorem 3.3. *Let $\varphi(x) \in H^{l+4}(\mathbb{R}^n)$, $\varphi(x) \le 1$, $h(x',t) \in H^{l+4,\frac{l+4}{2}}\left(\overline{\mathbb{R}}_T^{n-1}\right)$, $h(x',t) \ge h_0 > 0$, $h_0 = const$, $0 < l < 1$ and $h(x',0) = \varphi(x',0)$. Then there exists $T^* \in (0,T]$ such that equation (3.8) has a unique solution on the set $\mathcal{K}(T^*,K)$.*

Proof. First, we note that the functions from $\mathcal{K}(T,K)$ satisfy the inequalities

$$\left|u\right|_T^l \le 2K, \quad \left|v\right|_T^l \le 2K, \quad \left|q\right|_T^l \le 2K. \tag{3.12}$$

Let us first show that there exists some $T_3 \in (0,T]$ for which the operator L maps the set $\mathcal{K}(T_3,K)$ to itself. Using estimates (3.12) and equation (2.1), we obtain

$$u(x,t) \ge \Gamma(x,t) * \varphi(x) - \int_0^t \Gamma(x,t-\tau) * \left[\left|q(x',\tau)\right|\left|u\right|^{1+\alpha}(x,\tau)\right] d\tau$$

$$\ge 1 - (2K)^{2+\alpha}\,T, \quad (x,t) \in \mathbb{R}^n_T.$$

It follows that if $T < (2K)^{-2-\alpha}$, then $u(x,t) > 0$ for $(x,t) \in \mathbb{R}^n_T$. Further we will assume that this condition on T is satisfied.

Using (2.1), (3.5), (3.6), we find estimates for $\left|L_k(U) - U_k^0\right|_T^l$, $k = 1,2,3$, $(x,t) \in \mathbb{R}^n_T$:

$$\left|L_1(U) - U_1^0\right|_T^l = \left|-\int_0^t \Gamma(x,t-\tau) * \left[q(x',\tau)u^{1+\alpha}(x,\tau)\right] d\tau\right|_T^l \le m_1(T)\,(2K)^{2+\alpha},$$

$$\left|L_2(U) - U_2^0\right|_T^l = \left|-(1+\alpha)\int_0^t \Gamma(x,t-\tau) * \left[q(x',t)u^{\alpha}(x,t)v(x,t)\right] d\tau\right|_T^l \le m_1(T)(1+\alpha)\,(2K)^{2+\alpha},$$

$$\left|L_3(U) - U_3^0\right|_T^l = \left|-\frac{1+\alpha}{h^{1+\alpha}(x',t)}\int_0^t \Gamma_{x_n}(x,t-\tau)\big|_{x_n=0} * \left[q(x',t)u^{\alpha}(x,t)v(x,t)\right]_{x_n=0} d\tau\right|_T^l$$

$$\le m_2(T)\frac{1+\alpha}{h_0^{1+\alpha}}\,(2K)^{2+\alpha},$$

where $m_i(T) \to 0$, $i = 1,2$, for $T \to 0$.

We choose $T_3 \in (0,T)$ from the conditions $T_3 = \min\{T_3^0, T_3^{00}\}$, where T_3^0, T_3^{00} are the smallest positive roots of the equations

$$m_1(T)(1+\alpha)\,(2K)^{1+\alpha} = \frac{1}{2}, \qquad m_2(T)\frac{1+\alpha}{h_0^{1+\alpha}}\,(2K)^{1+\alpha} = \frac{1}{2},$$

respectively. Then we obtain the inequalities

$$\left|U - U^0\right|_T^l = \max_{1\le i\le 3}\left\{\left|L_i(U) - U_i^0\right|_T^l\right\} \le K, \quad u(x,t) > 0, \quad (x,t) \in \mathbb{R}^n_{T_3},$$

from which it follows that the operator $L(U)$ maps the set $\mathcal{K}(T_3,K)$ into itself.

Let us now show that $L(U)$ is a contraction operator on $\mathcal{K}(T^*,K)$ for some $T^* \in (0,T_3]$. Take two elements $U(x,t) = \left(U_1,\,U_2,\,U_3\right) \in \mathcal{K}(T_3,K)$ and $\widetilde{U}(x,t) = \left(\widetilde{U}_1,\,\widetilde{U}_2,\,\widetilde{U}_3\right) \in \mathcal{K}(T_3,K)$. Then we have

$$\left|qu^{1+\alpha} - \widetilde{q}\widetilde{u}^{1+\alpha}\right|_T^l = \left|q\left(u^{1+\alpha} - \widetilde{u}^{1+\alpha}\right) + \widetilde{u}^{1+\alpha}\left(q - \widetilde{q}\right)\right|_T^l.$$

Using relations similar to (2.7)-(2.10) to estimate $u^{1+\alpha}-\widetilde{u}^{1+\alpha}$, we continue

$$\Big|qu^{1+\alpha}-\widetilde{q}\widetilde{u}^{1+\alpha}\Big|_T^l \le 2\big|U-\widetilde{U}\big|_T^l \max\Big\{(1+\alpha)(2K)^\alpha\big|U_3\big|_T^l,\ \big|U_1^{1+\alpha}\big|_T^l\Big\}$$

$$\le 2(1+\alpha)(2K)^{1+\alpha}\big|U-\widetilde{U}\big|_T^l; \tag{3.13}$$

Such reasoning will lead to inequalities

$$\Big|qu^\alpha v-\widetilde{q}\widetilde{u}^\alpha\widetilde{v}\Big|_T^l=\Big|qv\big(u^\alpha-\widetilde{u}^\alpha\big)+\widetilde{u}^\alpha q\big(v-\widetilde{v}\big)+\widetilde{u}^\alpha\widetilde{v}(q-\widetilde{q})\Big|_T^l$$

$$\le 3\big|U-\widetilde{U}\big|_T^l\max\Big\{\alpha(2K)^\alpha\big|U_2U_3\big|_T^l,\ \big|\widetilde{U}_1^\alpha U_3\big|_T^l,\ \big|\widetilde{U}_1^\alpha\widetilde{U}_2\big|_T^l\Big\}$$

$$\le 3\max\Big\{\alpha(2K)^{2+\alpha},\ (2K)^{1+\alpha}\Big\}\big|U-\widetilde{U}\big|_T^l. \tag{3.14}$$

We will use (3.13) and (3.14) to estimate the norm of the differences $\big(L(U^1)-L(U^2)\big)_i$, $i=1,2,3$. Then we get

$$\Big|\big(L(U)-L(\widetilde{U})\big)_1\Big|_T^l=\Bigg|\int_0^t\Gamma(x,t-\tau)*\Big[q(x',\tau)u^{1+\alpha}(x,\tau)-\widetilde{q}(x',\tau)\widetilde{u}^{1+\alpha}(x,\tau)\Big]d\tau\Bigg|_T^l$$

$$\le 2m_1(T)(1+\alpha)(2K)^{1+\alpha}\big|U-\widetilde{U}\big|_T^l,$$

$$\Big|\big(L(U)-L(\widetilde{U})\big)_2\Big|_T^l=\Bigg|-(1+\alpha)\int_0^t\Gamma(x,t-\tau)*\Big[q(x',t)u^\alpha(x,t)v(x,t)-\widetilde{q}(x',t)\widetilde{u}^\alpha(x,t)\widetilde{v}(x,t)\Big]d\tau\Bigg|_T^l$$

$$\le 3m_1(T)\max\Big\{\alpha(2K)^{2+\alpha},\ (2K)^{1+\alpha}\Big\}\big|U-\widetilde{U}\big|_T^l,$$

$$\Big|\big(L(U)-L(\widetilde{U})\big)_3\Big|_T^l=\Bigg|-\frac{1+\alpha}{h^{1+\alpha}(x',t)}\int_0^t\Gamma_{x_n}(x,t-\tau)\big|_{x_n=0}*\Big[q(x',t)u^\alpha(x,t)v(x,t)$$

$$-\widetilde{q}(x',t)\widetilde{u}^\alpha(x,t)\widetilde{v}(x,t)\Big]_{x_n=0}d\tau\Bigg|_T^l\le 3m_2(T)\frac{1+\alpha}{h_0^{1+\alpha}}\max\Big\{\alpha(2K)^{2+\alpha},\ (2K)^{1+\alpha}\Big\}\big|U-\widetilde{U}\big|_T^l.$$

Let $\rho\in(0,1)$ and $T^*(0,T_3]$ be chosen as $T^*=\min\{T_4,\,T_5\}$, where T_i, $i=4,5$ are the smallest positive roots of the equations (with respect to T)

$$3m_1(T)\max\Big\{\alpha(2K)^{2+\alpha},\ (1+\alpha)(2K)^{1+\alpha}\Big\}=\rho,\ 3m_2(T)\frac{1+\alpha}{h_0^{1+\alpha}}\max\Big\{\alpha(2K)^{2+\alpha},\ (2K)^{1+\alpha}\Big\}=\rho,$$

respectively (this is possible due to the properties $m_i(T), i=1,2$). Then the inequality

$$\big|L(U)-L(\widetilde{U}\big|_T^l=\max_{1\le i\le 3}\Big\{\Big|\big(L(U)-L(\widetilde{U})\big)_i\Big|_T^l\Big\}\le\rho\big|U-\widetilde{U}\big|_T^l,$$

which means that $L(U)$ is a contraction operator on the set $\mathcal{K}(T^*,K)$. Then it follows from the Banach principle that the operator equation (3.8) has on the set $\mathcal{K}(T^*,K)$ unique solution. Theorem 3.3 is proved. □

Note that the found functions $u(x,t)$ and $q(x',t)$ will be solutions of the direct and inverse problems from the classes $H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_T\right)$ and $q(x',t)\in H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^{n-1}_T\right)$, respectively. To show this, we construct successive approximations for equation (3.8):

$$U^{n+1}=L(U^n),\quad n=0,1,2,\cdots. \tag{3.15}$$

Under the conditions of the lemma, $U^0(x,t)\in H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_T\right)$, then each iteration leads to a function from the class $H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_T\right)$. Due to the uniform convergence of successive approximations (3.15) to $U(x,t)$ (Theorem 3.3 guarantees convergence to a solution), we conclude that this vector function belongs to the class $H^{l+2,\frac{l+2}{2}}\left(\overline{\mathbb{R}}^n_{T^*}\right)$, and the first two components $U(x,t)$ are solutions of the direct and inverse problems.

References

[1] Beznoshchenko N.Ya. Determination of the coefficient q from the solution of the second boundary value problem for the equation $u_t-\triangle u+qu=f$ in a half space (existence in the large), Sibirsk. Mat. Zh., –1982.–23(1).–P. 3-11.

[2] Beznoshchenko N. Ya. Existence of a solution of the problem of determining the coefficient q in the equation $u_t-\triangle u+qu=F$, Differ. Uravn.,–1979.–15(1).–P. 10-17

[3] DuChateau P.; Monotonicity and invertibility of coefficient-to-data mappings for parabolic inverse problems. SIAM J. Math. Anal., –1995.–26.–P.1473–1487.

[4] DuChateau P., Thelwell R., Butters G.; Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient. Inverse Problems, –2004.–20.–P.601–625.

[5] Hanke M., Scherzer O.; Error analysis of an equation error method for the identification of the diffusion coefficient in a quasilinear parabolic differential equation. SIAM J. Math. Anal., –1999.–59.–P.1012–1027.

[6] Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N.; Linear and Quasi-linear Equations of Parabolic Type. Translations of Mathematical Monographs, Vol. 23. Providence, RI: American Mathematical Society, –1968(In Russian).

[7] Liu Z.H.; On the identification of coefficients of semilinear parabolic equations. Acta Math. Appl. Sinica, –1994.–10.–P.356–367.

[8] Liu Z.H.; Identification of parameters in semilinear parabolic equations. Acta Mathematica Scientia, English Series, –1999.–19.–P.175–180.

[9] Liu Z.H., Zou J.Z.; Strong convergence results for hemivariational inequalities. Science in China, Series A, Mathematics, –2006.–49(7).–P.893–901.

[10] Liu Z.H., Li J., Li Z.W.; Regularization method with two parameters for nonlinear ill-posed problems. Science in China, Series A, Mathematics, –2008.–51(1).–P.70–78.

[11] Ou Y.H., Hasanov A., Liu Z.H.; Inverse Coefficient Problems for Nonlinear Parabolic Differential Equations. Acta Mathematica Sinica, –2008.–24(10).–P.1617–1624.

[12] Kaltenbacher B., Klibanov M.V. An Inverse Problem for a Nonlinear Parabolic Equation with Applications in Population Dynamics and Magnetics, SIAM Journal on Mathematical Analysis, –2008.–39(6) https://doi.org/10.1137/070685907

[13] Roques L., Cristofol M. The inverse problem of determining several coefficients in a nonlinear Lotka-Volterra system. Inverse Problems,–2012.– 28(7), –16 p.

[14] Zeghal A. Existence results for inverse problems associated with a nonlinear parabolic equation, J. Math. Anal. Appl., –2002.–272–P. 240–248.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.77-85
DOI: 10.29229/uzmj.2025-2-7

Forward and inverse problems for a mixed type equation with Caputo fractional derivative and Bessel operator

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. This work is devoted to the investigation of initial-boundary value and inverse problems for mixed-type equations involving a time-fractional derivative of order ρ in the sense of Caputo, together with a Bessel operator. By applying the method of separation of variables, the problem is reduced to solving a system of ordinary differential equations for the coefficients of the series expansions of the unknown functions. These expansions are represented as Fourier-Bessel series in terms of orthonormal Bessel functions of the first kind and zero order. The conditions for the existence and uniqueness of the solution are established. Further, it is assumed that the order of the derivative ρ is unknown, and the inverse problem of determining this parameter is considered. An additional condition is found that guarantees the existence and uniqueness of a solution to this inverse problem.

Keywords: Mixed type equation, the Caputo derivatives, forward and inverse problems, Fourier method.

MSC (2020): 35R11, 34A12

1. Introduction

Let $\rho \in (0,1)$ be a fixed number. The fractional derivative of order ρ of a function f in the Caputo sense is defined as

$$D_t^\rho f(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\rho} d\tau,$$

where $\Gamma(\cdot)$ denotes the well-known gamma function.

In the domain $Q := \{(x,t)|\ 0 < x < 1, -\alpha < t < \infty\}$ consider the following mixed type equation:

$$f(x,t) = \begin{cases} D_t^\rho u - B_0 u, & 0 < x < 1, \quad 0 < t < \infty, \\ u_{tt} - B_0 u, & 0 < x < 1, \quad -\alpha < t < 0, \end{cases} \tag{1.1}$$

where $f(x,t)$ is a given sufficiently smooth function and $\alpha > 0$ is given real number and

$$B_0 u = u_{xx}(x,t) + \frac{1}{x} u_x(x,t) \tag{1.2}$$

is the Bessel operator.

Forward problem. Find a function $u(x,t)$ satisfying equation (1.1) and the boundary conditions

$$\lim_{x\to 0} x u_x(x,t) = 0, \quad u(1,t) = 0, \tag{1.3}$$

and gluing conditions

$$\lim_{t\to +0} u(x,t) = \lim_{t\to -0} u(x,t), \quad \lim_{t\to +0} D_t^\rho u(x,t) = \lim_{t\to -0} u_t(x,t), \tag{1.4}$$

and also a initial condition

$$u(x,-\alpha)=\varphi(x), \quad 0<x<1, \tag{1.5}$$

where $\varphi(x)$ is a given function.

Next, it is convenient to introduce the following designations: $Q_+ = Q\times(0,\infty)$, $Q_- = Q\times(-\alpha,0)$.

Let us present the definition of the solution of the problem (1.1)-(1.5).

Definition 1.1. A function $u(x,t)\in C(\overline{Q})$ and $u(x,t)\in AC(Q_+)$ is called the (classical) solution of the forward problem, if $u(x,t)\in C^2_x(Q)$, $D^\rho_t u(x,t)\in C(\overline{Q}_+)$, $u_t(x,t)\in C(\overline{Q}_-)$, $u_{tt}(x,t)\in C(Q_-)$, and $u(x,t)$ satisfies all conditions of forward problem.

Let $\tau-$ be an arbitrary real number. In the space $L_2(0,1)$, we introduce the operator A, which operates according to the rule

$$A^\tau f(x)=\sum_{k=1}^{\infty}\lambda_k^\tau f_k J_0(\lambda_k x),$$

here $f_k=(f,J_0(\lambda_k x))$ are the Fourier-Bessel coefficients of a element $f\in L_2(0,1)$ and λ_k and $J_0(\lambda_k x)$ are eigenvalues and eigenfunctions of the Bessel operator.

Obviously, this operator A with the domain of the definition

$$D\left(A^\tau\right)=\left\{f\in L_2(0,1):\ \sum_{k=1}^{\infty}\lambda_k^{2\tau}\left|f_k\right|^2<\infty\right\}$$

is selfadjoint.

Let us denote by B_0 the operator in $L_2(0,1)$ acting according to the rule (1.2) and with the domain of definition

$$D(B_0)=\left\{f\in C^2(0,1):\lim_{x\to 0}xf'(x)=0,\ f(1)=0\right\}.$$

The selfadjoint extension in $L_2(0,1)$ of operator B_0 is A (see [5], p. 139).

Inverse problem. Find a pair of fuctions $\{u(t),\rho\}$ such that $\rho\in[\rho_0,1]$, $0<\rho_0<1$ and $u(t)$ and ρ together satisfy the conditions of Definition 1.1 and the additional condition :

$$U(t_0,\rho)=||u(t_0)-A^{-2}f||^2=u_0,\ \ 0<t_0<\infty, \tag{1.6}$$

where $u_0\in L_2(0,1)$ is given vector. When solving the inverse problem, we will assume that $f(x,t)=f(x)$.

The similar problems to problem (1.1)-(1.5) for various operators have been considered by a number of authors. Let us mention only some of these works. For example, in works [6] - [4] forward and inverse problems for mixed-type equations with the Bessel operator of fractional Caputo order have been studied. It should be noted that in all these works , the inverse problem consisted of finding the right-hand side. The authors of the work [2] also study the inverse problem of determinig unknown parametr ρ for the following mixed type equations

$$\begin{cases} D_t^\rho u(t)-Au(t)=0,\ \ 0<t\le t_2,\\ u'(t)-Au(t)=0,\ \ \ -t_1\le t<0,\end{cases}$$

where A is positive selj-adjoint abstract operator. In this work additional condition (1.6) looks like

$$U(\rho)=||u(t_0)||^2=u_0,\ \ -t_1<t_0<0.$$

2. Preliminaries

The corresponding eigenvalue problem is given by

$$X^{''} + \frac{1}{x}X^{'} + \lambda^2 X = 0, \quad 0 < x < 1, \tag{2.1}$$

together with

$$X(1) = 0. \tag{2.2}$$

At the singular end point $x = 0$, from applications point of view, we impose a boundary condition of the form

$$\lim_{x\to 0} xX^{'}(x) = 0. \tag{2.3}$$

The corresponding eigenfunctions are the Bessel functions of the first kind of order zero are solutions of Eq.(2.1)

$$X_k(x) = J_0(\lambda_k x), \quad k = 1, 2, 3....$$

We find the eigenvalues using the boundary conditions (2.2) and the positive roots of the equation $J_0(\lambda_k) = 0$, i.e., the values of λ_k represent the zeros of the Bessel function $J_0(x)$, which has the following asymptotic representation (see [10], p.213):

$$J_0(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x + \frac{\pi}{4}\right) + \frac{r_0(x)}{x\sqrt{x}},$$

where the function $r_0(x)$ remains bounded as $x \to \infty$. Hence, the zeros of $J_0(x)$ are given by ([11], p.556):

$$\lambda_k \simeq k\pi - \frac{\pi}{4} + \frac{1}{8\pi k} + O\left(\frac{1}{k^2}\right).$$

The uniform convergence of Fourier-Bessel series are stated in the following theorems:

Theorem 2.1. *(see* [10], *p.230-231) Let $f(x)$ be a function defined on the interval $[0,1]$ such that $f(x)$ is differentiable $2s$, $s \in \mathbb{N}$ times $s \geq 1$ and such that*

(1) $f(0) = f^{'}(0) = ... = f^{2s-1}(0) = 0$;

(2) $f(1) = f^{'}(1) = ... = f^{2s-1}(1) = 0$;

(3) $f^{2s}(x)$ *are bounded (this derivative may not exist at certain points);*

Then the following inequality is satisfied by the Fourier-Bessel coefficients of $f(x)$:

$$|f_k| \leq \frac{C}{\lambda_k^{2s-1/2}} \quad (c = const).$$

Theorem 2.2. *(see* [10], *p.225) If $\nu \geq 0$, $M > 0$ and if*

$$|f_k| \leq \frac{M}{\lambda_k^{1+\varepsilon}}, \tag{2.4}$$

where ε is a positive constant, then the series

$$\sum_{k=1}^{\infty} f_k J_\nu(\lambda_k x),$$

converges absolutely and uniformly on $[0,1]$.

Next, we recall some properties of the Mittag-Leffler functions.

Let $0<\rho<1$ and μ be an arbitrary complex number. The function defined by the following equation

$$E_{\rho,\mu}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\rho k+\mu)}$$

is called the Mittag-Leffler function with two-parameters (see [7], p. 12).

Lemma 2.3. *(see [7], p. 61) For all positive t and $\rho>0$ one has*

$$\int_0^t \eta^{\rho-1}E_{\rho,\rho}\left(-\lambda\eta^{\rho}\right)d\eta=t^{\rho}E_{\rho,\rho+1}\left(-\lambda t^{\rho}\right). \tag{2.5}$$

3. Investigation of the forward problem

Now we expand functions $u(x,t)$ and $f(x,t)$ in the Fourier-Bessel series (see e.g. [10], p.211), writing them in the form

$$u(x,t)=\sum_{k=1}^{\infty}T_k(t)J_0(\lambda_k x), \tag{3.1}$$

$$f(x,t)=\sum_{k=1}^{\infty}f_k(t)J_0(\lambda_k x). \tag{3.2}$$

Substituting (3.1) and (3.2) into (1.1), and applying the gluing conditions (1.4), we obtain

$$T_k(t)=\begin{cases} A_kE_{\rho,1}\left(-\lambda_k^2t^{\rho}\right)+\int\limits_0^t(t-\tau)^{\rho-1}E_{\rho,\rho}(-\lambda_k^2(t-\tau)^{\rho})f_k(\tau)\,d\tau, & t>0,\\ A_k\left(\cos\lambda_kt-\lambda_k\sin\lambda_kt\right)+\dfrac{f_k(0)\sin\lambda_kt}{\lambda_k}+\dfrac{F_1(\lambda_k,t)}{\lambda_k}, & t<0,\end{cases} \tag{3.3}$$

where A_k are arbitrary constants and

$$F_1(\lambda_k,t)=\int\limits_0^t f_k\left(\eta\right)\sin\lambda_k\left(t-\eta\right)\,d\eta.$$

If $\Delta_\alpha(k)=\sqrt{1+\lambda_k^2}\sin\bigg(\lambda_k\alpha+\theta_k\bigg)\neq 0$ for all $k\in\mathbb{N}$, then for A_k one has

$$A_k=\frac{\varphi_k^*}{\Delta_\alpha(k)}, \tag{3.4}$$

where $\varphi_k^*=\varphi_k-F_1(\lambda_k,-\alpha)+\frac{f_k(0)\sin\lambda_k\alpha}{\lambda_k}$.

Thus, if $\Delta_\alpha(k)\neq 0$ for all k, then the unknown coefficients A_k are found uniquely and problem (1.1)-(1.5) also has a unique solution. Thus, we arrive at the following criterion for the uniqueness of the solution to problem (1.1)-(1.5):

Theorem 3.1. *If there is a solution to problem* (1.1)-(1.5)*, then this solution is unique if and only if the condition $\Delta_\alpha(k)\neq 0$ is fulfilled for all $k\in\mathbb{N}$.*

When solving the problem by the Fourier method, $\Delta_\alpha(k)$ appears in the denominator. Therefore, we need to estimate $\Delta_\alpha(k)$ from below.

Lemma 3.2. *If α is a positive rational number, then there exists a positive constant C_0 and k_0,($k_0 \in \mathbb{N}$) such that for all $k > k_0$ the following estimate from below holds*

$$|\Delta_\alpha(k)| > C_0. \tag{3.5}$$

Proof. Case 1. Let $\alpha = 4l, l \in \mathbb{N}$. Then we obtain

$$\Delta_\alpha(k) = \cos 4\lambda_k l + \lambda_k \sin 4\lambda_k l =$$

$$= (-1)^l \left(1 - 16l^2 \left(\frac{1}{8\pi k} + O\left(\frac{1}{k^2}\right)\right)^2\right) + \left(k\pi - \frac{\pi}{4} + \frac{1}{8\pi k} + O\left(\frac{1}{k^2}\right)\right) (-1)^l \sin\left(\frac{l}{2\pi k} + O\left(\frac{1}{k^2}\right)\right).$$

From this, we get

$$|\Delta_\alpha(k)| > \left|1 - 16l^2 \left(\frac{1}{8\pi k} + O\left(\frac{1}{k^2}\right)\right)^2 + \frac{2}{\pi}\left(k\pi - \frac{\pi}{4} + \frac{1}{8\pi k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{l}{2\pi k} + O\left(\frac{1}{k^2}\right)\right)\right|.$$

Thus, there exists a $k_2 \in \mathbb{N}$ such that, for all $k > \max\{k_1, k_2\}$ we have

$$|\Delta_\alpha(k)| > \left|1 + O\left(\frac{1}{k}\right) + \frac{l}{\pi}\right| > \frac{1}{2} + \frac{l}{\pi} \geq C_1 > 0. \tag{3.6}$$

Case 2. Let $\alpha = 4p/q$ $p, q \in \mathbb{N}$ is a rational number, where p and q are coprime numbers, i.e, $GCF(p, q) = 1$. Dividing $(4k-1)p$ by q with remainder $(4k-1)p = sq+r, \quad s, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leq r < q$, we have

$$|\Delta_\alpha(k)| = \left|(-1)^s \sqrt{1 + \lambda_k^2} \sin\left(\frac{r\pi}{q} + \frac{p}{2\pi qk} + O\left(\frac{1}{k^2}\right) + \theta_k\right)\right|. \tag{3.7}$$

If $r = 0$ then, (3.7), we obtain

$$|\Delta_\alpha(k)| = \sqrt{1 + \lambda_k^2} \left|\sin\left(\frac{p}{2\pi qk} + O\left(\frac{1}{k^2}\right) + \theta_k\right)\right|. \tag{3.8}$$

θ_k satisfies the estimate

$$\frac{1}{\sqrt{1 + \lambda_k^2}} \leq \theta_k \leq \frac{\pi}{2} \frac{1}{\sqrt{1 + \lambda_k^2}} < \frac{1}{k}. \tag{3.9}$$

Using (3.8) and (3.9) we have

$$|\Delta_\alpha(k)| > \frac{2}{\pi}\sqrt{1 + \lambda_k^2}\left(\frac{p}{2\pi qk} + O\left(\frac{1}{k^2}\right) + \frac{1}{\sqrt{1 + \lambda_k^2}}\right) = \frac{p}{\pi^2 qk}\sqrt{1 + \lambda_k^2} + \frac{2}{\pi} + O\left(\frac{1}{k^2}\right).$$

There exists a $k_4 \in \mathbb{N}$ such that, for all $k > \max\{k_3, k_4\}$ we have

$$|\Delta_\alpha(k)| \geq \frac{p}{4\pi q} + \frac{2}{\pi} \geq C_2 > 0. \tag{3.10}$$

Now let $r > 0$. Then, obviously, $1 \leq r \leq q - 1$, $q \geq 2$. Since the expression

$$\widetilde{\Delta}_\alpha(k) = \sin\left(\frac{r\pi}{q} + \frac{p}{2\pi qk} + O\left(\frac{1}{k^2}\right) + \theta_k\right),$$

has a non zero positive limit as $k \to \infty$, there exists a $k_5 \in \mathbb{N}$ such that, for all $k > k_5$ we can write

$$|\Delta_\alpha(k)| \geq \frac{1}{2}\sqrt{1+\lambda_k^2}\left|\sin\frac{\pi r}{q}\right| \geq \frac{\pi k}{4}\left|\sin\frac{\pi}{q}\right| = kC_3 \geq C_3 > 0. \tag{3.11}$$

Then inequalities (3.6), (3.10) and (3.11) imply the validity of estimate (3.5) for all values of k satisfying the inequality $k > k_0 = \max\{k_1, k_2, k_3, k_4, k_5\}$.

If, for the numbers α from Lemma 3.1, for some $k = p = k_1, k_2, ..., k_m$, where $1 \leq k_1 < k_2 < ... < k_m \leq k_0$, $\Delta_\alpha(k) = 0$ then problem (1.1)-(1.5) has a solution if and only if $\varphi_k^* = 0$ (see (3.4)). Naturally, in this case the solution of forward problem is determined as the sum of the series

$$u(x,t) = \left(\sum_{k=1}^{k_1-1} + ... + \sum_{k=k_{m-1}+1}^{k_m-1} + \sum_{k=k_m+1}^{\infty}\right) T_k(t)v_k(x) + \sum_p A_p T_p(t) v_p(x), \tag{3.12}$$

where, in the last sum, p takes the values $k_1, k_2, ..., k_m$ and $T_p(t)$ is defined by formula (3.3); here k must be replaced by p and A_p is an arbitrary constant.

4. Justification of formal solution

The formal solution to problem (1.1)-(1.5) has the form (3.3). Now we show that the series (3.3) satisfies all conditions of Definition 1.1.

To estimate the coefficients of the Fourier-Bessel series of the function $u(x,t)$ and the series obtained by its differentiation, we calculate the general terms of these series. Consider the case for $t > 0$, and in the case $t < 0$ the absolute convergence of solution (3.1) is proved in a similar way. Let

$$u(x,t) = \sum_{k=1}^{\infty}\left(\left(\frac{\varphi_k}{\Delta_\alpha(k)} + \frac{f_k(0)\sin\lambda_k\alpha}{\lambda_k\Delta_\alpha(k)} + \frac{F_1(\lambda_k,-\alpha)}{\lambda_k\Delta_\alpha(k)}\right)E_{\rho,1}(-\lambda_k^2 t^\rho) + F_1(\lambda_k,t)\right)J_0(\lambda_k x), \quad t \geq 0. \tag{4.1}$$

According to Theorem 2.2, we conclude that the representation of the series of $u(x,t)$ converges absolutely and uniformly in Q^+.

Now we prove the uniform convergence of the series $D_t^\rho u(x,t)$. It is not hard to see that $D_t^\rho F_1(\lambda_k,t) = f_k(t) - \lambda_k^2 F_1(\lambda_k,t)$. By Theorem 2.2, we conclude that, the series representation of $D_t^\rho u(x,t)$ converges absolutely and uniformly in $\overline{Q}_+$.

Finally, it remains to show the uniform convergence of the series representation of $u_{xx}(x,t)$, which is given by

$$u_{xx}(x,t) = \sum_{k=1}^{\infty}\frac{\lambda_k^2}{2}\left(U(k,t) + F_1(\lambda_k,t)\right)\left[J_2(\lambda_k(x)) - J_0(\lambda_k(x))\right].$$

Hence, for convergence, we have the following estimate:

$$\left|\frac{\lambda_k^2}{2}\left(U(k,t) + F_1(\lambda_k,t)\right)\right| \leq \frac{M_{10}}{\lambda_k^{3/2}} + \frac{M_{11}}{\lambda_k^{5/2}} + \frac{M_{12}}{\lambda_k^{5/2}} + \frac{M_{13}}{\lambda_k^{3/2}}.$$

Therefore, by Theorem 2.2, we conclude that, the series representation of $u_{xx}(x,t)$ converges absolutely and uniformly in Q_+. The convergence of series $\frac{\partial^2 u(x,t)}{\partial t^2}$, $\frac{\partial^2 u(x,t)}{\partial x^2}$ in Q_- can be shown in a similar way.

Thus we have the following theorem

Theorem 4.1. *Let the functions $\varphi(x)$ and $f(x,t)$ are differentiable four times with respect to x, and for derivatives with respect to x the following equalities be satisfied:*

(1) $f(0,t) = f'(0,t) = f''(0,t) = f'''(0,t) = 0;\quad f(1,t) = f'(1,t) = f''(1,t) = 0;\ \forall t \in (-\alpha, \beta)$;

(2) $\varphi(0) = \varphi'(0) = \varphi''(0) = \varphi'''(0) = 0;\quad \varphi(1) = \varphi'(1) = \varphi''(1) = 0$;

(3) $\partial^4 f(x,t)/\partial x$ *and* $\partial^4\varphi(x)/\partial x$, $\forall t \in (-\alpha, \beta)$, $x \in (0,1)$, *are bounded;*

(4) $f(x,t)$ *is continuous and continuously differentiable with respect to* t.

and let α be a positive rational number. In that case, if $\Delta_\alpha(k) \neq 0$ for $k = 1, ..., k_0$, then there exists a unique solution of problem (1.1)-(1.5), *which is determined by the series* (3.1). *If $\Delta_\alpha(k) = 0$ for some $k = 1, ..., k_0 \le k_0$, then problem* (1.1)-(1.5) *is solvable if and only if conditions*

$$\varphi_k = F_1(\lambda_k, -\alpha) - \frac{f_k(0)\sin\lambda_k\alpha}{\lambda_k}, \tag{4.2}$$

hold, and, in this case, the solution is determined as the sum of the series (3.12).

Remark 4.2. We especially emphasize that conditions (4.2) are both necessary and sufficient. However, these conditions are somewhat difficult to check. Therefore, we can formulate the following sufficient conditions

$$\varphi_k = (\varphi, X_k) = 0,\ \ f_k(t) = (f(t), X_k) = 0,$$

which, on the one hand, are easily verified, and, on the other hand, when they are satisfied, the necessary and sufficient conditions (4.2) are also satisfied.

5. Existence and uniqueness of the solution of the inverse problem

In order to find a formal solution to the inverse problem, we repeat all the actions that were used in the forward problem, and by virtue of (3.3) and (2.5) we have

$$T_k(t) = \begin{cases} a_{1k}E_{\rho,1}(-\lambda_k^2 t^\rho) + f_k t^\rho E_{\rho,\rho+1}(-\lambda_k^2 t^\rho), & t > 0, \\ b_{1k}\cos\lambda_k t + c_{1k}\sin\lambda_k t + \dfrac{f_k}{\lambda_k^2}\left(1 - \cos\lambda_k t\right), & t < 0. \end{cases} \tag{5.1}$$

After applying the conditions (1.4) we find the unknown coefficients a_{1k}, b_{1k}, c_{1k} as follows

$$a_{1k} = b_{1k} = \frac{\lambda_k\varphi_k + f_k(\Delta_\alpha(k) - 1)}{\Delta_\alpha(k)}, c_{1k} = \frac{f_k - \lambda_k a_{1k}}{\lambda_k}. \tag{5.2}$$

Taking (5.2) into account (5.1) we get

$$T_k(t) = \begin{cases} E_{\rho,1}(-\lambda_k t^\rho)\left(\dfrac{\varphi_k\lambda_k^2}{\Delta_\alpha(k)} - \dfrac{f_k}{\lambda_k^2\Delta_\alpha(k)}\right) + \dfrac{f_k}{\lambda_k^2}, & t > 0, \\ f_k\left(\Delta_\alpha + \sin\lambda_k t - \lambda_k\cos\lambda_k t\right) + \lambda_k^2\varphi_k, & t < 0. \end{cases} \tag{5.3}$$

For the solution the forward problem (3.1), function $U(t_0, \rho)$, defined in the additional condition (1.6), has the form

$$U(t_0,\rho) = \sum_{k=1}^{\infty}\left|E_{\rho,1}(-\lambda_k t_0^\rho)\right|^2\left|\frac{\varphi_k\lambda_k^2}{\Delta_\alpha(k)} - \frac{f_k}{\lambda_k^2\Delta_\alpha(k)}\right|^2, \quad 0 < t_0 < \infty.$$

Lemma 5.1. *Let $\rho_0 \in (0,1)$ and conditions of Theorem 2.2 be satisfied. There exists a number $T_0 = T_0(\rho_0) > 0$, such that for all $t_0 \geq T_0$, function $U(t_0, \rho)$ decreases monotonically with respect to $\rho \in [\rho_0; 1]$.*

It is easy to see that the following main result on the inverse problem is an immediate consequence of this lemma.

Theorem 5.2. *Let $\rho_0 \in (0,1)$ and let number T_0 be from Lemma 5.1. Then for all $t_0 \geq T_0$ the solution of the inverse problem $\{u(t), \rho\}$, $\rho \in [\rho_0, 1]$, exists and is unique if and only if*

$$U(t_0, 1) \leq u_0 \leq U(t_0, \rho_0). \tag{5.4}$$

To prove Theorem 5.2 we need the following lemma.

Lemma 5.3. *Let $\lambda_1 > 0$ be the first eigenvalue of Bessel operator. Given ρ_0 from the interval $0 < \rho_0 < 1$, there exists a number $T_0 = T_0(\lambda_1, \rho_0)$, such that for all $t_0 \geq T_0$ and $\lambda \geq \lambda_1$ the function $e_\lambda(\rho) = E_{\rho,1}(-\lambda^2 t_0^\rho)$ is positive and monotonically decreasing with respect to $\rho \in [\rho_0, 1]$ and*

$$e_\lambda(1) \leq e_\lambda(\rho) \leq e_\lambda(\rho_0).$$

Proof of Lemma 5.3 can be found in [2].

The proof of the main Lemma 5.1 follows easily from Lemma 5.3.

6. Proof of the main Lemma 5.1

Let us denote

$$C_k = \left| \frac{\varphi_k \lambda_k^2 - f_k/\lambda_k^2}{\Delta_\alpha(k)} \right|^2 \geq 0,$$

so that the function $U(t_0, \rho)$ can be written as

$$U(t_0, \rho) = \sum_{k=1}^{\infty} \left| E_{\rho,1}(-\lambda_k t_0^\rho) \right|^2 C_k.$$

From Lemma 5.3, we know that there exists $T_0 = T_0(\lambda_1, \rho_0)$ such that for all $t_0 \geq T_0$ and all $\lambda_k \geq \lambda_1$, the function

$$e_{\lambda_k}(\rho) = E_{\rho,1}(-\lambda_k t_0^\rho)$$

is positive and strictly decreasing with respect to $\rho \in [\rho_0, 1]$. Therefore, for each k, the term $\left| E_{\rho,1}(-\lambda_k t_0^\rho) \right|^2$ is decreasing with respect to ρ.

Since $C_k \geq 0$ and independent of ρ, each summand in the series defining $U(\rho)$ is monotonically decreasing in ρ on the interval $[\rho_0, 1]$. As a sum of monotonically decreasing nonnegative functions, the function $U(\rho)$ is also monotonically decreasing on this interval. Hence, for all $t_0 \geq T_0$, $U(t_0, \rho)$ is decreasing with respect to $\rho \in [\rho_0, 1]$.

As mentioned above, the Theorem 5.2 is a consequence of Lemma 5.1.

7. Acknowledgement.

The author is grateful to R.R. Ashurov and Z.A. Sobirov for discussions of these results and acknowledges financial support from the Ministry of Innovative Development of the Republic of Uzbekistan, Grant No F-FA-2021-424.

References

[1] Al-Musalhi F., Al-Salti N., Kerbal S.; Inverse problems of a fractional differential equation with Bessel operator. Math.Model.Nat.Phenom.–2017.–12.–3.–P.105-113.

[2] Ashurov R.R., Murzambetova M.B.; Inverse problem of determining the fractional derivative order in mixed-type equations. Uzbek mathematical journal.–2023.–2.–P.67-77.

[3] Durdiev D.K., Merajova Sh.B.; Inverse problem for an equation of mixed parabola-hyperbolic type with a Bessel operator. Jour. of Applied and Industrial Mathematics.–2022.–25.–3.–P.14-24.

[4] Durdiev D.K.; Inverse source problem for mixed parabola-hyperbolic equation with fractional time derivative in cylindrical domain. Vestn. Sam. gos. tehn. un-t. Ser. Fiz.-mat. nauki.–2022.–No2.–P.355-367.

[5] Il'in V.A.; The solvability of mixed problems for hyperbolic and parabolic equations. Russian Mathematical Surveys.–1960.–15.–2.–P.85-142.

[6] Karimov E., Mamchuev M., Ruzhanskiy M.; Non-local initial problem for second order time-fractional and space-singular equation. Hokkaido Mathematical Journal. –2020.–49.–P.349-361.

[7] Kilbas A.A., Srivastava H.M., Trujillo J.J.; Theory and applications of fractional defferential equations. Elsevier, North-Holland, Mathematics studies.–2006.

[8] Masood K., Messaoudi S., Zaman F.D.; Initial inverse problem in heat equation with Bessel operator. Int.J. of Heat and Mass Transfer.–2012.–45.–P.2959-2965.

[9] Pskhu A.V.; Fractional Partial Differential Equations.M., Nauka.(in Russian).–2005.

[10] Tolstov G.P.; Fourier series (translated by R.A. Silverman). Prentice Hall, Inc., Englewood Cliffs, N.J.–1962.

[11] Watson G. N.; A treatise on the theory of Bessel functions. Cambridge University Press. Second Edition.–1966.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.86-95
DOI: 10.29229/uzmj.2025-2-8

On regular solvability of one nonlocal boundary value problem for a mixed type equation of the second kind of the fourth order

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. This article discusses the single-valued solvability of a nonlocal boundary value problem for a mixed type equation of the second kind of the fourth order in the multidimensional case using the Faedo-Galerkin method and a priori estimates.

Keywords: three-dimensional equation of mixed type of the second kind of the fourth order, linear inverse problem with nonlocal boundary conditions of periodic type, correctness of the problem, Fourier method, methods of " $\varepsilon-$regularization", a priori estimates, sequence of approximations and Fourier transforms.

MSC (2020):35M10

1. Introduction

We consider the solvability of one class of boundary value problems for a fourth order mixed type equation of the form

$$Lu = Pu + Mu + \lambda u = f(x,t),\ (x,t) \in Q = \Omega \times (0,T), \tag{1.1}$$

where λ – is some real parameter and Ω is a bounded domain with boundary $\Gamma \in C^5$. Here

$$Pu = \sum_{i=0}^{4} K_i(x,t)\partial_t^i u,$$

where $\partial_t^i u = \dfrac{\partial^i u}{\partial t^i}$, $\partial_t^0 u = u$, $(i = 0,1,2,3,4)$. The operator M is defined by the differential expression

$$Mu = \sum_{|\alpha|+|\beta|\le 4} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta),\ a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$$

and the boundary conditions

$$B_j u|_\Gamma = 0,\quad B_j u = \sum_{|\beta|\le m_j} b_{j\beta}(x) D^\beta u,\ (j = 1,2),\ m_1 < m_2. \tag{1.2}$$

2. Boundary value problem

Find a solution $u(x,t)$ of equation (1.1), satisfying boundary conditions (1.2) and conditions

$$\left.\partial_t^p\ u\right|_{t=0} = \alpha \left.\partial_t^p\ u\right|_{t=T};\ \ p = 0,1,2, \tag{2.1}$$

where $\alpha \neq 0$ – is some real number.

In A.V. Bitsadze [3] it is shown that the Dirichlet problem for a second order mixed type equation is incorrect. The question naturally arises: is it not possible to replace the conditions of the Dirichlet problem by other conditions defined on the entire boundary, which ensure the correctness of the problem? For the first time such boundary value problems (nonlocal boundary value problems) for a mixed type equation of the second order were proposed and studied in the work of F.I. Frankl [12]. As close in formulation to the studied ones, the problem for a mixed-type equation of the second order was studied in bounded domains in [13], [6], [7], [8], [17], [22]. Nonlocal boundary value problems for a high-order partial derivative equation without derivation have been studied by many scientists, a complete bibliography is available in the book by A. Dezin [5]. For high order mixed type equations with local boundary conditions have been studied in different spaces in [25], [11], [4] and with nonlocal boundary conditions such problems have been very little studied [9], [18], [10]. In this paper, using the results of [25], [11], [7], [8] and applying the modified Galerkin method and a priori estimates, we study the single-valued solvability and smoothness of the regular generalized solution of the boundary value problem (1.1)-(2.1). We consider a model case, a 4th order equation; however, the reasoning is general and can be easily transferred to the case of higher order equations.

Problem statement and main results

Let us give conditions on the data. We assume that M, with the domain of definition $D(M) = \{u \in W_2^4(\Omega) : \; B_j u|_\Gamma = 0, \; j = 1, \ldots, 4\}$ is self-adjoint and positively defined in $L_2(\Omega)$, i.e., there exists a constant δ_0 such that $(Mu, u)_0 \geq \delta_0 \|u\|_0^2$ for all $u \in D(M)$, where $\|\cdot\|_0$, $(\cdot,\cdot)_0$ – are the norm and scalar product in the space $L_2(\Omega)$. The conditions when positive definiteness of the operator M takes place are discussed, for example, in [21]. We may not specify specific smoothness conditions on the coefficients of the operators M, B_j, however, from the point of view of the general theory [1], [21] it is enough to assume that

$$a_{\alpha\beta} \in C^{\max(|\alpha|,|\beta|)}(\overline{\Omega}), \; b_{j\beta} \in C^{\beta_j}(\overline{\Omega}), \; \beta_j > 2m - m_j - 1/2.$$

We first assume that

$$K_{4t}, K_4, K_3 \in L_\infty(Q), \; K_i \in L_\infty(0, T; L_{p_i}(\Omega)), i = 0, 1, 2, \tag{2.2}$$

where $p_i > n/2$ for $n \geq 4$, $p_i = 2$ for $n < 4$, $i = 0, 1$; $p_2 > n$ for $n \geq 2$ and $p_2 = 2$ for $n = 1$. Equation (1.1) belongs to mixed type equations of the second kind, since no restrictions are imposed on the sign of the function $K_4(x, t)$ on the variable t inside the interval $[0, T]$ [26], [8]. However, we still need some sign-conditioning. We assume that the following inequalities are satisfied

$$K_4(x, T) \geq 0, \; K_4(x, 0) \leq 0, \; K_3 - \frac{(\gamma K_4 + K_{4t})}{2} - K_4\beta \geq \delta_1, \; \gamma = \frac{2}{T} \ln|\alpha|, \tag{2.3}$$

for all $(x, t) \in Q$ and some constants $\beta < 3\gamma/2$, $\delta_1 > 0$. When obtaining various a priori estimates, we often use the inequality of the

$$|ab| \leq \varepsilon \frac{|a|^p}{p} + \varepsilon^{-1/(p-1)} \frac{|b|^q}{q}, \; 1/p + 1/q = 1. \tag{2.4}$$

We will need the following auxiliary statement.

Lemma 2.1. *Let $P_0 u = e^{\gamma t}(\partial_t^3 u + \beta \partial_t^2 u)$. Then there exists a number $\lambda_0 \geq 0$ such that,*

$$\int_0^T (P_0 u + \lambda_0 u) u \, dt \geq \delta_2 \|u\|^2_{W_2^1(0,T)}, \quad \gamma = \frac{2}{T} \ln|\alpha|, \; \beta < 3\gamma/2.$$

for all $u \in W_2^3(0, T)$ satisfying condition (1.2).

The statement of the lemma is a consequence of Theorem 1 and Remark 1 in [9]. Let us give some properties of the operator M.

The operators M^s ($s \in \mathbb{R}$) under our assumptions on Γ and on the coefficients of the boundary operators B_j, belong to the class $C^4(\Gamma)$. So we have (a consequence of the equality $D(M^s) =$

$(D(M), L_2(\Omega))_{1-s,2} = [D(M), L_2(\Omega)]_{1-s}$ (in Section 1.18.2 in [24] and in Theorem 8.1 of [15]) (See also the descriptions of the space $(D(M), L_2(\Omega))_{\theta,2}$ in Section 4.3.3 of [24] and in Theorem 7.5 of [16].))

$$D(M^s) = \{u \in W_2^{4s}(\Omega) : B_j u|_\Gamma = 0 \text{ for all } j \text{ such that } m_j < 4s - 1/2,$$

$$\int\limits_\Omega \frac{1}{\rho}|B_j u|^2\, dx < \infty \text{ if } m_j = 4s - 1/2\}, \quad \rho(x) = dist\,(x, \Gamma).$$

Apparently, there are also results where the previous equations are obtained under weaker conditions on the coefficients of the operators B_j.Let us assume that the two previous equations are satisfied. As a consequence, we have that either $D(M^{1/4}) = W_2^1(\Omega)$ if the first of the boundary operators B_1 does not impose the boundary condition Dirichlet $u|_\Gamma = 0$, or $D(M^{1/4}) = \overset{\circ}{W}{}_2^1(\Omega)$ otherwise. Let us also give another description of $D(M^s)$. Let λ_k, ω_k $(k = 1, 2, \dots)$ – be the eigennumbers and eigenvalues of the operator M. Without restriction of generality, we may assume that the eigenfunctions form an orthonormalized basis in $L_2(\Omega)$. Then

$$D(M^s) = \{u \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2s}|(u, \omega_k)_0|^2 < \infty\}, \ s > 0,$$

respectively, $M^s u = \sum_{k=1}^{\infty} \lambda_k^s (u, \omega_k)_0 \omega_k$. We will also use the equality

$$(W_2^s(0, T; L_2(\Omega)), L_2(0, T; W_2^r(\Omega)))_{\theta,2} = W_2^{s(1-\theta)}(0, T; W_2^{r\theta}(\Omega)), \ \theta \in (0, 1). \tag{2.5}$$

Next, let us put $D(M^s) = \tilde{W}_2^{4s}(\Omega)$. The dual space to $\tilde{W}_2^s(\Omega)$, constructed using the scalar product in $L_2(\Omega)$ denote by $\tilde{W}_2^{-s}(\Omega)$ $(s > 0)$. The following two theorems are actually corollaries of the corresponding results in [9] (see Theorems 3,4).

Theorem 2.2. *Let the above conditions on the operator be satisfied M and conditions* (2.3), (2.2) *for the coefficients of the operator P. Let $f(x, t) \in L_2(Q)$, then there is a constant λ_0 such that at $\lambda \geq \lambda_0$ the problem* (1.1)–(1.2) *has generalized solution $u(x, t)$ from the class $u \in L_2(0, T; \tilde{W}_2^3(\Omega)) \cap W_2^3(0, T; L_2(\Omega))$, $\partial_t(K_4 u^{(2n-1)}) \in L_2(0, T; W_2^{-1}(\Omega))$.*

Here, as usual, by the space $W_2^{-1}(\Omega))$ we mean the dual space to $\overset{\circ}{W}{}_2^1(\Omega)$ with respect to the duality defined by the scalar production in $L_2(\Omega)$. If we additionally demand that

$$K_i = K_i(t) \ (i = 3, 4), \ K_4 \in C^1([0, T]), \tag{2.6}$$

then the following theorem holds.

Theorem 2.3. *Let the conditions of Theorem 4.1 and condition* (2.6) *be satisfied. Then the solution constructed in Theorem 4.1 is unique.*

The assertion follows from Theorem 4 in [9]. Let us give additional conditions on the coefficients K_j:

$$K_i \in L_\infty(0, T; L_{r_i}(\Omega)), \ \nabla_x K_i \in L_\infty(0, T; L_{q_i}(\Omega)), \tag{2.7}$$

where at i=0,1, $r_i > n/2$ at $n \geq 4$, $r_i = 2$ at $n < 4$, $q_i > n/3$ at $n \geq 6$, $p_i = 2$ at $n < 6$, and at $i = 2$ $r_2 > n$ at $n \geq 2$, $r_2 = 2$ at $n < 2$, $q_2 > n/2$ at $n \geq 4$, $q_2 = 2$ at $n < 4$.

Let $(u, v) = \int_Q u(x, t)v(x, t)\, dxdt$ – be a scalar product in $L_2(Q)$. The following theorem is the main result of our work. Here we show we obtain regular solvability of the problem, and the obtained solution possesses all generalized derivatives included in the equation.

Theorem 2.4. *Suppose that conditions* (2.2), (2.3), (2.6), (2.7) *are satisfied and* $f \in L_2(0,T;\tilde{W}_2^1(\Omega))$. *Then we find* $\lambda_1 \geq 0$ *such that, given* $\lambda \geq \lambda_1$ *there exists a single solution to problem* (1.1)-(2.1) *such that* $u \in L_2(0,T;\tilde{W}_2^4(\Omega)) \cap W_2^3(0,T;L_2(\Omega))$, $\partial_t(K_4\partial_t^3 u) \in L_2(0,T;L_2(\Omega))$.

Proof. The proof is in general similar to the proof of Theorem 3 in [9], but it requires a number of additional considerations related to obtaining a priori estimates and the limit transition. Let us take the basis $\psi_j(t)$ in the space $W_2^1(0,T)$ and the orthonormalized basis $\{\omega_i\}$ in the space $L_2(\Omega)$ of the eigenfunctions of the operator M. It will also be and orthogonal basis in $\tilde{W}_2^4(\Omega)$, with scalar product $< u, v >= (Mu, Mv)_0$.

Then the system $\{\psi_j\omega_i\}$ – is a basis in the space $W_2^1(0,T;L_2(\Omega)) \cap L_2(0,T;\tilde{W}_2^4(\Omega))$. Consider the operator $P_0 + \lambda_0$. Let us construct the function $\varphi_j(t)$ as a solution of $(P_0 + \lambda_0)\varphi_j = \psi_j$ satisfying boundary conditions (1.2). We look for an approximation the solution of our problem in the form $u^N = \sum_{i,j=1}^N c_{i,j}\varphi_j\omega_i$, where the constants $c_{i,j}$ are determined by the from the system of equations

$$(Lu_N, \psi_j\omega_i) = (f, \psi_j\omega_i),\ i,j = 1,\dots,N. \tag{2.8}$$

Let us show the solvability of this system for suitable λ. The proof of the first a priori estimate is quite similar to the corresponding proof of Theorem 3 of [9]. Therefore, we will only give its scheme. Multiply (2.8) by $c_{i,j}$ and summarize the result for i,j. We obtain the equality

$$(Lu_N, (P_0 + \lambda_0)u_N) = (f, (P_0 + \lambda_0)u_N). \tag{2.9}$$

We have by the force of Lemma 2.5 (we consider that $\lambda > 0$)

$$(Mu_N + \lambda u_N, (P_0 + \lambda_0)u_N) = (M^{1/2}u_N, (P_0 + \lambda_0)M^{1/2}u_N) + \lambda\,(u_N, (P_0 + \lambda_0)u_N) \geq \delta_1(\|u_N\|^2_{W_2^1(0,T;W_2^2(\Omega))} + \lambda\|u_N\|^2_{W_2^1(0,T;L_2(\Omega))}), \tag{2.10}$$

where δ_1 – is some positive constant. We can visualize the expression Pu_N in the form $Pu_N = K_4\partial_t^4 u_N + K_3\partial_t^3 u_N + lu_N$. Repeating the reasoning from the proof of Theorem 3 in [10], we arrive at the estimate

$$(K_4\partial_t^4 u_N + K_3\partial_t^3 u_N, (P_0 + \lambda_0)u_N) \geq \delta_2(\|\sqrt{|K_4(T)|}\partial_t^3 u_N(x,T)\|^2_{L_2(\Omega)} + \|\sqrt{|K_4(0)|}\partial_t^3 u_N(x,0)\|^2_{L_2(\Omega)} + \|u_N\|^2_{W_2^3(0,T;L_2(\Omega))}) - c_1\|u_N\|^2_{L_2(Q)}. \tag{2.11}$$

Finally, the expression $(lu_N, (P_0 + \lambda_0)u_N)$ admits an estimate of

$$|(lu_N, (P_0+\lambda_0)u_N)| \leq \varepsilon_0(\|u\|^2_{W_2^3(0,T;L_2(\Omega))} + \|u\|^2_{W_2^1(0,T;W_2^2(\Omega))}) + c(\varepsilon_0)\|u_N\|^2_{L_2(Q)},\ \varepsilon_0 > 0, \tag{2.12}$$

where the parameter ε_0 is arbitrary (see the proof of Theorem 3 in [10]). The embedding theorems, interpolation inequalities and inequality (2.4) were used to obtain this estimate. We do not specify the deductions; below we will give analogous deductions in the proof of the second a priori estimate. Adding (2.10), (2.11), we obtain

$$(K_4\partial_t^4 u_N + K_3\partial_t^3 u_N + Mu_N + \lambda u_N, (P_0 + \lambda_0)u_N) \geq \delta_2(\|\sqrt{|K_4(T)|}\partial_t^3 u_N(x,T)\|^2_{L_2(\Omega)} + \|\sqrt{|K_4(0)|}\partial_t^3 u_N(x,0)\|^2_{L_2(\Omega)} + \|u_N\|^2_{W_2^3(0,T;L_2(\Omega))}) + \delta_1(\|u_N\|^2_{W_2^1(0,T;W_2^2(\Omega))} + \lambda\|u_N\|^2_{W_2^1(0,T;L_2(\Omega))}) - c_2\|u_N\|^2. \tag{2.13}$$

Using (2.12), we obtain

$$(Lu_N, (P_0+\lambda_0)u_N) \geq \delta_1(\|u_N\|^2_{W_2^1(0,T;W_2^2(\Omega))} + \lambda\|u_N\|^2_{W_2^1(0,T;L_2(\Omega))}) + \delta_2(\|\sqrt{|K_4(T)|}\partial_t^3 u_N(x,T)\|^2_{L_2(\Omega)} + \|\sqrt{|K_4(0)|}\partial_t^3 u_N(x,0)\|^2_{L_2(\Omega)} + \|u_N\|^2_{W_2^3(0,T;L_2(\Omega))}) - \varepsilon_0(\|u\|^2_{W_2^3(0,T;L_2(\Omega))} + \|u\|^2_{W_2^1(0,T;W_2^2(\Omega))}) - c_1(\varepsilon_0)\|u_N\|^2_{L_2(Q)}. \quad (2.14)$$

Choosing the parameter $\varepsilon_0 < \min(\delta_1,\delta_2)/2$ and then a sufficiently large parameter λ_1, ($\lambda_1 \geq 2c_1(\varepsilon_0)/\delta_1$) we find that at $\lambda \geq \lambda_1$ the following estimate is valid

$$(Lu_N, (P_0+\lambda_0)u_N) \geq \delta_3(\|u_N\|^2_{W_2^1(0,T;W_2^2(\Omega))} + \lambda\|u_N\|^2_{W_2^1(0,T;L_2(\Omega))}) + \|\sqrt{|K_4(T)|}\partial_t^3 u_N(x,T)\|^2_{L_2(\Omega)} + \|\sqrt{|K_4(0)|}\partial_t^3 u_N(x,0)\|^2_{L_2(\Omega)} + \|u_N\|^2_{W_2^3(0,T;L_2(\Omega))}), \quad (2.15)$$

where δ_3 – is some real constant and $\lambda \geq \lambda_1$. Using inequality (2.4), we have

$$|(f, (P_0+\lambda_0)u_N)| \leq c(\varepsilon)\|f\|^2_{L_2(Q)} + \varepsilon\|u_N\|_{L_2(0,T;W_2^3(\Omega))},\ \varepsilon > 0.$$

Then, choosing a sufficiently small parameter ε and using equality (2.9) and estimation (2.15), we arrive at the first a priori estimation

$$\|u_N\|^2_{W_2^1(0,T;W_2^2(\Omega))} + \lambda\|u_N\|^2_{W_2^1(0,T;L_2(\Omega))} + \|\sqrt{|K_4(T)|}\partial_t^3 u_N(x,T)\|^2_{L_2(\Omega)} + \|\sqrt{|K_4(0)|}\partial_t^3 u_N(x,0)\|^2_{L_2(\Omega)} + \|u_N\|^2_{W_2^3(0,T;L_2(\Omega))} \leq c_1\|f\|^2_{L_2(Q)}, \quad (2.16)$$

where the constant c_1 does not depend on N. The solvability of the system (2.8) follows from this estimate and the sharp angle lemma. Let us proceed to the proof of the second a priori estimate. Multiply (2.8) by $\lambda_i^{1/2}c_{i,j}$ and summarize the result for i, j. We obtain equality

$$(Lu_N, M^{1/2}(P_0+\lambda_0)u_N) = (f, M^{1/2}(P_0+\lambda_0)u_N). \quad (2.17)$$

We have that Lemma 2.5 holds (assuming that $\lambda \geq \lambda_1$)

$$(Mu_N + \lambda u_N, M^{1/2}(P_0+\lambda_0)u_N) = (M^{3/4}u_N, (P_0+\lambda_0)M^{3/4}u_N) + \lambda\,(M^{1/4}u_N, (P_0+\lambda_0)M^{1/4}u_N) \geq \delta_1(\|u_N\|^2_{W_2^1(0,T;W_2^3(\Omega))} + \lambda\|u_N\|^2_{W_2^1(0,T;W_2^1(\Omega))}). \quad (2.18)$$

Then, as before, we have an estimate (see inequality (2.15))

$$(K_4\partial_t^4 u_N + K_3\partial_t^3 u_N, M^{1/2}(P_0+\lambda_0)u_N) = (K_4\partial_t^4 M^{1/4}u_N + K_3\partial_t^3 M^{1/4}u_N, (P_0+\lambda_0)M^{1/4}u_N) \geq \delta_2(\|\sqrt{|K_4(T)|}M^{1/4}\partial_t^3 u_N(x,T)\|^2_{L_2(\Omega)} + \|\sqrt{|K_4(0)|}M^{1/4}\partial_t^3 u_N(x,0)\|^2_{L_2(\Omega)} + \|M^{1/4}\partial_t^3 u_N\|^2) - c_4\|M^{1/4}u_N\|^2. \quad (2.19)$$

Adding the estimates (2.18), (2.19), we arrive at the inequality

$$(K_4\partial_t^4 u_N + K_3\partial_t^3 u_N + Mu_N + \lambda u_N, M^{1/2}(P_0+\lambda_0)u_N) \geq \delta_3(\|u_N\|^2_{W_2^1(0,T;W_2^3(\Omega))} + \lambda\|u_N\|^2_{W_2^1(0,T;W_2^1(\Omega))}) + \|\sqrt{|K_4(T)|}M^{1/4}\partial_t^3 u_N(x,T)\|^2_{L_2(\Omega)} + \|\sqrt{|K_4(0)|}M^{1/4}\partial_t^3 u_N(x,0)\|^2_{L_2(\Omega)} + \|u_N\|^2_{W_2^3(0,T;W_2^1(\Omega)}) - c_5\|u_N\|^2_{L_2(0,T;W_2^1(\Omega))}. \quad (2.20)$$

Note that the estimate for the last summand follows from the first a priori estimate. Let us evaluate the expression $|(lu_N, M^{1/2}(P_0+\lambda_0)u_N)|$. Consider the senior slavable, e.g., the expression $J_1 = |(K_2(x,t)\partial_t^2 u_N, M^{1/2}\partial_t^3 u_N)|$. Note that if the first boundary condition $B_1 u|_\Gamma = 0$ is the Dirichlet condition, then the function $K_2(x,t)\partial_t^2 u_N$ by construction also satisfies this condition. Then if $K_2(x,t)\partial_t^2 u_N \in L_2(0,T;W_2^1(\Omega))$, then in either case $K_2(x,t)\partial_t^2 u_N \in L_2(0,T;\tilde{W}_2^1(\Omega))$ and we can write that $J_1 = |(K_2(x,t)\partial_t^2 u_N, M^{1/2}\partial_t^3 u_N)| = |(M^{1/4}K_2(x,t)\partial_t^2 u_N, M^{1/4}\partial_t^3 u_N)|$. Let us show that $K_2(x,t)\partial_t^2 u_N \in L_2(0,T;W_2^1(\Omega))$. Taking into account the above, we obtain the following estimate

$$\|K_2(x,t)\partial_t^2 u_N\|^2_{L_2(0,T;W_2^1(\Omega))} \le \|K_2(x,t)\partial_t^2 u_N\|^2_{L_2(Q)} + 2\|\nabla_x K_2(x,t)\partial_t^2 u_N\|^2_{L_2(Q)} + 2\|K_2(x,t)\nabla_x \partial_t^2 u_N\|^2_{L_2(Q)}. \quad (2.21)$$

Let's estimate each summand, for example, using Holder's inequality we have

$$\|K_{2x_i}\partial_t^2 u_N\|^2_{L_2(Q)} \le \int_0^T \|K_{2x_i}\|^2_{L_{q_2}(\Omega)}\|\partial_t^2 u_N\|^2_{L_{2q_2'}(\Omega)}\,dt \le \|K_{2x_i}\|^2_{L_\infty(0,T;L_{q_2}(\Omega))}\|\partial_t^2 u_N\|^2_{L_2(0,T;L_{2q_2'}(\Omega))},\ q_2' = q_2/(q_2-2).$$

Let, for example, $n \ge 4$. By convention $q_2 > n/2$. Then $2q_2' < 2n(n-4)$ at $n > 4$ and $2q_2' < \infty$ at $n = 4$. Hence, there is $s < 2$ such that $2q_2' = 2n/(n-2s)$ and hence there is an embedding $W_2^s(\Omega) \subset L_{2r_2'}(\Omega)$ Then, using the interpolation inequalities from [24] of the form

$$\|u\|_{W_2^s(\Omega)} \le c\|u\|^\theta_{W_2^{s_1}(\Omega)}\|u\|^{1-\theta}_{W_2^{s_2}(\Omega)},\ s_1\theta + s_2(1-\theta) = s,\ s_1 < s < s_2$$

and inequality (2.4), we obtain

$$\|\partial_t^2 u_N\|^2_{L_2(0,T;L_{2q_2'}(\Omega))} \le \varepsilon\|\partial_t^2 u_N\|^2_{L_2(0,T;W_2^2(\Omega))} + c(\varepsilon)\|u_N\|^2_{L_2(Q)}, \quad (2.22)$$

where $\varepsilon -$ is an arbitrary positive number. The final estimate is

$$\|K_{2x_i}\partial_t^2 u_N\|^2_{L_2(Q)} \le c_0\varepsilon\|\partial_t^2 u_N\|^2_{L_2(0,T;W_2^2(\Omega))} + c_1(\varepsilon)\|\partial_t^2 u_N\|^2_{L_2(Q)}, \quad (2.23)$$

where ε is an arbitrary positive number and c_0 ε and N. Let now $n < 4$. Then we have the estimation

$$\|K_{2x_i}\partial_t^2 u_N\|^2_{L_2(Q)} \le \int_0^T \|K_{2x_i}\|^2_{L_2(\Omega)}\|\partial_t^2 u_N\|^2_{L_\infty(\Omega)}\,dt \le c_2\|\partial_t^2 u_N\|^2_{L_2(0,T;L_\infty(\Omega)} \le c_3\|\partial_t^2 u_N\|^2_{L_2(0,T;W_2^s(\Omega))}, \quad (2.24)$$

where $2 > s > n/2$. Then, using again the interpolation inequality and inequality (2.4), we arrive at an inequality of the form (2.23). Let us now consider the norm

$$\|K_2\partial_t^2 u_{Nx_i}\|^2_{L_2(Q)} \le \int_0^T \|K_2\|_{L_{r_2}(\Omega)}\|\partial_t^2 u_{Nx_i}\|^2_{L_{2r_2'}(\Omega))}\,dt \le c_2\|\partial_t^2 u_{Nx_i}\|^2_{L_2(0,T;L_{2r_2'}(\Omega))}\,dt \le c_3\|\partial_t^2 u_N\|^2_{L_2(0,T;W_2^{1+s}(\Omega))}, \quad (2.25)$$

where $r_2' = r_2/(r_2-2)$, $2r_2' = 2n/(n-2s)$, $s < 1$. We have $r_2 > n$, then $2r_2' < 2n/(n-2)$ and so we find such $s < 1$, that the equality $2r_2' = 2n/(n-2s)$ is satisfied. Again applying the interpolation inequality and inequality (2.4),we obtain

$$\|K_2\partial_t^2 u_{Nx_i}\|^2_{L_2(Q)} \le c_4\varepsilon\|\partial_t^2 u_N\|^2_{L_2(0,T;W_2^2(\Omega))} + c_5(\varepsilon)\|u_N\|^2_{L_2(Q)}, \quad (2.26)$$

where the constant c_4 is independent of ε and ε – is an arbitrary positive number. Estimation

$$\|K_2\partial_t^2 u_N\|^2_{L_2(Q)} \le c_6\varepsilon\|\partial_t^2 u_N\|^2_{L_2(0,T;W_2^2(\Omega))} + c_7(\varepsilon)\|u_N\|^2_{L_2(Q)}, \tag{2.27}$$

is now obvious. The estimates are similarly proved

$$\|K_j(x,t)\partial_t^j u_N\|^2_{L_2(0,T;W_2^1(\Omega))} \le \|K_j(x,t)\partial_t^j u_N\|^2_{L_2(Q)} + 2\|\nabla_x K_j(x,t)\partial_t^j u_N\|^2_{L_2(Q)} + 2\|K_j(x,t)\nabla_x\partial_t^j u_N\|^2_{L_2(Q)}, \tag{2.28}$$

$$\|K_j\partial_t^j u_{Nx_i}\|^2_{L_2(Q)} \le c_7\varepsilon\|\partial_t^j u_N\|^2_{L_2(0,T;W_2^3(\Omega))} + c_8(\varepsilon)\|\partial_t^j u_N\|^2_{L_2(Q)}, \tag{2.29}$$

$$\|K_{jx_i}\partial_t^j u_N\|^2_{L_2(Q)} \le c_9\varepsilon\|\partial_t^j u_N\|^2_{L_2(0,T;W_2^3(\Omega))} + c_{10}(\varepsilon)\|\partial_t^j u_N\|^2_{L_2(Q)}, \tag{2.30}$$

where constants c_7, c_9 are independent of ε and ε – is an arbitrary positive number, $i = 1,2,\dots,n$, $j = 0,1$. Using the estimates (2.21), (2.23), (2.26)-(2.29), we obtain the inequality

$$\sum_{j=0}^{2}\|K_j(x,t)\partial_t^j u_N\|^2_{L_2(0,T;W_2^1(\Omega))} \le c_8\varepsilon(\|u_N\|^2_{L_2(0,T;W_2^3(\Omega))} + \|\partial_t u_N\|^2_{L_2(0,T;W_2^3(\Omega))} + \|\partial_t^2 u_N\|^2_{L_2(0,T;W_2^2(\Omega))}) + c_9(\varepsilon)\sum_{j=0}^{2}\|\partial_t^j u_N\|^2_{L_2(Q)}. \tag{2.31}$$

Thus, we have evaluated all the summands in the expression $|(lu_N, M^{1/2}(P_0+\lambda_0)u_N)|$. Finally, using (2.31) and inequality (2.4), we have that

$$\begin{aligned}|(lu_N, M^{1/2}(P_0+\lambda_0)u_N)| &= |(M^{1/4}lu_N, M^{1/4}(P_0+\lambda_0)u_N)| \le \\ &C(\varepsilon_1)\|M^{1/4}lu_N\|_{L_2(Q)} + \varepsilon_1\|M^{1/4}(P_0+\lambda_0)u_N\|_{L_2(Q)} \le \\ &C_1(\varepsilon_1)\|lu_N\|_{L_2(0,T;W_2^1(\Omega))} + c_{10}\varepsilon_1\|(P_0+\lambda_0)u_N\|_{L_2(0,T;W_2^1(\Omega))} \le \\ &c_{11}(\varepsilon C(\varepsilon_1)+\varepsilon_1)(\|u_N\|^2_{L_2(0,T;W_2^3(\Omega))} + \|\partial_t u_N\|^2_{L_2(0,T;W_2^3(\Omega))} + \|\partial_t^2 u_N\|^2_{L_2(0,T;W_2^2(\Omega))} \\ &+ \|\partial_t^3 u_N\|^2_{L_2(0,T;W_2^1(\Omega))}) + c_{12}(\varepsilon,\varepsilon_1)\sum_{j=0}^{2}\|\partial_t^j u_N\|^2_{L_2(Q)},\end{aligned} \tag{2.32}$$

where the constant c_{11} is independent of, $\varepsilon, \varepsilon_1 > 0$. Note that the last summand, by virtue of the first a priori estimate, is estimated by a constant independent of N. There is also an estimation

$$|(f, M^{1/2}(P_0+\lambda_0)u_N)| = |(M^{1/4}f, M^{1/4}(P_0+\lambda_0)u_N)| \le \varepsilon_2\|u_N\|_{W_2^3(0,T;W_2^1(\Omega))} + c(\varepsilon_2)\|f\|_{L_2(0,T;W_2^1(\Omega))}. \tag{2.33}$$

From equality (2.17) and (2.33) we have

$$(Lu_N, M^{1/2}(P_0+\lambda_0)u_N) \le \varepsilon_2\|u_N\|_{W_2^3(0,T;W_2^1(\Omega))} + c(\varepsilon_2)\|f\|_{L_2(0,T;W_2^1(\Omega))}. \tag{2.34}$$

On the other hand, the left part of the inequality, using (2.18), (2.19) is evaluated from below through

$$\begin{aligned}&\delta_3(\|u_N\|^2_{W_2^1(0,T;W_2^3(\Omega))} + \lambda\|u_N\|^2_{W_2^1(0,T;W_2^1(\Omega))} + \|\sqrt{|K_4(T)|}M^{1/4}\partial_t^3 u_N(x,T)\|^2_{L_2(\Omega)} + \\ &\|\sqrt{|K_4(0)|}M^{1/4}\partial_t^3 u_N(x,0)\|^2_{L_2(\Omega)} + \|u_N\|^2_{W_2^3(0,T;W_2^1(\Omega))}) - \\ &c_5\|u_N\|^2_{L_2(0,T;W_2^1(\Omega))} - |(lu_N, M^{1/2}(P_0u_N+\lambda_0)u_N)\end{aligned}$$

In view of the estimate (2.32) the last inequality, if you choose a small parameter ε_1 and then a sufficiently small parameter ε, is estimated from below via

$$\frac{\delta_3}{2}(\|u_N\|^2_{W_2^1(0,T;W_2^3(\Omega))}+\lambda\|u_N\|^2_{W_2^1(0,T;W_2^1(\Omega))}+\|\sqrt{|K_4(T)|}M^{1/4}\partial_t^3u_N(x,T)\|^2_{L_2(\Omega)} + \|\sqrt{|K_4(0)|}M^{1/4}\partial_t^3u_N(x,0)\|^2_{L_2(\Omega)}+\|u_N\|^2_{W_2^3(0,T;W_2^1(\Omega))}) = c_{13}\|u_N\|^2_{L_2(0,T;W_2^1(\Omega))}-c_{14}(\varepsilon,\varepsilon_1)(\sum_{j=0}^{2}\|\partial_t^ju_N\|^2_{L_2(Q)}. \quad (2.35)$$

Using the already obtained estimate (2.16), from (2.34), (2.35) we obtain an estimate of

$$\|u_N\|^2_{W_2^1(0,T;W_2^3(\Omega))}+\lambda\|u_N\|^2_{W_2^1(0,T;W_2^1(\Omega))})+\|\sqrt{|K_4(T)|}M^{1/4}\partial_t^3u_N(x,T)\|^2_{L_2(\Omega)}+ \|\sqrt{|K_4(0)|}M^{1/4}\partial_t^3u_N(x,0)\|^2_{L_2(\Omega)}+\|u_N\|^2_{W_2^3(0,T;W_2^1(\Omega))})\le c\|f\|^2_{L_2(0,T;W_2^1(\Omega))}. \quad (2.36)$$

Estimates (2.16), (2.36) guarantee that a subsequence u_{N_k} such that, there is convergence of $D^\alpha\partial_t^iu_{N_k}\to D^\alpha\partial_t^iu$ weakly in $L_2(Q)$, $Mu_{N_k}\to Mu$ weakly in $L_2(0,T;\tilde W_2^{-1}\Omega))$, where $|\alpha|\le 3$ at $i=0,1$, $|\alpha|\le 2$ at $i=2$, $|\alpha|\le 1$ at $i=3$,
$u\in W_2^3(0,T;\tilde W_2^1(\Omega))\cap W_2^1(0,T;\tilde W_2^3(\Omega))$; there is also weak convergence

$$K_4(T)\partial_t^3u_{N_k}(x,T)\to u_T\in\tilde W_2^1(\Omega),\ K_4(0)\partial_t^3u_{N_k}(x,0)\to u_0\in\tilde W_2^1(\Omega).$$

Consider the equality (2.35). Let us rewrite it in the form

$$(-\partial_t^3u_{N_k},(\psi_jK_4)_t\omega_i)+(K_3\partial_t^3u_{N_k},\psi_j\omega_i)+(M^{1/2}u_{N_k},M^{1/2}\psi_j\omega_i)+(lu_{N_k}+\lambda u_{N_k},\psi_j\omega_i) + (K_4(T)\partial_t^3u_{N_k}(x,T),\psi_j(T)\omega_i)-(K_4(0)\partial_t^3u_{N_k}(x,0),\psi_j(0)\omega_i)=(f,\psi_j\omega_i), \quad (2.37)$$

where $i,j=1,\ldots,N_k$. Multiplying (2.37) by the constants $\gamma_{i,j}$ and summing over i,j, we obtain

$$(-\partial_t^3u_{N_k},(vK_4)_t)+(K_3\partial_t^3u_{N_k},v)+(M^{1/2}u_{N_k},M^{1/2}v)+(lu_{N_k},v)+(\lambda u_{N_k},v) + (K_4(T)\partial_t^3u_{N_k}(x,T),v(x,T))-(K_4(0)\partial_t^3u_{N_k}(x,0),v(x,0))=(f,v), \quad (2.38)$$

where $v=\sum_{i,j=1}^N\gamma_{i,j}\psi_j\omega_i$. Passing to the limit on k, we obtain the integral identity

$$(-\partial_t^3u,(vK_4)_t)+(K_3\partial_t^3u,v)+(M^{1/2}u,M^{1/2}v)+(lu,v)+(\lambda u,v) + (u_T(x),v(x,T))-(u_0(x),v(x,0))=(f,v), \quad (2.39)$$

by virtue of the arbitrariness of the constants $\gamma_{i,j}$, we conclude that equality (2.39) is satisfied for all functions $v\in L_2(0,T;\tilde W_2^1(\Omega))\cap W_2^1(0,T;L_2(\Omega))$. From the definition of generalized we obtain, that there exists a generalized derivative $\partial_t(K_4\partial_t^3u)\in L_2(0,T;\tilde W_2^{-1}(\Omega))$. Moreover the traces $K_4\partial_t^3u(x,t)|_{t=T}=u_T\in\tilde W_2^1(\Omega)$, $K_4\partial_t^3u(x,t)|_{t=0}=u_0\in\tilde W_2^1(\Omega)$ are defined. Then equation (1.1) is fulfilled in space $L_2(0,T;\tilde W_2^{-1}(\Omega))$. Note that, by virtue of the estimates obtained in the proof above, $lu\in L_2(0,T:\tilde W_2^1(\Omega))$. Integrating (2.39) from zero to t, we obtain

$$Mv+\lambda v=\int_0^t f(x,\tau)\,d\tau+\int_0^t(K_{4t}-K_3)\partial_t^3u(x,\tau)-lu(x,\tau)\,d\tau - K_4\partial_t^3u(x,t)+u_0(x)\in L_2(0,T;\tilde W_2^1(\Omega)),\quad v=\int_0^t u(x,\tau)\,d\tau. \quad (2.40)$$

Since the right-hand side in (2.40) belongs to $L_2(0,T;\tilde{W}_2^1(\Omega))$, $v \in L_2(0,T;W_2^5(\Omega))$. Since $v \in W_2^3(0,T;W_2^1(\Omega))$, we have from (2.5) that $\partial_t^i v \in L_2(0,T;W_2^{5(1-i/4)+i/4}(\Omega))$. In particular, $v' = u \in L_2(0,T;W_2^4(\Omega))$.Then it follows from the equation and the definition of the generalized Sobolev derivative that there exists a generalized derivatived $\partial_t(K_4\partial_t^3 u) \in L_2(Q)$.

□

Remark 2.5. The methodology of the proof is sufficiently general and transferable to arbitrary high-order mixed-type equations of the form

$$\sum_{i=0}^{2n} K_i(x,t)\partial_t^i u + Mu = f,$$

where M a self-adjoint positive elliptic operator of order $2m$. To this equation can also be added the minor terms depending on all variables of the form $a_{\beta,k}D^\beta\partial_t^k u$.

Remark 2.6. The question arises regarding the further improvement of the smoothness of solutions with respect to the variable t. This question can be easily studied using the same Galerkin method, but with the use of a special basis $\{\psi_j\}$ and additional a priori estimates. As such a basis it is most convenient to take the eigenfunctions of the problem

$$v' = \lambda v,\ v(0) = \alpha v(T),$$

which coincide with the functions $e^{\beta t}e^{2\pi jt/T}$ (j=0,1,2,...), forming a basis in $L_2(0,T)$ ($\beta = \beta(\alpha)$ - is a numerical parameter). The solution itself in this case is sought in the form $u = \sum_{ij} c_{ij}\psi_j\omega_i$. Since the eigenvalues are complex numbers, in this case it is natural to consider complex function spaces, which will not really affect the estimates. When obtaining additional estimates in the proof, additional conditions such as periodicity on the coefficients of K_i will arise.

References

[1] Agmon S.; On the eigenfunctions and eigenvalues of general elliptic boundary value problems. Communications in Pure and Applied Mathematics, New York, 1962, pp. 119–147.

[2] Berezansky Y.M.; Decomposition by eigenfunctions of self-adjoint operators. Kyiv: Naukova Dumka, 1965, 512 p.

[3] Bitsadze A.V.; Incorrectness of the Dirichlet problem for equations of mixed type. Doklady Akademii Nauk SSSR, Moscow, 1953, pp. 167–170.

[4] Chueshev A.V.; On a linear high-order mixed-type equation. Siberian Mathematical Journal, Novosibirsk, 2002, pp. 454–472.

[5] Dezin A.A.; General questions of the theory of boundary value problems. Moscow: Nauka, 1980, 208 p.

[6] Dzhamalov S.Z.; On the correctness of a nonlocal boundary value problem with constant coefficients for a second-kind mixed-type equation of second order. Mathematical Notes of SVFU, Yakutsk, 2017, pp. 17–28.

[7] Dzhamalov S.Z.; On the smoothness of a nonlocal boundary value problem for a multivariate second-kind mixed-type equation in space. Journal of the Middle Volga Mathematical Society, Saratov, 2019, pp. 24–33.

[8] Dzhamalov S.Z.; Nonlocal boundary and inverse problems for equations of mixed type. Tashkent: Fan va Texnologiya, 2020, 173 p.

[9] Dzhamalov S.Z., Pyatkov S.G.; On some classes of boundary value problems for multivariate high-order mixed-type equations. Siberian Mathematical Journal, Novosibirsk, 2020, pp. 777–795.

[10] Dzhamalov S.Z., Khalkhadzhaev B.B., Yusupov Sh.B.; On the unique solvability of a semi-nonlocal boundary value problem for a second-kind fourth-order mixed-type equation. Bulletin of the Institute of Mathematics, Tashkent, 2024, pp. 35–42.

[11] Egorov I.E., Fedorov V.E.; Nonclassical equations of mathematical physics of high order. Novosibirsk: Nauka, 1995, 133 p.

[12] Frankl F.I.; Selected works on gas dynamics. Moscow: Nauka, 1973, 711 p.

[13] Glazatov S.N.; Nonlocal boundary value problems for equations of mixed type in a right-angled rectangle. Siberian Mathematical Journal, Novosibirsk, 1985, pp. 162–164.

[14] Grisvard P.; Commutativite de deux foncteurs d'interpolation et applications. Journal de Mathmatiques Pures et Appliques, Paris, 1966, pp. 143–206.

[15] Grisvard P.; Caracterisation de quelques espaces d'interpolation. Archive for Rational Mechanics and Analysis, Berlin, 1967, pp. 40–63.

[16] Grisvard P.; quations diffrentielles abstraites. Annales Scientifiques de l'cole Normale Suprieure, Paris, 1969, pp. 311–395.

[17] Karatopraklieva M.G.; On a nonlocal boundary value problem for a mixed-type equation. Differential Equations, Moscow, 1991, pp. 68–79.

[18] Khalkhadzhaev B.B., Yusupov Sh.B., Mambetsapaev K.A.; On a weak generalized solution of a second-kind fourth-order mixed-type equation. Scientific Reports of Bukhara State University, Bukhara, 2023, pp. 9–15.

[19] Kozhanov A.I.; Boundary value problems for equations of mathematical physics of odd order. Novosibirsk: Novosibirsk State University Press, 1990, 180 p.

[20] Ladyzhenskaya O.A.; Boundary value problems of mathematical physics. Moscow: Nauka, 1973, 407 p.

[21] Lions J.L., Magenes E.; Non-homogeneous boundary value problems and applications, vol. 1. Berlin: Springer-Verlag, 1972, 357 p.

[22] Terekhov A.N.; Nonlocal boundary value problems for equations of variable type. Institute of Mathematics, Siberian Branch, Academy of Sciences of the USSR, Novosibirsk, 1985, pp. 148–158.

[23] Trenogin V.A.; Functional analysis. Moscow: Nauka, 1980, 494 p.

[24] Triebel H.; Interpolation theory, function spaces, differential operators. Moscow: Mir, 1986, 512 p.

[25] Vragov V.N.; On the statement and solvability of boundary value problems for equations of mixed-composite type. Institute of Mathematics, Siberian Branch, Academy of Sciences of the USSR, Novosibirsk, 1978, pp. 5–13.

[26] Vragov V.N.; Boundary value problems for nonclassical equations of mathematical physics. Novosibirsk: Novosibirsk State University Press, 1983, 200 p.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.96-108
DOI: 10.29229/uzmj.2025-2-9

Oscillatory properties of the forth-order differential operator on a metric graph

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. We consider a boundary value problem for a network beam equation with a forcing term. The main aim of the paper is to study oscillatory property of the corresponding forth-order differential operator. The definition of the oscillation of an operator is given in terms of a special fundamental system of solutions of the corresponding homogeneous equation on a graph. We study sign-regularity properties of solutions of the corresponding homogeneous equations on a graph and show that the positivity of a special system of solutions implies disconjugacy of the operator on each edge of the graph.

Keywords: network equation, beam equation, sign-regularity, disconjugacy, oscillation.
MSC (2020): 34C10, 34B27, 34B24, 34L05

1. Introduction

The aim of this paper is to develop the disconjugacy theory for a forth-order differential equation on a graph. In recent years, differential equations on metric graphs (networks) and boundary value problems for such equations have attracted much attention from researchers. This is due to the fact that network equations arise in models describing multi-link flexible engineering structures, the propagation of waves in thin channels, the propagation of nerve impulses, in the hydrodynamics of a network of thin vessels, etc. [5, 16, 20, 21, 23, 22, 27, 28]. We also mention papers [1, 3, 26] in which inverse spectral problems for network equations were studied.

In this paper we study the linear differential operator L_r of the fourth-order boundary value problem on a graph that arises in modeling Euler-Bernoulli beam systems. We briefly write the corresponding boundary value problem as

$$L_r u \equiv L_0 u + r(x)u = f(x), \quad x \in \Gamma, \tag{1.1}$$

$$u|_{\partial\Gamma} = (\vartheta u' - \beta u'')|_{\partial\Gamma} = 0. \tag{1.2}$$

Here L_r is a differential operator with the forcing term $r(x)u$; the operator L_0 is defined by Euler-Bernoulli beam equations on the edges of Γ, and by sets of the rigid joint conditions at the junction nodes (see Section 2.2); $\partial\Gamma$ is the boundary of Γ.

We study sign regularity property of the differential operator L_r. The sign regularity of a differential operator has a significant importance in the qualitative theory of ordinary differential equations [2, 4, 7, 18, 24] and differential equations on metric graphs [21, 19]. This property is mainly used for the study of oscillation properties of spectra of boundary value problems and the positivity of the Green function for some classes of boundary value problems [6, 13, 14].

It is known that the positivity of the Green's function is closely related to the property of the disconjugacy of the differential operator, and the disconjugacy is related to the sign-regularity of special fundamental systems of solutions of the corresponding homogeneous differential equation. In the papers [2, 10, 11, 12], it was shown that for fourth-order equations on graphs the positivity of the Green's function is equivalent to the positivity of a special system of solutions of the homogeneous equation. In the present paper we show that the positivity of a special system of solutions implies disconjugacy of the operator on each edge of the graph.

Notice that the properties of the fourth-order equation on the graph have fundamental differences from the one-dimensional case. The one-dimensional operator $\ell u = (p(x)u'')''$ (p is uniformly positive) on the finite interval with the boundary conditions (1.2) is inverse-positive, whereas in general the operator on a graph with the same differential expression ℓu on the graph edges is not inverse positive [6, 7]. The properties of the operator on a graph essentially depend on the connection conditions at the nodes [10, 11, 14, 16]. The properties of the model of a planar system consisting of beam elements rigidly connected to each other ([6, 12]) differ significantly from the properties of the model with elastic hinged joints ([21, 9, 14]). The properties of the operator with elastic hinge joints are similar to the properties of the Sturm-Liouville operator. In particular, it was shown (see [21]) that for a model of beams with elastic hinge joints the maximum principle is valid and the corresponding differential operator is inverse-positive. At the same time, the differential operator L_0 with rigid joints may not be inverse-positive [6, 10, 12].

In the paper [6] we studied the operator L_0 with rigid joints corresponding to the boundary value problem (1.1), (1.2). It was shown in [6] that oscillatory property, i.e. positivity of a special system of solutions of the homogeneous equation $L_0u = 0$, is equivalent to the inverse positivity of L_0. This result made it possible to formulate the maximum principles for solutions to the equation $L_0u = 0$ with rigid joints [11, 12]. In the present paper we consider the perturbed operator $L_r = L_0 + rI$ with the forcing term rI.

Taking our cue from the one-dimensional homogeneous equation

$$u^{IV} + r(x)u = 0, \quad x \in (a, b) \subset \mathbb{R},$$

whose oscillatory behavior is very different according as r is positive or negative (see [15, 25]), we shall distinguish between two cases $r \leq 0$ and $r \geq 0$ on Γ. In the paper [13], we showed that the oscillatory of the operator L_r guarantees the positive invertibility of the perturbed operator L_r, provided $-\lambda_0 < r(x) \leq 0$, where λ_0 is the lowest eigenvalue of the operator L_0. In the present paper we consider the case $r \geq 0$ on Γ. We show that the oscillatory implies disconjugacy of the operator on each edge of the graph. We hope that this result will allow us to obtain conditions for the inverse positivity of the operator L_r in the case $r \geq 0$ on Γ.

2. Main notations and statement of the problem

2.1. **Terminology of networks.** Throughout the paper, $\Gamma \subset \mathbb{R}^2$ denotes a simple connected and finite metric graph without loops, with a finite set of vertices $V(\Gamma)$ and a set of edge points $E(\Gamma)$. An edge of a graph is an open interval of finite length, and a vertex of a graph is an endpoint of one or more edges. The edges of a graph are labeled by γ_i, the vertices are denoted by a, b, c, etc. Each edge γ_i is is an open interval (a_i, b_i) of finite length $l_i = \|b_i - a_i\|_{\mathbb{R}^2}$ and is assumed to be parameterized such that

$$\gamma_i = \left\{x \in \mathbb{R}^2 \middle|\, x = a_i + t(b_i - a_i)/l_i,\ 0 < t < l_i\right\}.$$

For any $a \in V(\Gamma)$ by $I(a)$ we denote the index set of the edges incident to a vertex a, and by $|I(a)|$ denote the number of elements of the set $I(a)$. Elements of the sets $J(\Gamma) = \{a \in V(\Gamma) : |I(a)| \geq 2\}$ and $\partial\Gamma = \{a \in V(\Gamma) : |I(a)| = 1\}$ are called *interior* and *boundary* vertices, respectively. For the boundary edge adjacent to the vertex $a \in \partial\Gamma$, we also use the notation γ_a.

We assume that $\Gamma = E(\Gamma) \cup J(\Gamma)$ and $\partial\Gamma \neq \varnothing$. Note that the boundary vertices are not included in the graph. Let Γ_0 and Γ be two graphs. The graph Γ_0 is said to be a *subgraph* of a graph Γ if Γ_0 is a connected subset of Γ such that $J(\Gamma_0) \subset J(\Gamma)$.

Let us define the concept of a neighborhood of a point on a graph. The distance $d(x_1, x_2)$ between two points $x_1, x_2 \in E(\Gamma) \cup V(\Gamma)$ is defined as the length of a shortest directed polygonal path from x_1 to x_2 within the graph. By definition, the ε-neighborhood of a point $x_0 \in \Gamma \cup \partial\Gamma$ is the set $\mathcal{N}_\varepsilon(x_0) = \{x \in \Gamma \,|\, d(x_0, x) < \varepsilon\}$. With this definition, the neighborhood of a boundary vertex does not contain the boundary vertex itself.

We further use the function spaces:

$$C[\Gamma] = \{u : \Gamma \to \mathbb{R} \,|\, u \text{ is uniformly continuous on each edge } \gamma \subset E(\Gamma)\};$$
$$C[E(\Gamma)] = \{u : E(\Gamma) \to \mathbb{R} \,|\, u \text{ is uniformly continuous on each edge } \gamma \subset E(\Gamma)\}.$$

By u_i we denote the restriction of a function $u \in C[\Gamma]$ (or $C[E(\Gamma)]$) to the edge $\gamma_i \subset \Gamma$. Every function $u \in C[\Gamma]$ (or $C[E(\Gamma)]$) has a limit $\lim\limits_{\gamma_i \ni x \to a} u_i(x)$ at each vertex $a \in V(\Gamma)$ for $i \in I(a)$; we denote it by $u_i(a)$. Note that $u_k(a)$ are not necessarily equal to $u_i(a)$ or $u(a)$, where $k, i \in I(a)$ $(k \neq i)$.

Identifying the edges γ_i with intervals $(0, l_i)$ on the real line, we define derivative of the function $u(x)$ on the edge γ_i as the derivative on an oriented manifold and denote it by $u_i'(x)$. Higher-order derivatives are defined analogously.

By $C^n[\Gamma]$ (or $C^n[E(\Gamma)]$) we denote the space of functions $u \in C[\Gamma]$ (or $C[E(\Gamma)]$) whose derivatives up to and including order n exist and belong to the space $C[E(\Gamma)]$. For a function $u \in C^n[\Gamma]$ (or $C^n[E(\Gamma)]$), at any vertex $a \in V(\Gamma)$, the set of derivatives $u_i^{(j)}(a)$, $1 \leq j \leq n$, along the edges adjacent to that vertex is defined.

For the sake of simplicity, we shall write

$$\int\limits_\Gamma u(x)\,dx = \sum_{\gamma_i \subset E(\Gamma)} \int\limits_0^{l_i} u_i(x(t))\,dt.$$

2.2. **Data and framework.** We are concerned with boundary value problem (1.1). By *a differential equation on a graph*, following [13, 21], we understand the set of differential equations on the edges and the set of consistency conditions at the interior vertices. The equations on the edges have the form

$$(p(x)u'')'' - (q(x)u')' + r(x)u = f(x), \quad x \in E(\Gamma), \tag{2.1}$$

where $p \in C^2[E(\Gamma)]$, and $r, f \in C[\Gamma]$.

At each junction node $c \in J(\Gamma)$, $|I(c)| \geq 3$, we impose the following transmission conditions:

$$\begin{gathered} u_i(c) = u_k(c), \quad \forall i \in I(c) \setminus k, \\ u_i'(c) = \alpha_{ki}(c)u_k'(c) + \alpha_{ji}(c)u_j'(c), \quad \forall i \in I(c) \setminus \{j, k\}, \end{gathered} \tag{2.2}$$

$$\begin{gathered} (p_k u_k'')(c) + \sum_{i \in I(c) \setminus \{j,k\}} \alpha_{ki}(c)(p_i u_i'')(c) = 0, \\ (p_j u_j'')(c) + \sum_{i \in I(c) \setminus \{j,k\}} \alpha_{ji}(c)(p_i u_i'')(c) = 0 \end{gathered} \tag{2.3}$$

$$\sum_{i \in I(c)} D^3 u_i(c) + r(c)u(c) = f(c), \quad c \in J(\Gamma). \tag{2.4}$$

Here k and j are fixed indices from $I(c)$; $\alpha_{ki}(c)$, $\alpha_{ji}(c)$ are given real numbers, $D^3u = (pu'')' - qu'$. In conditions (2.2)-(2.4) all the derivatives are calculated in the direction moving away from the vertex $c \in J(\Gamma)$.

If an interior vertex c is incident to just two edges γ_i and γ_k, then in the system of conditions (2.2)-(2.4) one needs to delete all the terms with the index j. In this case conditions (2.2)-(2.4) become

$$\begin{gathered} u_i(c) = u_k(c), \quad u_i'(c) = \alpha_{ki}(c)u_k'(c), \\ (p_k u_k'')(c) + \alpha_{ki}(c)(p_i u_i'')(c) = 0, \\ D^3u_i(c) + D^3u_k(c) + r(c)u(c) = f(c). \end{gathered} \tag{2.5}$$

The system of relations (2.1)-(2.4) is called *the differential equation on the graph* Γ and is denoted by $L_0 u = f(x)$. A *solution* of this equation is any function $u(x) \in C^4[\Gamma]$ continuous on Γ that satisfies the corresponding ordinary differential equation (2.1) on $E(\Gamma)$ and obeys conditions (2.2)-(2.4) at each inner vertex.

Thus, the differential operator $L_0 : D \to C[\Gamma]$ is defined by relations

$$\begin{gathered} D = \{u \in C^4[\Gamma] : \ u \text{ satisfies } (2.2), (2.3) \text{ on } J(\Gamma)\}, \\ L_0 u(x) := \begin{cases} (p(x)u'')'' - (q(x)u')', & x \in E(\Gamma), \\ \displaystyle\sum_{i \in I(x)} [(p_i(x)u_i'')' - q_i(x)u_i'], & x \in J(\Gamma); \end{cases} \\ L_r u(x) := L_0 u(x) + r(x)u(x), \quad x \in \Gamma. \end{gathered} \tag{2.6}$$

Remark 2.1. Note carefully that by definition a solution of equation $L_r u = f(x)$ is continuous on Γ, i.e. $u_i(c) = u(c)$ for all $c \in J(\Gamma)$ and $i \in I(c)$.

Remark 2.2. Relations (2.1)-(2.4) have a natural physical interpretation (see [13, 17]). They appear in a simulating small deformations of a planar framework, a network structure of thin straight beams. We assume that in the equilibrium the beams are placed in a plane forming graph Γ. The nodes of the system are the endpoints of three or more distinct beams which are rigidly coupled together. It is also assumed that at some points (not necessarily nodal or boundary points) the system is elastically supported.

Throughout we assume that the following conditions hold:

- the differential equation $L_r u = f$ is an equality generated by the system of relations (2.1)-(2.4), where f is the union of right-hand sides of relations (2.1) and (2.4);
- $p \in C^2[\Gamma]$, $\inf_{x \in \Gamma} p(x) > 0$, $q \in C^1[\Gamma]$, $r \in C[\Gamma]$, and $q, r \geqslant 0$ on Γ;
- for any vertex $c \in J(\Gamma)$ and any index $i \in I(c)$ at least one of the constants $\alpha_{ji}(c), \alpha_{ki}(c)$ is nonzero;
- for any vertex $c \in J(\Gamma)$, $|I(c)| \geq 3$, we can define basis indices $k, j \in I(c)$ such that the inequalities $\alpha_{ki_1}(c) < 0$, $\alpha_{ji_1}(c) \leq 0$ both hold for some index $i_1 \in I(c) \setminus \{j, k\}$ and $\alpha_{ki_2}(c) \leq 0$, $\alpha_{ji_2}(c) < 0$ both hold for some index $i_2 \in I(c) \setminus \{j, k\}$; and for any vertex $c \in J(\Gamma)$ with $|I(c)| = 2$, we assume $\alpha_{ki_1}(c) < 0$;
- in the boundary and transmission conditions all the derivatives are calculated in the direction moving away from the vertex;
- the graph Γ is a tree and $J(\Gamma) \neq \varnothing$.

Here, assumptions regarding the coefficients $\alpha_{ki}(c)$ and $\alpha_{ki}(c)$ ensure the unique solvability of the boundary value problem (1.1), (1.2) (see [8]).

3. Some qualitative properties of solutions of equation (1.1)

In this section we study sign-regularity properties of solutions of the homogeneous equation (1.1) generated by relations (2.1)-(2.4).

The following results are needed for the sequel.

Lemma 3.1. *For each $f \in C[\Gamma]$ the linear boundary value problem*

$$L_r u = f(x), \quad x \in \Gamma, \tag{3.1}$$

$$(u + \alpha D^3 u)|_{\partial\Gamma} = (\vartheta u' - \beta u'')|_{\partial\Gamma} = 0, \tag{3.2}$$

where $\alpha, \vartheta, \beta : \partial\Gamma \to [0, +\infty)$ and $\vartheta + \beta > 0$, is uniquely solvable.

Proof. Let u be a solution to the homogeneous problem (3.1), (3.2). Using (3.4), we have

$$0 = \int\limits_\Gamma u \cdot L_r u\, dx + \sum_{c \in J(\Gamma)} u(c) L_r u(c) = \int\limits_\Gamma (pu''^2 + qu'^2 + ru^2) dx$$
$$+ \sum_{c \in \partial\Gamma} (pu'')(c) u'(c) - \sum_{c \in \partial\Gamma} D^3 u(c)\, u(c) + \sum_{c \in J(\Gamma)} r(c) u^2(c).$$

It follows from (3.2) and from $p > 0$, $r \geq 0$ that all terms in the right-hand side are nonnegative and therefore equal to zero. Hence $u'' \equiv 0$ on $E(\Gamma)$. By virtue of the boundary conditions (3.2), $u|_{\partial\Gamma} = 0$. Thus, u solves the homogeneous problem (1.1), (1.2). It was shown in the paper [8] that this problem is uniquely solvable. Hence $u \equiv 0$ on Γ. The lemma is proved. □

Definition 3.2. The homogeneous equation

$$Lu \equiv (p(x)u'')'' - (q(x)u')' + r(x)u = 0, \quad x \in [a, b] \subset \mathbb{R}, \tag{3.3}$$

is said to be *disconjugate* on the real interval $[a, b]$ if every non-trivial solution has at most three zero on $[a, b]$.

Recall (see, for instance, [18]) that the disconjugacy is of great importance in the qualitative theory of ordinary differential equations. In particular, if the operator L is disconjugate, then L can be represented in the form of the decomposition

$$Lu = h_0(x) \frac{d}{dx}\big(h_1(x) \frac{d}{dx}\big(h_2(x) \frac{d}{dx}\big(h_1(x) \frac{d}{dx}\big(h_0(x) u\big)\big)\big)\big), \quad x \in [a, b].$$

Lemma 3.3. *Suppose that the homogeneous equation (3.3) is not disconjugate on $[a, b]$. If u is a non-trivial solution of the homogeneous equation (3.3) satisfying conditions $u(a) = 0$ and $u'(a)u''(a) \geq 0$, then u has a simple zero in $(a, b]$.*

Proof. Let us consider a solution v of the homogeneous equation (3.3) that has a triple zero at the point a. Since equation (3.1) is not disconjugate on $[a, b]$, we have that v has a zero $x_0 \in (a, b]$ (see [15, Lemma 2.4]) such that $v > 0$ on (a, x_0). Now, assuming that u has no zeros in $(a, b]$, we can find a linear combination of solutions $\lambda u + \mu v$, which has a double zero in (a, x_0). By virtue of Lemma 3.1 there is no a nontrivial solution of equation (3.3) satisfying $u(a) = 0$ and $u'(a)u''(a) \geq 0$ that has a double zero $x^* \in (a, b)$. Therefore, $\lambda u + \mu v \equiv 0$. This, however, contradicts linear independence of u and v. □

An extreme point of a function is called *nontrivial* if the function is different from the identically constant in any neighborhood of this point.

Let $u, z \in D$. By multiplying the function u by $L_r z$ and by integrating by parts, we obtain

$$\int\limits_\Gamma u \cdot L_r z\, dx = \int\limits_\Gamma (pz''u'' + qu'z' + rzu)dx + \sum_{c\in\partial\Gamma} (pz'')(c)u'(c)$$
$$- \sum_{c\in\partial\Gamma} D^3 z(c)\, u(c) + \sum_{c\in J(\Gamma)} \sum_{i\in I(c)} (p_i z_i'')(c) u_i'(c) - \sum_{c\in J(\Gamma)} u(c) \sum_{i\in I(c)} D^3 z_i(c).$$

For the sake of generality of notation we set $\alpha_{kk}(c) = \alpha_{jj}(c) = 1$, $\alpha_{kj}(c) = \alpha_{jk}(c) = 0$ for each vertex $c \in J(\Gamma)$. Then using (2.2)-(2.3), we get

$$\int\limits_\Gamma u \cdot L_r z\, dx = \int\limits_\Gamma (pz''u'' + qu'z' + rzu)dx + \sum_{c\in\partial\Gamma} (pz'')(c)u'(c) - \sum_{c\in\partial\Gamma} D^3 z(c)\, u(c)$$
$$- \sum_{c\in J(\Gamma)} u(c) \left(\sum_{i\in I(c)} D^3 z_i(c) + r(c)z(z) \right) + \sum_{c\in J(\Gamma)} r(c)u(c)z(c)$$
$$+ \sum_{c\in J(\Gamma)} u_k'(c) \sum_{i\in I(c)} \alpha_{ki}(c)(pz)_i''(c) + \sum_{c\in J(\Gamma)} u_j'(c) \sum_{i\in I(c)} \alpha_{ji}(c)(pz)_i''(c).$$

Hence

$$\begin{aligned} &\int\limits_\Gamma u \cdot L_r z\, dx + \sum_{c\in J(\Gamma)} u(c) L_r z(c) = \int\limits_\Gamma (pz''u'' + qz'u' + rzu)dx \\ &\quad + \sum_{c\in\partial\Gamma} (pz'')(c)u'(c) - \sum_{c\in\partial\Gamma} D^3 z(c)\, u(c) + \sum_{c\in J(\Gamma)} r(c)u(c)z(c). \end{aligned} \tag{3.4}$$

Integrating by parts twice again, we have

$$\int\limits_\Gamma u \cdot L_r z\, dx + \sum_{c\in J(\Gamma)} u(c) L_r z(c) - \int\limits_\Gamma z \cdot L_r u\, dx$$
$$= \sum_{c\in\partial\Gamma} p(c)(z''(c)u'(c) - u''(c)z'(c)) - \sum_{c\in\partial\Gamma} D^3 z(c)\, u(c) + \sum_{c\in\partial\Gamma} D^3 u(c)\, z(c)$$
$$- \sum_{c\in J(\Gamma)} z_k'(c) \sum_{i\in I(c)} \alpha_{ki}(c)(pu)_i''(c) - \sum_{c\in J(\Gamma)} z_j'(c) \sum_{i\in I(c)} \alpha_{ji}(c)(pu)_i''(c)$$
$$+ \sum_{c\in J(\Gamma)} z(c) \left(\sum_{i\in I(c)} D^3 u_i(c) + r(c)u(c) \right).$$

Now we finally get

$$\begin{aligned} &\int\limits_\Gamma u \cdot L_r z\, dx + \sum_{c\in J(\Gamma)} u(c) L_r z(c) - \int\limits_\Gamma z \cdot L_r u\, dx - \sum_{c\in J(\Gamma)} z(c) L_r u(c) \\ &= \sum_{c\in\partial\Gamma} p(c)(z''(c)u'(c) - u''(c)z'(c)) - \sum_{c\in\partial\Gamma} D^3 z(c)\, u(c) + \sum_{c\in\partial\Gamma} D^3 u(c)\, z(c). \end{aligned} \tag{3.5}$$

For each vertex $a \in \partial\Gamma$, we define a pair of functions w_a, v_a that are solutions of the equation $L_r u(x) = 0$, $x \in \Gamma$, satisfying the following boundary condition

$$\begin{aligned} &w_a(a) = 1, \quad \vartheta(a)w_a'(a) - \beta(a)w_a''(a) = 0, \quad w_a|_{\partial\Gamma\setminus a} = (\vartheta w_a' - \beta w_a'')|_{\partial\Gamma\setminus a} = 0; \\ &v_a(a) = 0, \quad \vartheta(a)v_a'(a) - \beta(a)v_a''(a) = 1, \quad v_a|_{\partial\Gamma\setminus a} = (\vartheta v_a' - \beta v_a'')|_{\partial\Gamma\setminus a} = 0. \end{aligned} \tag{3.6}$$

It follows from Lemma 3.1 that, for any vertex $a \in \partial\Gamma$, the set of all $2|\partial\Gamma|$ solutions of these problems forming a fundamental system of solutions of the homogeneous equation (1.1).

Denote by $B[u, v]$ the bilinear form

$$B[u,v] = \int\limits_{\Gamma} (pu''v'' + qu'v' + ruv)\,dx + \sum_{c\in J(\Gamma)} r(c)u(c)v(c).$$

Using (3.4) and the boundary conditions from (3.6), we get

$$\begin{aligned} 0 &= \int\limits_{\Gamma} v_a \cdot L_r v_a\,dx + \sum_{c\in J(\Gamma)} v_a(c) L_r v_a(c) \\ &= p(a)v_a''(a)v_a'(a) + B[v_a, v_a] + \sum_{\substack{c\in\partial\Gamma\setminus a\\ \vartheta(c)\neq 0}} \frac{p\beta}{\vartheta}(c) v_a''^2(c), \end{aligned} \tag{3.7}$$

$$\begin{aligned} 0 &= \int\limits_{\Gamma} w_a \cdot L_r w_a\,dx + \sum_{c\in J(\Gamma)} w_a(c) L_r w_a(c) \\ &= -D^3 w_a(a) + B[w_a, w_a] + \sum_{\substack{c\in\partial\Gamma\setminus a\\ \vartheta(c)\neq 0}} \frac{p\beta}{\vartheta}(c) w_a''^2(c). \end{aligned} \tag{3.8}$$

Similarly, using (3.4) and (3.5), we have the following equalities

$$\begin{aligned} 0 &= \int\limits_{\Gamma} v_a \cdot L_r w_a\,dx + \sum_{c\in J(\Gamma)} v_a(c) L_r w_a(c) \\ &= p(a)w_a''(a)v_a'(a) + B[w_a, v_a] + \sum_{c\in\partial\Gamma\setminus a} (pw_a'')(c) v_a'(c). \end{aligned} \tag{3.9}$$

$$\begin{aligned} 0 = \int\limits_{\Gamma} w_a \cdot L_r v_a\,dx + \sum_{c\in J(\Gamma)} w_a(c) L_r v_a(c) - \int\limits_{\Gamma} v_a \cdot L_r w_a\,dx - \\ \sum_{c\in J(\Gamma)} v_a(c) L_r w_a(c) = -D^3 v_a(a) + (p(a)v_a''(a)w_a'(a) - p(a)w_a''(a)v_a'(a)) \\ + \sum_{c\in\partial\Gamma\setminus a} ((pv_a'')(c)w_a'(c) - (pw_a'')(c)v_a'(c)). \end{aligned} \tag{3.10}$$

If $c \in \partial\Gamma \setminus a$ and $\vartheta(c) = 0$, then $v_a''(c) = w_a''(c) = 0$, and so $(pv_a'')(c)w_a'(c) - (pw_a'')(c)v_a'(c) = 0$. Otherwise, $\vartheta(c)w_a'(c) = \beta(c)w_a''(c)$, and by (3.6),

$$(pw_a'')(c)\left(v_a'(c) - v_a''(c)\frac{\beta(c)}{\vartheta(c)}\right) = 0.$$

It follows from (3.10) that

$$\begin{aligned} 0 &= \int\limits_{\Gamma} w_a \cdot L_r v_a\,dx + \sum_{c\in J(\Gamma)} w_a(c) L_r v_a(c) - \int\limits_{\Gamma} v_a \cdot L_r w_a\,dx - \sum_{c\in J(\Gamma)} v_a(c) L_r w_a(c) \\ &= -D^3 v_a(a) + (p(a)v_a''(a)w_a'(a) - p(a)w_a''(a)v_a'(a)). \end{aligned} \tag{3.11}$$

Lemma 3.4. *Let u be a solution of equation (1.1). If u vanishes at an inner vertex $c \in J(\Gamma)$ and does not change sign in some neighborhood of it, then $u_i'(c) = 0$ for all $i \in I(c)$.*

Proof. Suppose that $u(x) \geq 0$ in some neighborhood of c. Since $u(c) = 0$, we have $u_i'(c) \geq 0$ for all $i \in I(c)$. Let us consider conditions (2.2) at the vertex $c \in J(\Gamma)$. If $|I(c)| = 2$, then $u_i'(c) = \alpha_{ki}(c) u_k'(c)$, where $\alpha_{ki}(c) < 0$. Therefore $u_i'(c) = u_k'(c) = 0$.

Now consider the case $|I(c)| > 2$. It follows from properties of coefficients α_{ki}, α_{ji} (see Section 2.2) that there exist indices $i_1, i_2 \in I(c) \setminus \{k, j\}$ such that $\alpha_{ji_1}(c) < 0$, $\alpha_{ki_1}(c) \leq 0$ and $\alpha_{ji_2}(c) \leq 0$, $\alpha_{ki_2}(c) < 0$. Therefore

$$\begin{aligned} 0 \leq u_{i_1}'(c) = \alpha_{ki_1}(c) u_k'(c) + \alpha_{ji_1}(c) u_j'(c) \leq 0, \\ 0 \leq u_{i_2}'(c) = \alpha_{ki_2}(c) u_k'(c) + \alpha_{ji_2}(c) u_j'(c) \leq 0. \end{aligned}$$

Consequently, the derivatives $u_k'(c)$, $u_j'(c)$ are simultaneously zero, and so $u_i'(c) = 0$ for all indices $i \in I(c)$. □

Corollary 3.5. *Let $c \in J(\Gamma)$ and $i_0 \in I(c)$. Assume that u solves the homogeneous equation (1.1), and $u_i \equiv 0$ for all $i \in I(c) \setminus i_0$. Then $u_{i_0} \equiv 0$ as well.*

An extreme point of a function is called *nontrivial* if the function is different from the identically constant in any neighborhood of this point.

Lemma 3.6. *For any $a \in \partial\Gamma$ the following inequalities hold:*

(i) $v_a'(a) > 0$ *and* $v_a''(a) < 0$;

(ii) $D^3 w_a(a) > 0$;

(iii) $D^3 v_a(a) = \begin{cases} -(p w_a')(a)/\beta(a), & \textit{if } \vartheta(a) = 0, \\ -(p w_a'')(a)/\vartheta(a), & \textit{if } \vartheta(a) \neq 0; \end{cases}$

Proof. Since $p(x) > 0$ and $q(x), r(x) \geqslant 0$, we deduce that $B[v_a, v_a]$ is nonnegative, and so $v_a''(a) v_a'(a) \leqslant 0$. Assume that $v_a''(a) v_a'(a) = 0$. Then the function $v_a(x)$ is a solution of the homogeneous equation (1.1) on Γ satisfying the conditions $v_a(a) = 0$, $v_a''(a) = 0$ (or $v_a'(a) = 0$) at the boundary vertex $a \in \partial\Gamma$. But boundary problem (1.1), (1.2) is unique solvable. Therefore v_a is trivial on Γ, a contradiction. This contradiction proves the property (i).

Let us proceed to the proof of property 2). It follows from (3.8) and $B[w_a, w_a] \geqslant 0$ that $D^3 w_a(a) \geqslant 0$. Let us show that in fact there is a strict inequality.

Assume by way of contradiction that $D^3 w_a(a) = 0$. It follows from (3.8) that $B[w_a, w_a] = 0$. Therefore the function $w_a(x)$ is linear on every edge of the graph Γ and $w_a = 0$ on $V(\Gamma) \setminus a$. Therefore $w_a \equiv 0$ on $\Gamma \setminus \gamma_a$. Let $c \in J(\Gamma)$ be the endpoint of the boundary edge γ_a. Then c is a nontrivial minimum of the function w_a. However, this contradicts Lemma 3.4. The proof of property (ii) is finished.

Let's now prove property (iii). If $\vartheta(a) = 0$, then $v_a''(a) = -1/\beta(a)$, and $w_a''(a) = 0$. So

$$(p v_a'')(a) w_a'(a) - (p w_a'')(a) v_a'(a) = -(p w_a')(a)/\beta(a).$$

Otherwise, $\vartheta(c) w_a'(c) = \beta(c) w_a''(c)$, and by (3.6),

$$\begin{aligned} (p v_a'')(a) w_a'(a) - (p w_a'')(a) v_a'(a) &= -(p w_a'')(a) \left(v_a'(a) - v_a''(a) \frac{\beta(a)}{\vartheta(a)} \right) \\ &= -(p w_a'')(a)/\vartheta(a). \end{aligned}$$

Now property (iii) follows from (3.11). The lemma is proved. □

A point $x_0 \in \Gamma \cup \partial\Gamma$ is said to be a *trivial zero* of a function $u \in C[\Gamma]$ if $u(x_0) = 0$ and there exists $\varepsilon > 0$ such that $u \equiv 0$ in $\mathcal{N}_\varepsilon(x_0)$.

If a function $u \in C[\Gamma]$ has a finite number of zeros in the edge γ_i adjacent to a vertex $c \in V(\Gamma)$, then we shall denote by $\operatorname{sgn} u_i(c+0)$ the limit $\lim\limits_{\gamma_i \ni x \to c} \operatorname{sgn} u_i(x)$, assuming that the edge γ_i is directed away from the vertex c. In the case $u_i(x) \equiv 0$, $x \in \gamma_i$, we put, by definition, $\operatorname{sgn} u_i(c+0) = 0$.

Let us consider an arbitrary vertex $c \in J(\Gamma)$. Since the graph Γ is a tree, the point c splits the graph Γ into several subgraphs for which c is a boundary vertex. Hence we can say that each vertex $c \in J(\Gamma)$ generates a finite set of disjoint branches $\{\Gamma_i(c)\}_{i \in I(a)}$ of Γ, and the branch $\Gamma_i(c)$ containing the edge γ_i.

4. Oscillation of solutions of the equation $L_r u = 0$

In the present section we define an oscillatory operator and show that if the operator L_r is oscillatory on a graph, then it is disconjugate on every edge of the graph.

We still assume $r \geq 0$ in Γ. Denote by $S_L[\Gamma]$ the space of all solutions of the homogeneous equation $L_r u(x) = 0$, $x \in \Gamma$, generated by (2.1)-(2.4). From Lemma 3.1 it follows that $\dim S_L[\Gamma] = 2|\partial\Gamma|$ and $\{w_a, v_a\}_{a \in \partial\Gamma}$ is a basis of $S_L[\Gamma]$.

Definition 4.1. The differential operator L_r is said to be nonoscillatory on a graph Γ if for each vertex $a \in \partial\Gamma$ the corresponding solution v_a is positive on Γ.

Before proceeding further, we need the following lemma that will be used in the sequel.

Lemma 4.2. *If graph Γ is a tree, then*

$$|\partial\Gamma| = 2 + \sum_{c \in J(\Gamma)} (I(c) - 2). \tag{4.1}$$

Proof. The proof is by induction over the number N of inner vertices of the graph Γ.

For $N = 1$ the proof is trivial. Now for the induction argument, assume the result to be true for all natural numbers not exceeding N.

Consider a graph Γ such that $|J(\Gamma)| = N + 1$. Take an arbitrary vertex $c_0 \in J(\Gamma)$. Delete the vertex c_0 from the graph Γ, and it splits into $|I(c_0)|$ branches $\Gamma_i(c_0)$ for which the point c_0 is a boundary vertex. It is clear that $|J(\Gamma_i(c_0))| \leq N$ for each $i \in I(c_0)$. It follows from our induction hypothesis that

$$\begin{aligned} |\partial\Gamma| &= \sum_{i \in I(c_0)} \Big(2 + \sum_{c \in J(\Gamma_i(c_0))} (I(c) - 2)\Big) - |I(c_0)| \\ &= \sum_{i \in I(c_0)} 2 + \sum_{c \in J(\Gamma) \setminus c_0} (I(c) - 2) - |I(c_0)| = 2 + \sum_{c \in J(\Gamma)} (I(c) - 2). \end{aligned}$$

The lemma is proved. □

Lemma 4.3. *Let $m = |\partial\Gamma| - 1$, and let $\{u^{[n]}\}_{n=1}^{2m-1}$ be a set of nontrivial functions from $S_L[\Gamma]$. If $u^{[n]} \equiv 0$ on the boundary edge incident to the vertex $a \in \partial\Gamma$ for all $1 \leq n \leq 2m - 1$, then the functions $u^{[1]}, u^{[2]}, \ldots, u^{[2m-1]}$ are linearly dependent.*

Proof. The plan is to construct a nontrivial linear combination of the functions $u^{[1]}, \ldots, u^{[2m-1]}$, which is identically zero on Γ.

If $m = 1$, then $|I(c)| = 2$. From Corollary (3.5) it follows that $u^{[1]}$ is trivial on Γ.

Suppose $m \geq 2$. Let $\gamma_k = (a,c)$ be a boundary edge incident to the vertex $a \in \partial\Gamma$, where $c \in J(\Gamma)$. Without loss of generality we can assume that k is a basis index in $I(c)$ (see Section 2.2). As usual, we denote by $j \in I(c) \setminus k$ the second basis index in $I(c)$.

1. Passing to linear combinations, from the set $\{u^{[n]}\}_{n=1}^{2m-1}$ we can obtain a set $\{v^{[n]}\}_{n=1}^{2m-2}$ of functions such that $\left(v_j^{[n]}\right)'(c) = 0$ for all $1 \leq n \leq 2m-2$. To do this we take a function $u^{[n_0]}$ such that $\left(u_j^{[n_0]}\right)'(c) \neq 0$ and construct linear combinations

$$u^{[n]}(x) - \left(u_j^{[n]}\right)'(c) \cdot u^{[n_0]}(x) / \left(u_j^{[n_0]}\right)'(c), \quad n \in \{1, \ldots, 2m-1\} \setminus n_0.$$

In the case $\left(u_j^{[n_0]}\right)'(c) = 0$ for all $1 \leq n \leq 2m-1$, we can set, for example, $v^{[n]} = u^{[n]}$ for $1 \leq n \leq 2m-2$.

Since $v^{[n]}(x) \equiv 0$ on γ_k, it follows from (2.2) that $v_i^{[n]}(c) = \left(v_i^{[n]}\right)'(c) = 0$ for each $i \in I(c)$ and for each $1 \leq n \leq 2m-2$.

2. Using the set $\{v^{[n]}\}_{n=1}^{2m-2}$ we can obtain (as above) a set $\{w^{[n]}\}_{n=1}^{2m-|I(c)|+1}$ of functions from $S_L[\Gamma]$ such that $\left(w_i^{[n]}\right)''(c) = 0$ for each $i \in I(c) \setminus \{j,k,i_0\}$, where $i_0 \in I(c) \setminus \{j,k\}$.

Since $w^{[n]}(x) \equiv 0$ on γ_k, it follows from (2.3) that $\left(w_{i_0}^{[n]}\right)''(c) = 0$ for each $1 \leq n \leq 2m - |I(c)| + 1$, and so

$$w_i^{[n]}(c) = \left(w_i^{[n]}\right)'(c) = \left(w_i^{[n]}\right)''(c) = 0, \quad \forall i \in I(c), \quad 1 \leq n \leq 2m - |I(c)| + 1.$$

3. Finally, we can construct a set $\{z^{[n]}\}_{n=1}^{2m-2|I(c)|+3}$ of functions from $S_L[\Gamma]$ such that $\left(z_i^{[n]}\right)'''(c) = 0$ for all $i \in I(c) \setminus \{k, i_1\}$, where $i_1 \in I(c) \setminus \{k\}$. Since $z^{[n]}(x) \equiv 0$ on γ_k, it follows from (2.4) that $\left(z_i^{[n]}\right)'''(c) = 0$ for all $i \in I(c)$ and $1 \leq n \leq 2m - 2|I(c)| + 3$. It follows from Lemma 4.1 that $2m - 2|I(c)| + 3 = 2m - 1 - 2(|I(c)| - 2) \geq 1$.

Thus, we have the set of functions $\{z^{[n]}\}_{n=1}^{2m-1-2(|I(c)|-2)}$ those are nontrivial linear combinations of functions $\{u^{[n]}\}_{n=1}^{2m-1}$. It is easy to see that each function $z^{[n]}$ is identically zero on the star $\bigcup_{i \in I(c)} \gamma_i$.

If $J(\Gamma) = \{c\}$, then the theorem is proved. Otherwise, consider an arbitrary vertex $c_1 \in J(\Gamma)$ adjacent to c. Repeating the steps 1–3 for the functions $\{z^{[n]}\}_{n=1}^{2m-1-2(|I(c)|-2)}$ we can obtain the set of $2m - 1 - 2(|I(c)| - 2) - 2(|I(c_1)| - 2)$ functions from $S_L[\Gamma]$, all of which are identically zero on $\bigcup_{i \in I(c) \cup I(c_1)} \gamma_i$, and so on.

It follows from (4.1) that there is at least one nontrivial linear combination of the functions $u^{[1]}, \ldots, u^{[2m-1]}$ identically equal to zero on Γ. This completes the proof of the lemma. □

Corollary 4.4. *Let $a_0 \in \partial\Gamma$. There are at least two functions from the set $\{w_a, v_a\}_{\partial\Gamma \setminus a_0}$ that are linearly independent on the boundary edge γ_{a_0}.*

Proof. Indeed, suppose by contradiction that any two functions from the set $\{w_a, v_a\}_{\partial\Gamma \setminus a_0}$ are linearly dependent on the edge γ_{a_0}. Then it is possible to construct a system of $2m-1$ linearly independent functions identically equal to zero on γ_{a_0}, which is a contradiction. □

Theorem 4.5. *If the operator L_r is nonoscillatory on the graph Γ, then its restriction to the closure of any edge $\gamma \subset E(\Gamma)$ is disconjugate on $\overline{\gamma}$.*

Proof. Since L_r is nonoscillatory on the graph Γ then L_r is disconjugate on the closure $\overline{\gamma}_a$ of each boundary edge γ_a, $a \in \partial\Gamma$. Otherwise, by Lemma 3.3, w_a has a zero in $\overline{\gamma}_a \setminus a$.

Now suppose to the contrary that L_r is disconjugate on the closure $\overline{\gamma}_k$ of the edge $\gamma_k = (c_1, c_2) \subset E(\Gamma)$, where $c_1, c_2 \in J(\Gamma)$. As usual, denote by $\partial\Gamma_k(c_1)$ the branch containing the edge γ_k, $k \in I(c_1)$. Using Corollary 4.3, it is easy to show that at least two functions from the set $\{w_a, v_a\}_{a\in\partial\Gamma_k(c_1)\setminus c_1}$ are linearly independent on the edge γ_k. Indeed, suppose by contradiction that there exists the vertex $a_0 \in \partial\Gamma_k(c_1) \setminus c_1$ such that each function from the set $\{w_a, v_a\}_{a\in\partial\Gamma_k(c_1)\setminus c_1}$ is a constant multiple of w_{a_0} (or v_{a_0}) on the edge γ_k. Then it is possible to construct a system of $2|\partial\Gamma_k(c_1)| - 3$ linearly independent on $\Gamma_k(c_1)$ functions those identically equal to zero on the boundary edge γ_k of the branch $\partial\Gamma_k(c_1)$. But this fact implies a contradiction to Lemma 4.2.

Take two linearly independent on the edge γ_k functions from the set $\{w_a, v_a\}_{a\in\partial\Gamma_k(c_1)\setminus c_1}$ and denote them by w and by v. Since L is nonoscillatory on Γ, we have $w > 0$ and $v > 0$ on Γ. Consider the nontrivial solution $u(x) = w(x)v(c_1) - w(c_1)v(x)$. It is clear that $u(c_1) = 0$ and

$$u(b) = \vartheta(b)u'(b) - \beta(b)u''(b) = 0 \quad \forall b \in \partial\Gamma \setminus \partial\Gamma_k(c_1). \tag{4.2}$$

Let us delete the point c_1 from the graph Γ, and it splits into several connected components $\Gamma_i(c_1)$ for which the point c_1 is a boundary vertex. We claim $u_k'(c_1)u_k''(c_1) \geq 0$ (as usually, we assume that the edge γ_k is parameterized away from the vertex c_1 into the branch $\Gamma_k(c_1)$). Without loss of generality we can assume $\operatorname{sgn} u_k(c_1 + 0) > 0$. Fix now an arbitrary vertex $a \in \partial\Gamma_k(c_1) \setminus c_1$ and let consider the solution $z(x) = u(x) - \varepsilon w_a(x)$, where $\varepsilon > 0$. It is clear that $z(c_1) < 0$ and z has a zero $\xi \in \gamma_k \subset \Gamma_k(c_1)$, provided $\varepsilon > 0$ is sufficiently small. Denote by $\Gamma(\xi)$ the branch of $\Gamma \setminus \xi$ such that $c_1 \in \Gamma(\xi)$. It is clear that z satisfies the boundary conditions

$$z|_{\partial\Gamma(\xi)} = (\vartheta z' - \beta z'')|_{\partial\Gamma(\xi)\setminus\xi} = 0.$$

By virtue of Lemma 3.6 and $z(c_1) \neq 0$, we have $z_k'(\xi)z_k''(\xi) > 0$ for all sufficiently small $\varepsilon > 0$ (recall that the parameterization on γ_k is defined from the vertex c_1 to the point ξ). Moreover, by definition of the solution z, $\xi \to c_1$ as $\varepsilon \to 0$. Since $z \xrightarrow{C^2[\Gamma]} u$ as $\varepsilon \to 0$, we have $u_k'(c_1)u_k''(c_1) \geq 0$.

According to our assumption, L_r is not disconjugate on $\overline{\gamma}_k$. It follows from Lemma 3.3 that u has at least one zero $x_0 \in \gamma_k$. Fix an arbitrary vertex $a^* \in \partial\Gamma \cap \partial\Gamma(c_1)$. Since $u_k(c_1) = u_k(x_0) = 0$ and $v_{a^*}(x) > 0$ on Γ, we deduce that there exists μ such that $y(x) = v_{a^*}(x) - \mu u(x)$ has a double zero $\eta \in (c_1, x_0) \subset \gamma_k$. Denote by $\Gamma(\eta)$ the branch of $\Gamma \setminus \eta$ such that $c_1 \in \Gamma(\eta)$. It follows from (4.2) that y solves the boundary value problem

$$L_r y = 0, \quad x \in \Gamma(\eta),$$
$$y|_{\partial\Gamma(\eta)} = y'(\eta) = (\vartheta y' - \beta y'')|_{\partial\Gamma(\eta)\setminus\eta} = 0.$$

It follows from Lemma 3.1 that $y \equiv 0$ on $\Gamma(\eta)$, and so v_{a^*} is a constant multiple of u on $\Gamma(\eta)$. But $u(c_1) = 0$ and $v_{a^*}(c_1) > 0$, a contradiction. The theorem is proved. $\square$

Remark 4.6. Note that the disconjugacy of the operator L_r on each edge $\overline{\gamma} \subset \Gamma$ does not imply nonoscillatory on the graph Γ. As an example, we can consider the operator L_0, corresponding to a three-point boundary value problem

$$\begin{gathered} u^{IV} = f(x), \quad x \in (0,1) \cup (1,3.5), \\ u(1+0) - u(1-0) = 0, \quad u'(1+0) + u'(1-0) = 0, \quad u''(1+0) - u''(1-0) = 0, \\ u'''(1+0) - u'''(1-0) + \delta u(1) = 0, \\ u(0) = u'(0) = 0, \quad u(3.5) = u'(3.5) = 0. \end{gathered} \tag{4.3}$$

If $\delta \geqslant 16.5$, then the solution v_0 of the boundary value problem

$$L_0 u = 0, \quad u(0) = 0, \quad u'(0) = 1, \quad u(3.5) = u'(3.5) = 0$$

is not positive (see [7]). Nevertheless, if we consider the restriction $Lu = u^{IV}$ of L_0 to $[0, 1]$ and to $[1, 3.5]$, it is easy to see that L is disconjugate on $[0, 1]$ and on $[1, 3.5]$.

Acknowledgments. This work was supported by the Ministry of Science and Higher Education of the Russian Federation. Agreement No. 075-02-2025-1447.

References

[1] Bondarenko N. Inverse problems for the differential operator on the graph with a cycle with different orders on different edges. Tamkang Journal of Mathematics, 2015, 46(3), 229–243. https://doi.org/10.5556/j.tkjm.46.2015.1694

[2] Borovskikh A.V. Sign Regularity Conditions for Discontinuous Boundary-Value Problems. Math. Notes. 2003, 74(5), 607–618.

[3] Buterin S.A. On Damping a Control System of Arbitrary Order with Global Aftereffect on a Tree. Math Notes. 2024, 115, 877–896.

[4] Derr V.Ya. Disconjugacy of solutions of linear differential equations. Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 2009, no. 1, 4–89. (Russian)

[5] Exner P., Lipovský J. Topological bulk-edge effects in quantum graph transport. Physics Letters A, 2020, 384(18), 126390.

[6] Kulaev R. Ch. Necessary and sufficient condition for the positivity of the Green function of a boundary value problem for a fourth-order equation on a graph. Differential Equations. 2015, 51(3), 303–317.

[7] Kulaev R. Ch. Criterion for the positiveness of the Green function of a many-point boundary value problem for a fourth-order equation. Diff. Equ. 2015, 51(2), 1–14.

[8] Kulaev R. Ch. On the solvability of a boundary value problem for a fourth-order equation on a graph. Diff. Equ. 2014, 50(1), 27–34.

[9] Kulaev R. Ch. The qualitative theory of fourth-order differential equations on a graph. Mediterr. J. Math. 2022, 19(73), 1-15.

[10] Kulaev R. Ch. The Green function of the boundary value problem on a star-shaped graph. Russ. Math. (Iz. VUZ), 2013, 57(2), 48–57.

[11] Kulaev R. Ch. Disconjugacy of fourth-order equations on graphs. Sb. Math. 2015, 206(12), 1731–1770.

[12] Kulaev R. Ch. On the nonoscillation of an equation on a graph. Diff. Equ. 2015, 50(12), 1565–1566.

[13] Kulaev R., Karkuzaev S. On the positive invertibility of a differential operator on a graph. Applicable Analysis, 2024, 1–15.

[14] Kulaev R., Urtaeva A. Spectral properties of a fourth-order differential operator on a network. Math. Meth. Appl. Sci. 2023, 1–21.

[15] Kondrat'ev V. A. Oscillation of solutions of linear equations of third and fourth order. (Russian) Trudy Moskov. Mat. Obšč. 1959, 8, 259–281.

[16] Mercier D., Régnier V. Spectrum of a network of Euler-Bernoulli beams. J. Math. Anal. Appl., 2008, 337(1), 174–196.

[17] Kiik J.-C., Kurasov P., Usman M. On vertex conditions for elastic systems. Phys. Lett. A. 2015, 379(34-35), 1871–1876.

[18] Levin A.Ju. The non-oscillation of solutions of the equation $x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0$. Russian Math. Servey, 1969, 24(2), 44–99.

[19] Pokornyi Yu.V. Nonoscillation of Ordinary Differential Equations and Inequalities on Spatial Networks. Differential Equations, 2001, 37(5), 695–705.

[20] Panasenko G., Pileckas K. Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure, I,II. Nonlinear Analysis, 2015, 122, 125–168.

[21] Pokornyi Yu. V., Penkin O. M., Pryadiev V. L., et al. Differentsialnye uravneniya na geometricheskikh grafakh (Differential Equations on Geometric Graphs). Moscow, Fiz.-Mat. Lit. 2005.

[22] Sobirov Z. A. Inverse source problem for subdiffusion equation on metric star graph in Sobolev spaces. Lobachevskii J. Math. 2023, 44(12), 5416–5426.

[23] Sobirov Z. A. Cauchy Problem for Subdiffusion Equation on Metric Star Graph with Edge Dependent Order of Time-Fractional Derivative. Lobachevskii J. Math. 2022, 43, 3282–3291.

[24] Stepanov G. D., Effective criteria for the strong sign-regularity and the oscillation property of the Green's functions of two-point boundary-value problems. Sb. Math. 1997, 188(11), 1687–1728.

[25] Swanson C. A. Comparison theorems and oscillation theory of linear differential equations. New York and London: Academic Press 1968.

[26] Yurko V. A. Inverse spectral problems for differential operators on spatial networks. Russian Math. Surveys, 2016, 71(3), 539–584.

[27] Zvereva M. B. and Kamenskii M. I. Boundary Value Problem on a Geometric Star-Graph with a Nonlinear Condition at a Node. Math. Notes, 2023, 114(2), 275–279.

[28] Zvereva M. B. and Kamenskii M. I. Problem on string system vibrations on star-shaped graph with nonlinear condition at node. Ufa Math. J., 2024, 16(1), 34–52.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.109-118
DOI: 10.29229/uzmj.2025-2-10

On the Cauchy problem for the Langevin-type fractional equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this article, the Cauchy problem for the Langevin-type time-fractional equation $D_t^\beta(D_t^\alpha u(t)) + D_t^\beta(Au(t)) = f(t), (0 < t \leq T)$ is studied. Here $\alpha, \beta \in (0,1)$, D_t^α, D_t^β is the Caputo derivative and A is an unbounded self-adjoint operator in a separable Hilbert space. Under certain conditions, we establish the existence and uniqueness of the solution and provide an explicit representation of it using eigenfunction expansions.

Keywords: Cauchy problem; Langevin-type fractional equation; Caputo fractional derivative

MSC (2020): 35R11, 34A12

1. Introduction

The Langevin equation is an important mathematical physics equation used to model phenomena occurring in fluctuating environments such as Brownian motion ([18],[12]). The classical form of this equation was derived in terms of ordinary derivatives by P. Langevin (Paul Langevin, 1872-1946, Paris) in [18]. The Langevin equation has enormous applications, that is, cell migration in biology [25]; polymer and protein dynamics in chemistry ([24],[26]); signal processing with noise, diamagnetics in electrical engineering ([12], [3],[20]).

With the intensive development of fractional derivatives, a natural generalization of the Langevin equation is to replace the ordinary derivative with a fractional derivative to yield a fractional Langevin equation, which can be considered a particular case of the generalized Langevin equation. Mainardi introduced the fractional Langevin equation [23] in the early 1990s. Many different types of Langevin equations were studied in the works [11], [14], [19], [1],[2]. The usual fractional Langevin equation involving only one fractional order was studied in [22],[11], the Langevin equation containing both a frictional memory kernel and a fractional derivative was studied in [14], the nonlinear Langevin equation involving two fractional orders was studied in the articles [19], [1], [2]. The vast majority of these articles determined the existence and uniqueness of Langevin equation solutions, and some promising results have been obtained using the Banach contraction principle, Krasnoselskii's fixed point theorem, Schauder's fixed point theorem, Leray-Schauder nonlinear alternative, Leray-Schauder degree, and other techniques.

Let H be a separable Hilbert space and $A : D(A) \to H$ be an arbitrary unbounded positive self-adjoint operator with the domain of definition $D(A)$. We assume that the operator A has a complete orthonormal system of eigenfunctions $\{v_k\}$ and a countable set of positive eigenvalues $\lambda_k : 0 < \lambda_1 \leq \lambda_2 ... \to +\infty$. The sequence $\{\lambda_k\}$ has no finite limit points.

Let $C((a,b);H)$ be the set of continuous vector-valued functions $y(t)$ on $t \in (a,b)$ with values in H.

Let $AC[0,T]$ be the set of absolutely continuous functions defined on $[0,T]$ and let $AC([0,T];H)$ stand for a space of absolutely continuous functions $y(t)$ with values in H (see, [17] p.339).

The definitions of fractional integrals and derivatives for the function $h : \mathbb{R}_+ \to H$ are discussed in detail in [21]. The fractional integral of order σ for a function $h(t)$ defined on $\mathbb{R}_+$

is given by:

$$I_t^\sigma h(t) = \frac{1}{\Gamma(\sigma)} \int_0^t \frac{h(\xi)}{(t-\xi)^{1-\sigma}} d\xi, \quad t > 0,$$

where $\Gamma(\sigma)$ is the Euler gamma function. Using this definition, the Caputo fractional derivative of order $\rho \in (0,1)$ can be defined as:

$$D_t^\rho h(t) = I_t^{1-\rho} \frac{d}{dt} h(t).$$

In this article, we consider the following Cauchy problem for a Langevin-type fractional differential equation:

$$\begin{cases} D_t^\beta (D_t^\alpha u(t)) + D_t^\beta (Au(t)) = f(t), \quad 0 < t \leq T, \\ u(+0) = \varphi, \\ D_t^\alpha u(+0) = \psi, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $0 < \beta < 1$; $\varphi, \psi \in H$ and $f(t) \in C([0,T]; H)$.

Definition 1.1. A function $u(t) \in AC([0,T]; H)$ is called a solution of problem (1.1) if $D_t^\beta(Au(t))$, $D_t^\beta(D_t^\alpha u(t)) \in C((0,T]; H)$, $D_t^\alpha u(t) \in C([0,T]; H)$ and $u(t)$ satisfies all conditions of problem (1.1)

In the case $\beta = 0$, the Langevin-type fractional equation with two different orders coincides with the fractional subdiffusion equation. The forward and inverse problems for the fractional subdiffusion equation have been studied in the articles [4], [6], [7].

If $\alpha = \beta$, then a fractional Langevin equation with two fractional orders can coincide with the fractional telegraph equation under certain conditions. The non-local and inverse problems for the fractional telegraph equation are studied in the works [8], [9], [10].

In this article, we prove the following theorem:

Theorem 1.2. *Let $\varphi \in D(A)$, $\psi \in H$. Further, let $0 < \varepsilon < 1$ be any fixed number and $f(t) \in C([0;T]; D(A^\varepsilon))$. Then, problem (1.1) has a unique solution given by:*

$$u(t) = \sum_{k=1}^\infty \Big[\varphi_k E_{\alpha,1}(-\lambda_k t^\alpha) + [\psi_k + \lambda_k \varphi_k] t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha) +$$

$$+ \int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^\alpha) f_k(\eta) d\eta \Big] v_k, \tag{1.2}$$

where $f_k(t), \varphi_k$ and ψ_k are the Fourier coefficients of the elements $f(t)$, φ and ψ, respectively.

2. Preliminaries

In this section, we present several pieces of data about the Mittag-Leffler functions, which we will use below.

Let ε be an arbitrary real number. The power of the operator A is defined by the following:

$$A^\varepsilon h = \sum_{k=1}^\infty \lambda_k^\varepsilon h_k v_k,$$

where h_k is the Fourier coefficient of the function $h \in H$, i.e., $h_k = (h, v_k)$. The domain of this operator is defined as:

$$D(A^\varepsilon) = \{h \in H : \sum_{k=1}^\infty \lambda_k^{2\varepsilon} |h_k|^2 < \infty\}.$$

For elements of $D(A^{\varepsilon})$ we introduce the norm

$$||h||_{\varepsilon}^{2}=\sum_{k=1}^{\infty}\lambda_{k}^{2\varepsilon}|h_{k}|^{2}.$$

The function

$$E_{\alpha,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\mu)}$$

is called the Mittag-Leffler function with two parameters(see [13], p 134), where $0<\alpha<1$, $\mu\in\mathbb{C}$.

We present some asymptotic estimates for the Mittag-Leffler function:

Lemma 2.1. *Let $0<\alpha<1$ and $\mu\in\mathbb{C}$. For any $t\geq 0$ one has (see [13], p. 136)*

$$|E_{\alpha,\mu}(-t)|\leq\frac{C}{1+t},$$

where constant C doesn't depend on t and α.

Lemma 2.2. *The following relation holds:*

$$|t^{\alpha-1}E_{\alpha,\mu}(-\lambda t^{\alpha})|\leq C_{\varepsilon}\lambda^{\varepsilon-1}t^{\varepsilon\alpha-1},$$

where λ is a positive number and $0<\varepsilon<1$.

This lemma is proven in [5]

Lemma 2.3. *Let $\alpha>0$ and $\lambda\in\mathbb{C}$, then the following relation holds (see [16], p. 78):*

$$D_{t}^{\alpha}E_{\alpha,1}(\lambda t^{\alpha})=\lambda E_{\alpha,1}(\lambda t^{\alpha}).$$

Lemma 2.4. *For $Re\gamma>0$, $Re\mu>0$, $\lambda\in\mathbb{R}$, the following relation is valid (see [15], p. 87):*

$$D_{t}^{\gamma}\left(t^{\mu-1}E_{\alpha,\mu}(\lambda t^{\alpha})\right)=t^{\mu-\gamma-1}E_{\alpha,\mu-\gamma}(\lambda t^{\alpha}).$$

Lemma 2.5. *Let $f(t)\in C[0,T]$, $0<\alpha<1$ and $0<\beta<1$.Then, the solution of the following Cauchy problem*

$$\begin{cases}D_{t}^{\beta}(D_{t}^{\alpha}y(t))+\lambda D_{t}^{\beta}y(t)=f(t), & 0<t\leq T,\\ y(+0)=\varphi,\\ D_{t}^{\alpha}y(+0)=\psi\end{cases}\tag{2.1}$$

has the form

$$y(t)=\varphi E_{\alpha,1}(-\lambda t^{\alpha})+(\psi+\lambda\varphi)t^{\alpha}E_{\alpha,\alpha+1}(-\lambda t^{\alpha})+\int_{0}^{t}(t-\eta)^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(-\lambda(t-\eta)^{\alpha})f(\eta)d\eta.$$

Proof. Firstly, we apply the operator I_{t}^{β} to both sides of the equation and obtain the following equality:

$$D_{t}^{\alpha}y(t)-D_{t}^{\alpha}y(0)+\lambda y(t)-\lambda y(0)=I_{t}^{\beta}f(t).$$

Using the conditions of the problem (2.1), we have $D_{t}^{\alpha}y(t)=\psi+\lambda\varphi-\lambda y(t)+I_{t}^{\beta}f(t)$.

We apply the operator I_{t}^{α} to obtain an integral equation for $y(t)$:

$$y(t)=\frac{\psi+\lambda\varphi}{\Gamma(\alpha+1)}t^{\alpha}+\varphi-\lambda I_{t}^{\alpha}y(t)+I_{t}^{\alpha+\beta}f(t).$$

To solve this integral equation, we use the method of successive approximations (see [16], p.137)

$$y_m(t) = \frac{\psi + \lambda\varphi}{\Gamma(\alpha+1)} t^\alpha + \varphi - \lambda I_t^\alpha y_{m-1}(t) + I_t^{\alpha+\beta} f(t).$$

Let us assume that the zeroth approximation is $y_0(t) = \frac{\psi+\lambda\varphi}{\Gamma(\alpha+1)} t^\alpha + \varphi$. Then the first approximation can be written as follows:

$$y_1(t) = \frac{\psi + \lambda\varphi}{\Gamma(\alpha+1)} t^\alpha + \varphi - \frac{\lambda(\psi+\lambda\varphi)}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{\lambda\varphi}{\Gamma(\alpha+1)} t^\alpha + I_t^{\alpha+\beta} f(t).$$

Here we obtain the second approximation, which reads as

$$y_2(t) = (\psi + \lambda\varphi) t^\alpha \sum_{k=0}^{2} \frac{(-1)^k t^{\alpha k} \lambda^k}{\Gamma(\alpha k + \alpha + 1)} + \varphi \sum_{k=0}^{2} \frac{(-1)^k t^{\alpha k} \lambda^k}{\Gamma(\alpha k + 1)} +$$

$$+ \int_0^t \left[\sum_{k=0}^{2} \frac{(-1)^k \lambda^k}{\Gamma(\alpha k + \alpha + \beta} (t-\tau)^{\alpha k + \alpha + \beta - 1} f(\tau) d\tau \right].$$

Hence, continuing in this manner, we obtain

$$y_m(t) = (\psi + \lambda\varphi) t^\alpha \sum_{k=0}^{m-1} \frac{(-1)^k t^{\alpha k} \lambda^k}{\Gamma(\alpha k + \alpha + 1)} + \varphi \sum_{k=0}^{m-1} \frac{(-1)^k t^{\alpha k} \lambda^k}{\Gamma(\alpha k + 1)} +$$

$$+ \int_0^t \left[\sum_{k=0}^{m-1} \frac{(-1)^k \lambda^k}{\Gamma(\alpha k + \alpha + \beta)} (t-\tau)^{\alpha k + \alpha + \beta - 1} f(\tau) d\tau \right].$$

Taking this limit as $m \to \infty$, we have

$$y(t) = (\psi + \lambda\varphi) t^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k t^{\alpha k} \lambda^k}{\Gamma(\alpha k + \alpha + 1)} + \varphi \sum_{k=0}^{\infty} \frac{(-1)^k t^{\alpha k} \lambda^k}{\Gamma(\alpha k + 1)} +$$

$$+ \int_0^t \left[\sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{\Gamma(\alpha k + \alpha + \beta)} (t-\tau)^{\alpha k + \alpha + \beta - 1} f(\tau) d\tau \right].$$

According to the definition of the Mittag-Leffler function, the last equality can be written as:

$$y(t) = \varphi E_{\alpha,1}(-\lambda t^\alpha) + [\psi + \lambda\varphi] t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda (t-\tau)^\alpha) f(\tau) d\tau.$$

Lemma 2.5 has been proved. □

Lemma 2.6. *Let $0 < \varepsilon < 1$ be any fixed number, and $f(t) \in C([0,T]; D(A^\varepsilon))$. Then the following estimate holds:*

$$\sum_{k=1}^{\infty} \left| \lambda_k \int_0^t (t-\eta)^{\alpha-1} E_{\alpha,\mu}(-\lambda_k (t-\eta)^\alpha) f_k(\eta) d\eta \right|^2 \leq C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2. \tag{2.2}$$

Proof. By using Lemma 2.2 for any fixed number $0<\varepsilon<1$, we take

$$\sum_{k=1}^{n}\lambda_k^2\left|\int_0^t (t-\eta)^{\alpha-1}E_{\alpha,\mu}(-\lambda_k(t-\eta)^{\alpha})f_k(\eta)d\eta\right|^2\le C_\varepsilon^1\sum_{k=1}^{n}\left[\int_0^t (t-\eta)^{\varepsilon\alpha-1}\lambda_k^{\varepsilon}|f_k(\eta)|d\eta\right]^2.$$

Using the generalized Minkowski inequality, we have

$$C_\varepsilon^1\sum_{k=1}^{n}\left[\int_0^t (t-\eta)^{\varepsilon\alpha-1}\lambda_k^{\varepsilon}|f_k(\eta)|d\eta\right]^2\le C_\varepsilon^1\left(\int_0^t (t-\eta)^{\varepsilon\alpha-1}\left(\sum_{k=1}^{n}\lambda_k^{2\varepsilon}|f_k(\eta)|^2\right)^{\frac{1}{2}}d\eta\right)^2$$

$$\le C_\varepsilon^1 T^{\varepsilon\alpha}\max_{t\in[0,T]}||f||_\varepsilon^2=C_\varepsilon\max_{t\in[0;T]}||f||_\varepsilon^2.$$

Taking the limit as $n\to\infty$, we obtain the estimate (2.2).

Lemma 2.6 has been proved. □

3. Proof of Theorem 1.2

Assume that a solution to problem (1.1) exists. Then, due to the completeness of the system $\{v_k\}$ in H, the arbitrary solution can be written in the form:

$$u(t)=\sum_{k=1}^{\infty}T_k(t)v_k, \tag{3.1}$$

where $T_k(t)$ are the Fourier coefficients of the function $u(t)$. Then, by virtue of (3.1), we obtain the following problem:

$$\begin{cases} D_t^{\beta}(D_t^{\alpha}T_k(t))+\lambda_k D_t^{\beta}T_k(t)=f_k(t),\\ T_k(0)=\varphi_k,\\ D_t^{\alpha}T_k(0)=\psi_k.\end{cases} \tag{3.2}$$

By Lemma 2.5, the solution of problem (3.2) is given by

$$T_k(t)=\varphi_k E_{\alpha,1}(-\lambda_k t^{\alpha})+(\psi_k+\lambda_k\varphi_k)t^{\alpha}E_{\alpha,\alpha+1}(-\lambda_k t^{\alpha})+$$

$$+\int_0^t (t-\eta)^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(-\lambda_k(t-\eta)^{\alpha})f_k(\eta)d\eta. \tag{3.3}$$

Thus, according to the equalities (3.1) and (3.3), we find the formal solution of the problem (1.1) in the form (1.2).

To prove the uniqueness of the solution, we use the standard technique, that is, the solution of problem (3.2) with the homogeneous condition (i.e, $\varphi_k=0$, $\psi_k=0$ and $f_k(t)=0$) is identically zero. Then it follows that $T_k(t)\equiv 0$, for all $k\ge 1$. According to the equality (3.1) and the completeness of the system $\{v_k\}$, we obtain $u(t)\equiv 0$.

We now verify that the formal solution satisfies the conditions of Definition 1.1. Let $S_j(t)$ be the partial sum of the series in (1.2)

$$S_j(t)=\sum_{k=1}^{j}\left[\varphi_k E_{\alpha,1}(-\lambda_k t^{\alpha})+(\psi_k+\lambda_k\varphi_k)t^{\alpha}E_{\alpha,\alpha+1}(-\lambda_k t^{\alpha})+\right.$$

$$\left.+\int_0^t (t-\eta)^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(-\lambda_k(t-\eta)^{\alpha})f_k(\eta)d\eta\right]v_k.$$

Then, by applying the operator A on the partial sum $S_j(t)$,we have

$$AS_j(t) = \sum_{k=1}^{j} \Big[\varphi_k E_{\alpha,1}(-\lambda_k t^\alpha) + (\psi_k + \lambda_k\varphi_k) t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha) +$$

$$+ \int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^\alpha) f_k(\eta) d\eta\Big]\lambda_k v_k. \tag{3.4}$$

Using Parseval's identity, we can obtain

$$||AS_j(t)||^2 = \sum_{k=1}^{j} \lambda_k^2 \Big|\varphi_k E_{\alpha,1}(-\lambda_k t^\alpha) + (\psi_k + \lambda_k\varphi_k) t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha) +$$

$$+ \int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^\alpha) f_k(\eta) d\eta\Big|^2.$$

Now, we split the above sum into three terms concerning φ_k, ψ_k, and $f_k(\eta)$, and by using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we obtain:

$$||AS_j(t)||^2 \leq \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 \big| \left(E_{\alpha,1}(-\lambda_k t^\alpha) + \lambda_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)\right) \big|^2 + \sum_{k=1}^{j} \lambda_k^2 \big|\psi_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)\big|^2 +$$

$$+ \sum_{k=1}^{j} \lambda_k^2 \left| \int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^\alpha) f_k(\eta) d\eta \right|^2 = AS_j^1 + AS_j^2 + AS_j^3.$$

In the first sum, we split it into two terms:

$$AS_j^1 = \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 \left|\left(E_{\alpha,1}(-\lambda_k t^\alpha) + \lambda_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)\right)\right|^2 \leq AS_j^{11} + AS_j^{12},$$

where

$$AS_j^{11} = \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 |E_{\alpha,1}(-\lambda_k t^\alpha)|^2, \quad AS_j^{12} = \sum_{k=1}^{j} \lambda_k^4 |\varphi_k|^2 |t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)|^2.$$

By applying Lemma 2.1 and the inequality $\lambda_k t^\alpha (1+\lambda_k t^\alpha)^{-1} < 1$ for AS_j^{11} and AS_j^{12}, we have the following estimates:

$$AS_j^{11} \leq C \sum_{k=1}^{j} |\varphi_k|^2, \quad AS_j^{12} \leq C \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2.$$

Thus, we have $AS_j^1 \leq C \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2, \quad t > 0$. Similarly, using Lemma 2.1 and the inequality $\lambda_k t^\alpha (1+\lambda_k t^\alpha)^{-1} < 1$ for the second sum, we have

$$AS_j^2 = \sum_{n=1}^{j} \lambda_k^2 \big|\psi_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)\big|^2 \leq C \sum_{n=1}^{j} |\psi_k|^2, \quad t > 0.$$

Let us estimate the sum AS_j^3. According to Lemma 2.6, we have:

$$AS_j^3 = \sum_{n=1}^{j} \lambda_k^2 \left| \int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^\alpha) f_k(\eta) d\eta \right|^2 \leq C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2.$$

Thus, if $\varphi \in D(A)$, $\psi \in H$ and $f(t) \in C([0,T]; D(A^{\varepsilon}))$, then from estimates of AS_i^j we obtain $Au(t) \in C((0,T]; H)$.

From the above, we can prove uniform convergence of the Fourier series corresponding to the function $u(t)$. If $\varphi \in H$, $\psi \in H$ and $f(t) \in C([0,T]; D(A^{\varepsilon}))$, then $u(t) \in AC([0,T]; H)$.

Next, we prove that $D_t^{\beta}(Au(t)) \in C((0,T]; H)$. Let us apply D_t^{β} term by term to series (3.4). By applying Lemma 2.3 , Lemma 2.4 and Parseval's identity, we obtain have the following expression

$$||D_t^{\beta}(AS_j(t))||^2 = \sum_{k=1}^{j} \lambda_k^2 \Big| \varphi_k(-\lambda_k) E_{\alpha,1}(-\lambda_k t^{\alpha}) + (\psi_k + \lambda_k \varphi_k) t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-\lambda_k t^{\alpha}) +$$

$$+ D_t^{\beta} \left(\int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^{\alpha}) f_k(\eta) d\eta \right) \Big|^2. \tag{3.5}$$

We split this sum (3.5) into three parts and estimate each term separately:

$$||D_t^{\beta}(AS_j(t))||^2 \le \sum_{k=1}^{j} \lambda_k^2 \Big| \varphi_k(-\lambda_k) E_{\alpha,1}(-\lambda_k t^{\alpha}) + \lambda_k \varphi_k t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-\lambda_k t^{\alpha}) \Big|^2 +$$

$$+ \sum_{k=1}^{j} \lambda_k^2 \Big| \psi_k t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-\lambda_k t^{\alpha}) \Big|^2 + \sum_{k=1}^{j} \lambda_k^2 \Big| D_t^{\beta} \left(\int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^{\alpha}) f_k(\eta) d\eta \right) \Big|^2 =$$

$$= K_1(t) + K_2(t) + K_3(t), \tag{3.6}$$

where $K_1(t) = \sum_{k=1}^{j} \lambda_k^2 \left| \varphi_k(-\lambda_k) E_{\alpha,1}(-\lambda_k t^{\alpha}) + \varphi_k \lambda_k t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-\lambda_k t^{\alpha}) \right|^2$,

$$K_2(t) = \sum_{k=1}^{j} \lambda_k^2 \left| \psi_k t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-\lambda_k t^{\alpha}) \right|^2,$$

$$K_3(t) = \sum_{k=1}^{j} \lambda_k^2 \Big| D_t^{\beta} \left(\int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^{\alpha}) f_k(\eta) d\eta \right) \Big|^2.$$

For $K_1(t)$, using Lemma 2.1 and $\lambda_k t^{\alpha-\gamma}(1+\lambda_k t^{\alpha})^{-1} < t^{-\gamma}$, we arrive at

$$K_1(t) \le \left[\frac{C}{t^{2\alpha}} + \frac{C}{t^{2\beta}} \right] \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2, \quad t > 0.$$

For $K_2(t)$, by applying Lemma 2.1 and the inequality $\lambda_k t^{\alpha-\gamma}(1+\lambda_k t^{\alpha})^{-1} < t^{-\gamma}$, we have

$$K_2(t) \le \frac{C}{t^{2\beta}} \sum_{k=1}^{j} |\psi_k|^2, \quad t > 0.$$

We need to prove uniform convergence of $K_3(t)$. Firstly, we denote the integral of $K_3(t)$ by $F(t)$ and integrate by parts

$$F(t) = \int_0^t (t-\eta)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda_k (t-\eta)^{\alpha}) f_k(\eta) d\eta =$$

$$= f_k(0) t^{\alpha+\beta} E_{\alpha,\alpha+\beta+1}(-\lambda_k t^{\alpha}) + \int_0^t (t-\eta)^{\alpha+\beta} E_{\alpha,\alpha+\beta+1}(-\lambda_k (t-\eta)^{\alpha}) f_k'(\eta) d\eta.$$

Then, we compute the Caputo fractional derivative of order β according to its definition:

$$D_t^\beta F(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t\frac{F'(s)}{(t-s)^\beta}ds=\frac{f_k(0)}{\Gamma(1-\beta)}\int_0^t\frac{(s^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(-\lambda_k s^\alpha))'_s}{(t-s)^\beta}ds+$$

$$+\frac{1}{\Gamma(1-\beta)}\int_0^t\int_0^s\frac{f_k'(\eta)}{(t-s)^\beta}\frac{\partial}{\partial s}\Big[(s-\eta)^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(-\lambda_k(s-\eta)^\alpha)\Big]d\eta ds=$$

$$=f_k(0)t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)+\int_0^t f_k'(\eta)(t-\eta)^\alpha E_{\alpha,\alpha+1}(-\lambda_k(t-\eta)^\alpha)d\eta.$$

Using integration by parts, we have

$$D_t^\beta F(t)=f_k(0)t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)+f_k(\eta)(t-\eta)^\alpha E_{\alpha,\alpha+1}(-\lambda_k(t-\eta)^\alpha)\Big|_0^t+$$

$$+\int_0^t f_k(\eta)(t-\eta)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\eta)^\alpha)d\eta=\int_0^t f_k(\eta)(t-\eta)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\eta)^\alpha)d\eta.$$

Therefore, $K_3(t)$ takes the following form:

$$K_3(t)=\sum_{k=1}^j\lambda_k^2\Big|\int_0^t f_k(\eta)(t-\eta)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\eta)^\alpha)d\eta\Big|^2.$$

Applying Lemma 2.6, we get

$$K_3(t)=\sum_{k=1}^j\lambda_k^2\Big|\int_0^t f_k(\eta)(t-\eta)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\eta)^\alpha)d\eta\Big|^2\le C_\varepsilon\max_{t\in[0,T]}||f||_\varepsilon^2$$

Thus, if $\varphi\in D(A)$, $\psi\in H$ and $f(t)\in C([0,T];D(A^\varepsilon))$, then from (3.6) and estimates K_1,K_2,K_3 , we obtain $D_t^\beta(Au(t))\in C((0,T];H)$.

The equation of problem (1.1) can be written as $D_t^\beta(D_t^\alpha u(t))=f(t)-D_t^\beta(Au(t))$. Therefore, from the above reasoning, we have $D_t^\beta(D_t^\alpha u(t))\in C((0,T];H)$.

Likewise, using Lemma 2.3, Lemma 2.4 and the definition of the Caputo fractional derivative, if we compute $D_t^\alpha u(t)$ in the same manner as $D_t^\beta(Au(t))$ was calculated, we obtain the following expression:

$$D_t^\alpha S_j(t)=\sum_{k=1}^j\Big[\psi_k E_{\alpha,1}(-\lambda_k t^\alpha)+\int_0^t f_k(\eta)(t-\eta)^{\beta-1}E_{\alpha,\beta}(-\lambda_k(t-\eta)^\alpha)d\eta\Big]v_k. \tag{3.7}$$

Using Parseval's identity, we have the following expression

$$||D_t^\alpha S_j(t)||^2=\sum_{k=1}^j\Big|\psi_k E_{\alpha,1}(-\lambda_k t^\alpha)+\int_0^t f_k(\eta)(t-\eta)^{\beta-1}E_{\alpha,\beta}(-\lambda_k(t-\eta)^\alpha)d\eta\Big|^2\le B_1(t)+B_2(t),$$

where

$$B_1(t)=\sum_{k=1}^j|\psi_k|^2|E_{\alpha,1}(-\lambda_k t^\alpha)|^2,\quad B_2(t)=\sum_{k=1}^j\Big|\int_0^t f_k(\eta)(t-\eta)^{\beta-1}E_{\alpha,\beta}(-\lambda_k(t-\eta)^\alpha)d\eta\Big|^2.$$

Now, using $|E_{\alpha,1}(-z)| \leq C$ (see, [15], p.62), we have $B_1(t) \leq C\sum_{k=1}^{j}|\psi_k|^2$, where $C>0$ is constant. According to $|E_{\alpha,\beta}(-z)| \leq 1$ (see [15]) and the generalized Minkowski inequality, we obtain the following estimate for $B_2(t)$:

$$B_2(t) \leq \sum_{k=1}^{j}\left|\int_0^t f_k(\eta)(t-\eta)^{\beta-1}d\eta\right|^2 \leq \left(\int_0^t\left(\sum_{k=1}^{j}|f_k(\eta)|^2\right)^{\frac{1}{2}}(t-\eta)^{\beta-1}d\eta\right)^2 \leq CT^{2\beta}\max_{t\in[0,T]}||f||^2.$$

Hence, if $\psi \in H$ and $f(t) \in C([0,T];H)$, then we have $D_t^\alpha u(t) \in C([0,T];H)$.

Now, by considering the case $t=+0$ in the equalities (1.2) and (3.7), we can verify that the solution $u(t)$ satisfies the conditions of the problem (1.1)

$$u(t)\Big|_{t=+0} = \sum_{k=1}^{\infty}\varphi_k v_k = \varphi, \quad D^\alpha u(t)\Big|_{t=+0} = \sum_{k=1}^{\infty}\psi_k v_k = \psi.$$

Theorem 1.2 has been proved.

4. Acknowledgement.

The authors deeply thank Professor R.R. Ashurov for a useful discussion of the article.

References

[1] Ahmad B., Nieto J.; Solvability of nonlinear Langevin equation involving two fractional orders with Dirichlet boundary conditions. Int. J. Diff. Equ.,–2010–2.

[2] Ahmad B., Nieto J., Alsaedi A., El-Shahed M.; A study of nonlinear Langevin equation involving two fractional orders in different intervals. Nonlinear Anal Real World Appl.,–2012.–13.–P.599-606.

[3] Ahmadova A., Mahmudov I.; Langevin differential equations with general fractional orders and their applications to electric circuit theory. Journal of Computational and Applied Mathematics,–2021.–388.–113299.

[4] Alimov Sh., Ashurov R.; Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation. Journal of Inverse and Ill-posed Problems,–2020.–28.–No5.–P.651–658.

[5] Ashurov R., Fayziev Y.; On the Nonlocal Problems in Time for Time-Fractional Subdiffusion Equations. Fractal and Fractional,–2022.–6,–No1.–P.41-62.

[6] Ashurov R., Fayziev, Y.; Uniqueness and Existence for Inverse Problem of Determining an Order of Time-Fractional Derivative of Subdiffusion Equation. Lobachevskii J. Math.,–2021.– 42.–P.508-516.

[7] Ashurov R., Fayziev Y.; Determination of Fractional Order and Source Term in Subdiffusion Equations. Eurasian Mathematical Journal,–2022.– 13.–No1.–P.19-31.

[8] Ashurov R., Saparbayev R.; Forward and Inverse Problems for Fractional Telegraph Equation. Lobachevskii J. Math.,–2024.– 45.–P. 4459-4478.

[9] Ashurov R., Saparbayev R.; Time-dependent identification problem for a fractional Telegraph equation with the Caputo derivative. Fract. Calc. Appl. Anal.,–2024–27.– P. 652-676.

[10] Ashurov R., Saparbayev R.; Fractional Telegraph Equation with the Caputo Derivative. Fractal Fract.,–2023.–7.–No6.

[11] Burov S, Barkai E; Critical exponent of the fractional Langevin equation. Physics Rev. Lett.,–2008.–100.–P.601–610.

[12] Coffey W., Kalmykov Y., Waldron J.; The Langevin Equation. World Scientific,–2004.

[13] Dzherbashian M. [=Djrbashian]; Integral Transforms and Representation of Functions in the Complex Domain. Moscow: NAUKA, (in Russian)–1966.

[14] Fa K.; Fractional Langevin equation and RiemannLiouville fractional derivative. Eur. Phys. Jour. Eng.,–2007.–24.–P.139-143.

[15] Gorenflo R., Kilbas A., Mainardi F., Rogozin S.; Mittag-Leffler Functions, Related Topics and Applications. Springer, Berlin-Heidelberg, Germany.–2014.

[16] Kilbas A., Srivastava H., Trujillo J.; Theory and Applications of Fractional Differential Equations. Elsevier.Amsterdam,–2006.

[17] Kolmogorov A., Fomin, S.; Introductory real analysis. Courier Corporation,–1975.

[18] Langevin P.; Sur la theorie du mouvement brownien. C. R. Acad.Sci.,–1908.–146.–P.530-533.

[19] Lim S.,Li M.,Teo L.; Langevin equation with two fractional orders. Physics Letters A.–2008.–372.–P.63096320.

[20] Lis V., Tthov J., Glod L.; On the correlation properties of thermal noise in fluids. International Journal of Thermophysics,–2013.–34.–P. 629–641.

[21] Lizama C.; Abstract linear fractional evolution equations. Handbook of fractional calculus with applications,– 2019.–2.–P.465–98.

[22] Lutz E.; Fractional Langevin equation.Phys. Rev. E.,–2001.

[23] Mainardi F., Pironi P.; The fractional Langevin equation: Brownian motion revisited. Extracta Math.,–1996.–10.–P.140-154.

[24] Panja D.; Generalized Langevin equation formulation for anomalous polymer dynamics. Journal of Statistical Mechanics: Theory and Experiment,–2010.

[25] Schienbein M., Gruler H.; Langevin equation, Fokker-Planck equation and cell migration. Bltn Mathcal Biology,–1993.–55.–P. 585-608.

[26] Schluttig J., Alamanova D., Helms V., Schwarz S.; Dynamics of protein-protein encounter: A Langevin equation approach with reaction patches. The Journal of Chemical Physics,–2008.– 129.–No15.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.119-131
DOI: 10.29229/uzmj.2025-2-11

On the sharp estimates for convolution operators related to some non-convex hypersurfaces

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract: In this article, we study the convolution operators M_k with oscillatory kernel, which are related to solutions to the Cauchy problem for the strictly hyperbolic equations. The operator M_k is associated to the characteristic hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$ of a strictly hyperbolic equation and smooth amplitude function, which is homogeneous of order $-k$ for large values of the argument. We study the convolution operators assuming that the corresponding amplitude function is contained in a sufficiently small conic neighborhood of a given point $v \in \Sigma$ at which at least $n-1$ principal curvatures do not vanish. Such hyper-surfaces exhibit singularities of type A in the sense of Arnol'd's classification. Denoting by k_p the minimal number such that M_k is $L^p \mapsto L^{p'}$-bounded for any $k > k_p$, we show that the number k_p depends on some discrete characteristics of the hyper-surface Σ.

Keywords: Convolution operator, hypersurface, oscillatory integral, singularities
MSC (2020): 42B10, 42B20, 42B37

1. Introduction

It is well known that solution operator of the Cauchy problem for homogeneous constant coefficient strictly hyperbolic equation, up to a regularizing operator, can be written as a sum of convolution operators of the type:

$$\mathcal{M}_k = F^{-1}[e^{it\varphi(\xi)}a_k]F, \tag{1.1}$$

where F is the Fourier transform operator , $\varphi \in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ is a smooth function that is homogeneous of order one, $a_k \in C^\infty(\mathbb{R}^{n+1}_\xi)$ is a homogeneous function of order $-k$ for large ξ.

After the scaling arguments for time $t > 0$ the operator $\mathcal{M}_k$ is reduced to the following convolution operator (see [12]):

$$M_k = F^{-1}[e^{i\varphi(\xi)}a_k]F. \tag{1.2}$$

Since the PsDO (Pseudo-differential operators) with a symbol from the space $\mathcal{S}^0(\mathbb{R}^{n+1})$ (see [6] page. no. 94) is bounded on the space $L^p(\mathbb{R}^{n+1})$ (for $1 < p < \infty$), then the boundedness problem for the operator M_k mapping from $L^p(\mathbb{R}^{n+1})$ into $L^{p'}(\mathbb{R}^{n+1})$ (for $1 < p \leq 2$) with a smooth amplitude function a_k that is homogeneous of order $-k$ for large values of ξ, and the analogous problem for $a_k \in \mathcal{S}^{-k}(\mathbb{R}^{n+1})$, are equivalent. Here and throughout, we will use the notation $\mathcal{S}^\nu(\mathbb{R}^{n+1})$ to represent the class of classical symbols of PsDO with order $\nu \in \mathbb{R}$.

Let $1 \leq p \leq 2$ be a fixed number: We consider the problem: *find the minimal number $k(p)$ such that the operator $M_k : L^p(\mathbb{R}^{n+1}) \to L^{p'}(\mathbb{R}^{n+1})$ is bounded for any $k > k(p)$.*

Analogical problems have been considered by many authors including Strichartz [13], in the case when the characteristic hypersurface is the unit sphere, by Brenner [4], in the case when the characteristic hypersurface has non-vanishing Gaussian curvature. These results were extended by M. Sugimoto [10], [11], [12], in the case when the characteristic hypersurface is convex (but, not necessarily strictly convex) and also for some non-convex hypersurfaces [12] (see also [9] and [7]).

Nevertheless, the problem remains in general wide open. Also, the issue is related to many other open problems of harmonic analysis related to oscillatory integrals.

In this paper, we consider the hyper-surface in $\mathbb{R}^{n+1}$ having at least $n-1$ principal curvatures at any point. Then the corresponding phase function has singularities of type A. We obtain an upper bound for the critical exponent $k_p(v)$. The upper bound is sharp for wide class of hypersurfaces. We obtain the sharp value of $k_p(v)$ for the model case $n=3$. The main Theorem of the paper extend the obtained results of the papers [12] and [7].

The paper organized as follows, in the next section 2 we give the main results of the paper. Then we discuss the decay rate of oscillatory integrals and obtain upper estimates for the number $k_p(v)$ in the section 3. Then we obtain a lower bound for the number $k_p(v)$, in the section 3.2, which agree with the upper bounds. The results of the last section 4 are related to Model case of the convolution operators, which show that conditions of the main Theorem are essential.

Conventions: Throughout this article, we shall use the variable constant notation, i.e., many constants appearing in the course of our arguments, often denoted by $c, C, \varepsilon, \delta$; will typically have different values at different lines. Moreover, we shall use symbols such as $\backsim$, $\lesssim$ or $<<$ in order to avoid writing down constants. By χ_c we shall denote a non-negative smooth cut-off function on $\mathbb{R}^\nu (\nu \geq 1)$ with typically small compact support which is identically 1 on a small neighborhood of the point c.

2. The main results

Note that the problem becomes more complicated when the function φ vanishes at some non-zero point ξ_0. Next, we assume that $\varphi(\xi) \neq 0$ for any $\xi \neq 0$. Then, without loss of generality, we can assume that $\varphi(\xi) > 0$ for all $\xi \neq 0$. Since φ is a smooth homogeneous function of order one, then, due to the Euler's homogeneity relation we have:

$$\sum_{j=1}^{n+1} \xi_j \frac{\partial \varphi(\xi)}{\partial \xi_j} = \varphi(\xi),$$

and hence the set Σ defined by the following

$$\Sigma = \{\xi \in \mathbb{R}^{n+1} : \varphi(\xi) = 1\}$$

is a smooth or a real analytic hyper-surface provided φ is a smooth or a real analytic function on $\mathbb{R}^{n+1} \setminus \{0\}$ respectively. The hypersurface Σ is said to be a smooth (analytic) hyper-surface if it can be locally represented as the graph of a smooth (real analytic) function. The smoothness of the hyper-surface Σ follows from the classical implicit function Theorem.

Further, we use notation:

$$k_p := k_p(\Sigma) := \inf_{k>0} \{k > 0 : M_k \, \text{is} \, L^p(\mathbb{R}^{n+1}) \to L^{p'}(\mathbb{R}^{n+1}) \, \text{bounded for any} \, a_k \in \mathcal{S}^{-k}(\mathbb{R}^{n+1})\}. \tag{2.1}$$

where $\mathcal{S}^{-k}(\mathbb{R}^{n+1})$ represents the set of classical symbols of PsDO with order $-k$. It turns out that the number $k_p(\Sigma)$ depends on geometric properties of the hypersurface Σ. More precisely, the number depends on behavior of the Fourier transform of measures supported on Σ.

Since $\Sigma \subset \mathbb{R}^{n+1} \setminus \{0\}$ is a compact hypersurface, then following Sugimoto [12], it is enough to consider the local version of the problem. More precisely, we may assume that the amplitude function a_k is concentrated in a sufficiently small conic neighborhood Γ of a fixed point $v \in \Sigma$ and $\varphi \in C^\infty(\Gamma)$. The smallness of the conic (with the vertex at the origin) neighborhood Γ is determined by the smallness of $S^n \cap \Gamma$ (where S^n is the unit sphere in $\mathbb{R}^{n+1}$ centered at the origin). Fixing such a point $v \in \Sigma$, let us define the following local exponent $k_p(v)$ associated to this point:

$$k_p(v) := \inf_{k>0}\{k : \exists\Gamma,\ M_k : L^p(\mathbb{R}^{n+1}) \mapsto L^{p'}(\mathbb{R}^{n+1}) \text{ is bounded, whenever } supp(a_k) \subset \mathcal{S}_0^{-k}(\Gamma)\},$$

where $\mathcal{S}_0^{-k}(\Gamma)$ represents the set of classical symbols of PsDO with order $-k$ whose support is in Γ.

The definition of $k_p(v)$ implies that it is an upper semi-continuous function of v, for a fixed value of $p \in [1, 2]$.

Additionally, for the purpose of clarity, we assume that $v = (0, \dots, 0, 1)$ and $\varphi(v) = 1$. Actually, any point of Σ, by using rotation and scaling, can be reduced to that point v. After possible a linear transform in the space $\mathbb{R}^{n+1}_\xi$, which preserves the point v, we may assume that $\partial_j\varphi(v) = 0(j = 1, \dots, n)$. Thus, in a neighborhood of the point v the hypersurface Σ is given as the graph of a smooth function:

$$\Sigma \cap \Gamma = \{(\xi_1, \dots, \xi_n, 1 + \phi(\xi_1, \dots, \xi_n)) \in \mathbb{R}^{n+1} : (\xi_1, \dots, \xi_n) \in U\},$$

where $U \subset \mathbb{R}^n$ is a sufficiently small neighborhood of the origin and, $\phi \in C^\infty(U)$ is a smooth function satisfying the conditions: $\phi(0) = 0, \nabla\phi(0) = 0$.

2.1. **On a normal form of the phase function with respect linear transforms.** Before proceeding to the discussion of the results of our work, we will introduce the necessary definitions and notation. In this paper, we consider a class of smooth hypersurfaces $S \subset \mathbb{R}^{n+1}$ that have at least $n-1$ non-zero principal curvatures at a given point $X \in S$. Also, we consider a surface-carried measure concentrated in a sufficiently small neighborhood of that point.

Furthermore, let us suppose that in a sufficiently small neighbourhood of a fixed point, say the origin, the hypersurface S is given as the graph of a smooth function $x_{n+1} = \phi(x_1, ..., x_n)$, satisfying the conditions $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Then, the characteristic hypersurface, Σ, is defined by translating the surface, S, by the vector, v.

Suppose that exactly $n-1$ of the principal curvatures of the hypersurface S are non-zero at the origin. Then, the Hessian matrix $Hess\phi(x) := \{\partial_l\partial_j\phi(x)\}_{l,j=1}^n$ has rank $n-1$ at the origin of $\mathbb{R}^n$. Following [8], we will verify that there exists an orthogonal matrix A such that the Taylor expansion of the function $\phi_1(x) := \phi(Ax)$ has the form

$$\phi_1(x) = \frac{1}{2}\sum_{l,j=1}^{n-1} b_{lj}x_l x_j + R(x), \tag{2.2}$$

where $B = \{b_{ij}\}_{i,j=1}^{n-1}$ is a non-singular symmetric matrix, the eigenvalues of which match the non-zero eigenvalues of the Hessian matrix of ϕ at $x = 0$ and $R(x)$ is a remainder of order at least three at the origin. Further, without loss of generality, we assume that the Taylor expansion of ϕ is of the form (2.2).

Then the matrix $Hess\phi(0)$ has exactly $n-1$ non-zero eigenvalues. Moreover, in view of (2.2), we have $det\{\partial_l\partial_j\phi(0)\}_{l,j=1}^{n-1} \neq 0$. Consider the system of equations

$$\partial_l\phi(x) = 0, \quad l = 1, \dots, n-1. \tag{2.3}$$

Applying the implicit mapping theorem, we find that the system of equations (2.3) in a sufficiently small neighborhood of zero has a smooth solution $x_l = \psi_l(x_n)$ (where $l = 1, \dots, n-1$) satisfying the condition $\psi_l(0) = \psi_l'(0) = 0, l = 1, \dots, n-1$. Also, using the linear transformation of the sub- space $\mathbb{R}^n$, we can reduce the functions $\{\psi_l(x_n)\}_{l=1}^{n-1}$ to the form

$\psi_l(x_n) = x_n^{m_l}\omega_l(x_n), l = 1, \dots, n-1$, where $\omega_l(x_n)$ is a smooth function. Further, if ψ_l is not a flat function at the origin, then we can assume, $\omega_l(0) \neq 0$ and $m_l, l \in \{1, \dots, n-1\}$ is a finite natural number and also after a possible linear transformation, we may suppose $2 \leq m_1 < m_2 < \cdots < m_{n-1} \leq \infty$. If, for some l, the corresponding ψ_l is a flat function (i.e., it vanishes together with all its derivatives at the point $x_n = 0$), then we can formally put $m_l = \infty$.

The following statement was proved in the paper [1].

Lemma 2.1. *Let S be a smooth hypersurface in a neighborhood U of the origin in $\mathbb{R}^{n+1}$, given as the graph of a smooth function. More precisely, let*

$$S \cap U = \{(x_1, \dots, x_n, x_{n+1}) \in U : x_{n+1} = \phi(x_1, \dots, x_n), \phi(0) = 0, \nabla\phi(0) = 0\}. \tag{2.4}$$

If S has at least $n-1$ non-vanishing principal curvatures at the origin, then, possibly after a linear transformation of the space $\mathbb{R}^n$, in a sufficiently small neighborhood of zero, the function ϕ can be written in the form

$$\phi(x) = \frac{1}{2}\sum_{l,j=1}^{n-1} b_{lj}(x)(x_l - \psi_l(x_n))(x_j - \psi_j(x_n)) + b_n(x_n), \tag{2.5}$$

where $\{b_{lj}(x)\}_{l,j=1}^{n-1}$ is a symmetric matrix with smooth elements and $b_{lj}(0) = b_{lj}(l, j = 1, \dots, n-1)$; the $\{\psi_l(x_n)\}_{l=1}^{n-1}$ are smooth functions of the form $\psi_l(x_n) = x_n^{m_l}\omega(x_n)$ $l = 1, \dots, n-1$, for a smooth function ω_l satisfying the condition $\omega_l(0) \neq 0$ whenever ψ_l is not a flat function, and $2 \leq m_1 < m_2 < \cdots < m_{n-1} \leq \infty$. In (2.5), b_n is a smooth function defined by the relation $b_n(x_n) := \phi(\psi_1(x_n), \dots, \psi_{n-1}(x_n), x_n)$.

The main result of the paper is the following Theorem.

Theorem 2.2. *If ϕ has a singularity of type A_{N-1} at the origin, then the following inequality holds:*

$$k_p(v) \leq 2\left(\frac{n+3}{2} - \frac{1}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right), \tag{2.6}$$

In addition, if $2m_1 \geq N$, then the following relationship holds:

$$k_p(v) = 2\left(\frac{n+3}{2} - \frac{1}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right). \tag{2.7}$$

Corollary 2.3. *If $2\min\{m_1, \dots, m_{n-1}\} \geq N$, the following equality holds:*

$$k_p(v) = \left(n + 3 - \frac{2}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right). \tag{2.8}$$

As shown in the paper [7], the condition of the last corollary is both necessary and sufficient for relation (2.8) to hold in the case of $n = 2$.

3. Related oscillatory integrals

Let $S \subset \mathbb{R}^{n+1}$ be a smooth hypersurface, and let $\varphi \in C_0^\infty(S)$ be a smooth function with compact support. Consider the charge $d\mu(X) := \varphi(X)dS$, where dS is the induced Lebesgue measure on the hypersurface S. In particular, if φ is a nonnegative function, then we are dealing with a Borel measure. The Fourier transform of the charge $d\mu$ is defined by the following integral:

$$\widehat{d\mu}(\xi) := \int_S e^{ix\xi}\varphi(x)dS(x), \tag{3.1}$$

where $x\xi$ is the inner product of the vectors x and ξ. It is well known that behavior of the function $\widehat{d\mu}$ when $|\xi|$ gets large depends on geometric properties of S.

Further, we also define analogical functions associated to oscillatory integrals.

Let S be a smooth hypersurface, given as the graph of a function ϕ, with an A_{N-1} singularity at the origin. It means that the function b_n, defined by the equation (2.5), has a multiple root of order $N \geq 2$ at the origin. Using the method of stationary phase, we reduce the main part of the Fourier transform of the charge $d\mu$ to a one-dimensional oscillatory integral with multiplier $|\xi|^{-(n-1)/2}$ as $|\xi| \to +\infty$. Indeed, the Fourier transform of the charge $\widehat{d\mu}$ can be written as a multiple oscillatory integral. Since S is a smooth hypersurface, it can be represented as the graph of a smooth function $\phi(x)$ that satisfies the conditions $\phi(0) = 0$ and $\nabla\phi(0) = 0$. We also assume that ϕ has a singularity of type A_{N-1} at the origin. As a consequence, the surface integral reduces to the following multiple integral:

$$J(\lambda, s) := \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,s)}a(x)dx.$$

Here $\xi_{n+1} = \lambda$, $s_j := \xi_j/\lambda$, $j = 1, \dots, n-1$, $\Phi(x, s) = \phi(x) + s \cdot x$, where ϕ is a smooth function with $\phi(0) = 0$, $\nabla\phi(0) = 0$, and $rank(Hess\phi)(0)) \geq n-1$, which is equivalent to the condition Φ has a singularity of type A at the origin.

If $|s| > \varepsilon$ (where ε is a fixed positive number) and b is a smooth function with sufficiently small support contained in a neighborhood of the origin, then, for any natural number ν, integrating by parts, we obtain $J(s, \lambda) = O(|s\lambda|^{-\nu})$ as $|\lambda| \to +\infty$. Therefore, it suffices to consider the case of a vector s of sufficiently small length.

Since $det\{\partial_j\partial_k\phi(0,0)\}_{j,k=1}^{n-1} \neq 0$, we can use the Morse classical lemma (see [8]): there exists a neighborhood $V \times U$ and a diffeomorphic mapping $F : V \times U \mapsto V \times U$ of the form $x_n = y_n$, $x_j = F_j(s', y), j = 1, \dots, n-1$, such that, for the phase function $\Phi(x, s)$, the following relation holds:

$$\Phi(F(s', y), y_n, s) := \Phi_1(s', y_n) + s_n x_n + Q(y'),$$

where $\Phi_1(s', y_n)$, is a smooth function; moreover, $\Phi_1(0, y_n)$ has a singularity of type A_{N-1} at the point $y_n = 0$ and $Q(y') := \frac{1}{2}(By', y')$ is a non-degenerate quadratic form given by an invertible symmetric matrix B.

Thus, after the change of variables given by the mapping $(F(s', y), y_n)$, the oscillatory integral $J(\lambda, s)$ can be written as the repeated integral

$$J\lambda, s) = \int_{\mathbb{R}} e^{i\lambda\Phi(s',y_n)} \left(\int_{\mathbb{R}^{n-1}} e^{i\lambda Q(y')} b(y, s')dy_1 \dots dy_{n-1} \right) dy_n$$

with a smooth amplitude function $b_1 \in C_0^\infty(V \times U)$.

Then, by the classical method of stationary phase, we obtain the asymptotic relation (see [8]).

$$\int_{\mathbb{R}^{n-1}} e^{i\lambda Q(y')} b(y, s')dy_1 \dots dy_{n-1} = cb_1(0, \dots, 0, y_n, s')\lambda^{\frac{1-n}{2}} + O(\lambda^{-\frac{1+n}{2}}) \quad \text{as } |\lambda| \to +\infty.$$

where

$$c = \frac{\sqrt{(2\pi)^{n-1}} e^{isign(Q)(\pi/4)}}{\sqrt{|\det B|}},$$

where $sign(Q)$ represents the difference between the number of positive and negative eigenvalues. Hence the oscillatory integral $J(\lambda, s)$ can be written as

$$J(\lambda, s) = c\lambda^{\frac{1-n}{2}} \int_{\mathbb{R}} e^{i\lambda\Phi(s', y_n, s_n)} b_1(0, \dots, 0, y_n, s') dy_n + O(\lambda^{-\frac{1+n}{2}}) \quad \text{as } |\lambda| \to +\infty. \tag{3.2}$$

Therefore, the problem of the behavior of the oscillatory integral $J(\lambda, s)$ reduces to the problem of estimating the following one-dimensional oscillatory integral:

$$J_1(\lambda, s) = \int_{\mathbb{R}} e^{i\lambda\Phi(s', y_n, s_n)} b_1(0, \dots, 0, y_n, s') dy_n. \tag{3.3}$$

We can use Van der Corput type estimate [2] to the integral J_1 and obtain:

$$|J(\lambda, s)| \lesssim \frac{\|b(\cdot, s)\|_{C^n(U)}}{|\lambda|^{\frac{n-1}{2}+\frac{1}{N}}}. \tag{3.4}$$

3.1. **An upper estimate for $k_p(v)$.** Then we can use Proposition 2 of the paper [12] (page no. 386) and obtain the following upper bound for $k_p(v)$:

$$k_p(v) \leq 2\left(\frac{n+3}{2} - \frac{1}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right).$$

The last inequality proves the first part of Theorem 2.2. It should be noted that the last upper estimate is not dependent on the numbers $m_1, \dots, m_{n-1}$. Additionally, we demonstrate that the obtained estimate is sharp for sufficiently large values of these numbers (see Theorem 3.1).

3.2. **A lower estimate for $k_p(v)$.**

Theorem 3.1. *Let ϕ be a smooth function satisfying the condition $2m_1 \geq N$. Then the following lower estimate holds true:*

$$k_p(v) \geq \left(n + 3 - \frac{2}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right).$$

We remark that the lower bound agrees with the upper bound (2.6). So, we came to a proof of Corollary 2.3.

In this section we reduce a proof of the Theorem 3.1. The test functions, used in the course of the proof, are similar to Knapp type sequence .

Proof. Let ϕ be the phase function and the principal part is a weighted homogeneous polynomial with weight $\kappa := (\kappa', \kappa_n)$, where $\kappa' := (\frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^{n-1}$ and $\kappa_n := \frac{1}{N}$.

We can write the Taylor expansion:

$$\phi(x) = \sum_{(\kappa,\alpha)=1} c_\alpha x^\alpha + R(x), \quad \text{with} \quad c_\alpha := \frac{\partial^\alpha \phi(0)}{\alpha!},$$

under condition $2m_1 \geq N$, where (κ, α) is an inner product of vectors κ and $\alpha \in \mathbb{Z}_+^n$ and also R is the remainder term satisfying the condition:
$\|t^{-1}R(t^{\frac{1}{2}}x_1, \dots, t^{\frac{1}{2}}x_{n-1}, t^{\frac{1}{N}}x_n)\|_{C^M(U)} \to 0$ as $t \to +0$ for any natural number M.

Let us take a smooth function in $\mathbb{R}^{n+1}$ such that $a_k(\xi) = |\xi|^{-k}$ for large ξ. For example, we can take $a_k(\xi) = (1 - \chi_0(\xi))|\xi|^{-k}$, where χ_0 is a smooth function such that $\chi_0(\xi) \equiv 1$ in a neighborhood of the origin say for $|\xi| \le \varepsilon$ and $\chi_0(\xi) \equiv 0$ for $|\xi| \ge 2\varepsilon$ with a sufficiently small fixed positive number $\varepsilon > 0$.

Following, M. Sugimoto we introduce the function: $G(y) = 1 + \phi(y) - (y, \nabla\phi(y))$. Define a non-negative smooth function with $\chi_0(0) = 1$ concentrated in a sufficiently small neighborhood of the origin of $\mathbb{R}^{n-1}$, and a non-negative smooth function, satisfying $\chi_1(1) = 1$, with support in a sufficiently small neighborhood of the point 1 and $\chi_1 \equiv 0$ in a neighborhood of the origin of $\mathbb{R}$.

We set

$$u_j(x) = 2^{j(n+1-|\kappa|)\left(-\frac{1}{p'}\right)} F^{-1}(v_j(2^{-j}\cdot))(x),$$

where

$$v_j(\xi) = \frac{\chi_0\left(2^{\frac{j}{2}}\frac{\xi'}{\varphi(\xi)}\right)\chi_0\left(2^{\frac{j}{N}}\frac{\xi_n}{\varphi(\xi)}\right)\chi_1(\varphi(\xi))|\xi|^k}{\varphi(\xi)^n G\left(\frac{\xi'}{\varphi(\xi)}\right)} \in C_0^\infty(\mathbb{R}^{n+1}).$$

Note that $supp(v_j)$ does not contain the origin, because $\chi_1(\varphi(\xi)) \equiv 0$ in a neighborhood of the origin.

The sequence $\{u_j\}_{j=1}^\infty$ is bounded in $L^p(\mathbb{R}^{n+1})$. Indeed, we have:

$$u_j(x) = \frac{2^{\frac{(n+1)j}{p}+\frac{|\kappa|j}{p'}}}{\sqrt{(2\pi)^{n+1}}}\int_{\mathbb{R}^{n+1}} e^{-i2^j(\xi,x)}v_j(\xi)d\xi.$$

On the other hand following M. Sugimoto we use change of variables $\xi = (\lambda y, \lambda(1+\phi(y)))$ (where $y \in U \subset \mathbb{R}^n$) and get:

$$u_j(x) = \frac{2^{\frac{(n+1)j}{p}+\frac{|\kappa|j}{p'}}}{\sqrt{(2\pi)^{n+1}}}\int_{\mathbb{R}^{n+1}} e^{-i2^j\lambda(x'y+x_{n+1}(1+\phi(y)))}\chi_0(2^{\frac{j}{2}}y')\chi_0(2^{\frac{j}{k}}y_n)$$
$$\chi_1(\lambda)\lambda^k(|y|^2+(1+\phi(y)^2)^{\frac{k}{2}}d\lambda dy.$$

Finally, we use scaling $2^{\frac{j}{2}}y' \mapsto y'$, $2^{\frac{j}{N}}y_n \mapsto y_n$ in variables y and obtain:

$$u_j(x) = \frac{2^{\frac{(n+1)j}{p}-\frac{|\kappa|j}{p}}}{\sqrt{(2\pi)^{n+1}}}\int e^{-i2^j\lambda(2^{-\frac{j}{2}}x'\cdot y'+2^{-\frac{j}{N}}x_ny_n+x_{n+1}(1+\phi(\delta_{2^{-j}}(y))))}\chi_0(y)$$
$$\chi_1(\lambda)\lambda^k(2^{-j}|y'|^2+2^{-\frac{2j}{N}}y_n^2+(1+\phi(\delta_{2^{-j}}(y))^2)^{\frac{k}{2}}d\lambda dy.$$

Note that $|2^j\partial^\alpha\phi(\delta_{2^{-j}}(y))| << 1$ as $j >> 1$ for any $\alpha \in \mathbb{Z}_+^n$ provided the support of χ_0 are small enough. If $|x_{n+1}| > |x'2^{-\frac{j}{2}}| + |x_n2^{-\frac{j}{N}}|$ then we can use integration by parts formula in λ and get:

$$|u_j(x)| \lesssim_M \frac{2^{\frac{(n+1)j}{p}-\frac{|\kappa|j}{p}}}{(1+|x_{n+1}2^j|)^M},$$

for any natural $M \ge 1$, provided $|x_{n+1}2^j| >> 1$, otherwise e.g. if $|x_{n+1}2^j| \lesssim 1$, then the last estimate trivially holds, for the function $u_j(x)$.

Assume $|x_{n+1}| \le |x'2^{-\frac{j}{2}}|+|x_n2^{-\frac{j}{N}}|$. Then by using integration by parts formula in y variables, we get the following estimate:

$$|u_j(x)| \lesssim_M \frac{2^{\frac{(n+1)j}{p}-\frac{|\kappa|j}{p}}}{(1+|x'2^{\frac{j}{2}}|+|x_n2^{\frac{(N-1)j}{N}}|)^M},$$

Finally, combining the obtained estimates we obtain:

$$|u_j(x)| \lesssim_M \frac{2^{\frac{(n+1)j}{p}-\frac{|\kappa|j}{p}}}{(1+|2^j x_{n+1}|+|x' 2^{\frac{j}{2}}|+|x_n 2^{\frac{(N-1)j}{N}}|)^M}.$$

Consequently,

$$\|u_j\|_{L^p(\mathbb{R}^{n+1})} \lesssim 1, \quad \text{for} \quad j \geq 1.$$

So, the sequence $\{u_j\}_{j=1}^{\infty}$ is bounded in the space $L^p(\mathbb{R}^{n+1})$.

On the other hand we have the relation:

$$M_k u_j(x) = 2^{j(n+1-|\kappa|)(-\frac{1}{p'})-kj+nj} F^{-1}\left(e^{i\varphi(\xi)} \frac{\chi_0\left(2^{\frac{j}{2}}\frac{\xi'}{\varphi(\xi)}\right)\chi_0\left(2^{\frac{j}{N}}\frac{\xi_n}{\varphi(\xi)}\right)\chi_1(2^{-j}\varphi(\xi))}{\varphi(\xi)^n G\left(\frac{\xi'}{\varphi(\xi)}\right)}\right)(x).$$

We perform change of variables given by the scaling $2^{-j}\xi \mapsto \xi$ and obtain:

$$M_k u_j(x) = \frac{2^{j((n+1-|\kappa|)(-\frac{1}{p'})-k+n+1)}}{\sqrt{(2\pi)^{n+1}}} \int_{\mathbb{R}^{n+1}} e^{i2^j(\varphi(\xi)-x\xi)} \frac{\chi_0\left(\frac{2^{\frac{j}{2}}\xi'}{\varphi(\xi)}\right)\chi_0\left(\frac{2^{\frac{j}{N}}\xi_n}{\varphi(\xi)}\right)\chi_1(\varphi(\xi))}{\varphi^n(\xi) G\left(\frac{\xi}{\varphi(\xi)}\right)} d\xi.$$

Then following M. Sugimoto we use change of variables $\xi = (\lambda y, \lambda(1+\phi(y)))$ and gain the relation:

$$M_k u_j(x) = \frac{2^{j((n+1-|\kappa|)(-\frac{1}{p'})-k+n+1)}}{\sqrt{(2\pi)^{n+1}}} \int e^{i2^j\lambda(1-(x'y'+x_n y_n+x_{n+1}(1+\phi(y))))}$$
$$\chi_0(2^{\frac{j}{2}}y')\chi_0(2^{\frac{j}{N}}y_n)\chi_1(\lambda)d\lambda dy.$$

Finally, we use change of variables $2^{\frac{j}{2}}y' \mapsto y'$, $2^{\frac{j}{N}}y_n \mapsto y_n$ and obtain:

$$M_k u_j(x) = 2^{j((n+1-|\kappa|)(-\frac{1}{p'})-k-|\kappa|+n+1)} \int_{\mathbb{R}^{n+1}} e^{2^j i\lambda((x_{n+1}-1)-2^{-\frac{j}{2}}y'x'-2^{-\frac{j}{N}}y_n x_n - x_{n+1}\phi(\delta_{2^{-j}}(y)))}$$
$$\chi_0(y)\chi_1(\lambda)d\lambda dy.$$

If $|x_{n+1}-1| \ll 2^{-j}$, $|x'| \ll 2^{-\frac{j}{2}}$, $|x_n| \ll 2^{-\frac{j(N-1)}{N}}$, then the phase is the non-oscillating function, because $\lambda \backsim 1$ and

$$(x_{n+1}-1)-2^{-\frac{j}{2}}y'x'-2^{-\frac{j}{N}}y_n x_n - x_{n+1}\phi(\delta_{2^{-j}}(y)) = o(2^{-j})$$

provided the supports of χ_0 is small enough.

Consequently, we have the following lower bound:

$$\|M_k u_j\|_{L^{p'}} \gtrsim 2^{j\left(2(n+1-|\kappa|)\left(\frac{1}{p}-\frac{1}{2}\right)-k\right)}.$$

The last inequality is equivalent to

$$\|M_k u_j\|_{L^{p'}} \gtrsim 2^{j\left(\left(n+3-\frac{2}{N}\right)\left(\frac{1}{p}-\frac{1}{2}\right)-k\right)}.$$

Therefore, if

$$k < k_p(v) := \left(n+3-\frac{2}{N}\right)\left(\frac{1}{p}-\frac{1}{2}\right),$$

then

$$\|M_k u_j\|_{L^{p'}} \to \infty \quad (\text{as} \quad j \to +\infty).$$

Thus, the operator $M_k : L^p(\mathbb{R}^{n+1}) \to L^{p'}(\mathbb{R}^{n+1})$ is unbounded provided $k < k_p(v)$. Thus we get a lower bound for the number $k_p(v)$ which completes a proof of Theorem 3.1. The Theorem 3.1 completes a proof of the main Theorem 2.2.

□

Now, we consider a lower estimate for $k_p(v)$ for the case when $m_1, \dots, m_{n-1}$ are smaller than N.

Theorem 3.2. *If $2m_1 < 2m_2 < \cdots < 2m_{n-1} < N$, then the following estimate holds:*

$$k_p(v) \geq 2\left(n+1-\frac{\sum_{l=1}^{n-1} m_l + 1}{N}\right)\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{n-1}{2} + \frac{\sum_{l=1}^{n-1} m_l}{N}. \tag{3.5}$$

Proof of Theorem 3.2.

We slightly modified the Sugimoto [12] arguments and consider the sequence

$$u_j = 2^{-\frac{(n+1)j}{p'} + j\frac{\sum_{l=1}^{n-1} m_l + 1}{Np'}} F^{-1}(v_j(2^{-j}\cdot))(x),$$

where

$$v_j(\xi) = \chi_0\left(\frac{2^{\frac{j}{N}}\xi_n}{\varphi(\xi)}\right)\chi_0\left(2^{\frac{jm_1}{N}}\left(\frac{\xi_1}{\varphi(\xi)} - \left(\frac{\xi_n}{\varphi(\xi)}\right)^{m_1}\omega_1\left(\frac{\xi_n}{\varphi(\xi)}\right)\right)\right)\dots$$
$$\chi_0\left(2^{\frac{jm_{n-1}}{N}}\left(\frac{\xi_{n-1}}{\varphi(\xi)} - \left(\frac{\xi_n}{\varphi(\xi)}\right)^{m_{n-1}}\omega_{n-1}\left(\frac{\xi_n}{\varphi(\xi)}\right)\right)\right)\frac{\chi_1(\varphi(\xi))|\xi|^k}{\varphi^n(\xi)G\left(\frac{\xi'}{\varphi(\xi)}\right)},$$

where $\chi_0, \chi_1 \in C_0^\infty(\mathbb{R})$ are non-negative smooth functions satisfying the conditions: $\chi_0(0) = 1$ and support of function χ_0 lies in a sufficiently small neighborhood of the origin of $\mathbb{R}$ and χ_1 is a non-negative smooth function concentrated in a sufficiently small neighborhood of 1 and identically vanishes in a neighborhood of the origin and also $\chi_1(1) = 1$ (cf. [12]). We claim that that, for large j one has

$$\|u_j\|_{L^p(\mathbb{R}^{n+1})} \lesssim 1.$$

Indeed, by the definition of u_j, we have the following relationship:

$$F^{-1}(v_j(2^{-j}\cdot))(x) = \frac{1}{\sqrt{(2\pi)^{n+1}}}\int_{\mathbb{R}^{n+1}} e^{-i\xi\cdot x}v_j(2^{-j}\xi)d\xi = \frac{2^{(n+1)j}}{\sqrt{(2\pi)^{n+1}}}\int_{\mathbb{R}^{n+1}} e^{-i2^j\xi\cdot x}v_j(\xi)d\xi.$$

We use change of variables $\xi = \lambda(y, 1+\phi(y))$ and obtain:

$$V_j(x) := F^{-1}(v_j(2^{-j}\cdot))(x) = \frac{2^{(n+1)j}}{\sqrt{(2\pi)^{n+1}}}\int_{\mathbb{R}^{n+1}} e^{-i2^j\lambda(y'\cdot x' + y_n x_n + x_{n+1}(1+\phi(y))}$$
$$\chi_0(2^{\frac{j}{N}}y_n)\chi_0(2^{\frac{jm_1}{N}}(y_1 - y_n^{m_1}\omega_1(y_n))\dots\chi_0(2^{\frac{jm_{n-1}}{N}}(y_{n-1} - y_n^{m_{n-1}}\omega_{n-1}(y_n))\chi_1(\lambda)d\lambda dy.$$

Then, we use the change of variables provided by scaling:

$$2^{\frac{jm_1}{N}}y_1 \to y_1, \dots, 2^{\frac{jm_{n-1}}{N}}y_{n-1} \to y_{n-1},\ 2^{\frac{j}{N}}y_n \to y_n$$

and obtain:

$$V_j(x) = \frac{2^{\left(n+1-\frac{\sum_{l=1}^{n-1} m_l+1}{N}\right)j}}{\sqrt{(2\pi)^{n+1}}} \int_{\mathbb{R}^{n+1}} e^{-i2^j\lambda(\sum_{l=1}^{n-1} 2^{-\frac{jm_l}{N}} y_l x_l + 2^{-\frac{j}{N}} y_n x_n + x_{n+1}(1+\phi(\delta_{2^{-j}}(y))))}$$

$$\chi_0(y_n)\chi_0(y_1 - y_n^{m_1}\omega_1(2^{-\frac{1}{N}}y_n)\dots\chi_0(y_{n-1} - y_n^{m_{n-1}}\omega_{n-1}(2^{-\frac{1}{N}}y_n))\chi_1(\lambda)d\lambda dy.$$

Integration by parts arguments yield:

$$|V_j(x)| \lesssim_L \frac{2^{\left(n+1-\frac{\sum_{l=1}^{n-1} m_l+1}{N}\right)j}}{\left(1+\sum_{l=1}^{n-1}|2^{j-\frac{jm_l}{N}}x_l| + |2^{j-\frac{1}{N}}x_n| + |2^j x_{n+1}|\right)^L},$$

where L is any fixed positive integer number. We assume that $Lp > n+1$. The last inequality implies that

$$\|V_j\|_{L^p(\mathbb{R}^{n+1})} \lesssim_L 2^{\left(\frac{n+1}{p'}-\frac{\sum_{l=1}^{n-1} m_l+1}{Np'}\right)j}.$$

Consequently, $\|u_j\|_{L^p(\mathbb{R}^{n+1})} \lesssim 1$.

Now, we consider the lower estimate for $\|M_k u_j\|_{L^{p'}(\mathbb{R}^3)}$.

We have:

$$M_k u_j = F^{-1}e^{i\varphi(\xi)}a_k(\xi)Fu_j = 2^{\left(-\frac{n+1}{p'}+\frac{\sum_{l=1}^{n-1} m_l+1}{Np'}\right)j} F^{-1}(e^{i\varphi(\xi)}a_k(\xi)v_j(2^{-j}\xi))(x).$$

We perform change of variables given by the scaling $2^j\xi \to \xi$ and obtain:

$$M_k u_j(x) = \frac{2^{\left(\frac{n+1}{p}+\frac{\sum_{l=1}^{n-1} m_l+1}{Np'}-k\right)j}}{\sqrt{(2\pi)^{n+1}}} \int_{\mathbb{R}^{n+1}} e^{i2^j(\varphi(\xi)-\xi x)}a_k(\xi)v_j(\xi)d\xi.$$

Finally, we use the change of variables $\xi \to \lambda(y, 1+\phi(y))$ and we have:

$$M_k u_j(x) = \frac{2^{\left(\frac{n+1}{p}+\frac{\sum_{l=1}^{n-1} m_l+1}{Np'}-k\right)j}}{\sqrt{(2\pi)^{n+1}}} \int_{\mathbb{R}^{n+1}} e^{i2^j\lambda(1-y'\cdot x'-y_n x_n - x_{n+1}(1+\phi(y))}\chi_0(2^{\frac{j}{N}}y_n)$$

$$\chi_0(2^{\frac{jm_1}{N}}(y_1 - y_n^{m_1}\omega_1(y_n))\dots\chi_0(2^{\frac{jm_{n-1}}{N}}(y_{n-1} - y_n^{m_{n-1}}\omega_{n-1}(y_n))\chi_1(\lambda)d\lambda dy.$$

Now, we perform the change of variables

$$y_n = 2^{-\frac{j}{N}}z_n,\ y_2 = y_1^{m_1}\omega_1(y_1) + 2^{-j\frac{m_1}{N}}z_2, \dots, y_{n-1} = y_n^{m_{n-1}}\omega_{n-1}(y_1) + 2^{-j\frac{m_{n-1}}{N}}z_{n-1}.$$

Then we get

$$M_k u_j(x) = \frac{2^{\left(\frac{n+1}{p}-\frac{\sum_{l=1}^{n-1} m_l+1}{Np}-k\right)j}}{\sqrt{(2\pi)^{n+1}}} \int_{\mathbb{R}^{n+1}} e^{i2^j\lambda\Phi_3(z,x,j)}\chi_0(z_n)\chi_0(z_1)\dots\chi_0(z_{n-1})\chi_1(\lambda)d\lambda dy.$$

where

$$\Phi_3(z,x,j) := 1 - x_{n+1} - (2^{-\frac{j}{N}}x_n z_n + x_1 2^{-\frac{jm_1}{N}}z_n^{m_1}\omega_1(2^{-\frac{j}{N}}z_n) + \dots +$$

$$x_{n-1}2^{-\frac{jm_{n-1}}{N}}z_n^{m_{n-1}}\omega_{n-1}(2^{-\frac{j}{N}}z_n)) + x_{n+1}\sum_{l,k=1}^{n-1} b_{lk}2^{-\frac{j(m_l+m_k)}{N}}z_l z_k + 2^{-j}z_n^N\beta(2^{-\frac{j}{N}}z_n)).$$

We use the stationary phase method in z' assuming ,

$$|1-x_{n+1}|<<2^{-j},\ |x_n|<<2^{-\frac{N-1}{N}j},\ |x_l|<<2^{-\frac{j(N-m_l)}{N}},l=1,\dots,n-1. \tag{3.6}$$

and, reminding that $2m_1<\dots<2m_{n-1}<N$, to obtain:

$$M_ku_j(x)=\frac{2^{\left(\frac{n+1}{p}-\frac{\sum_{l=1}^{n-1}m_l+1}{Np}-k-\frac{n-1}{2}+\frac{\sum_{l=1}^{n-1}m_l}{N}\right)j}}{\sqrt{(2\pi)^{n+1}}}$$

$$\left(\int_{\mathbb{R}^2}e^{i2^j\lambda\Phi_4}\chi_0(z_n)g(z_2^c(z_n,x))\chi_1(\lambda)d\lambda dz_n+O(2^{j\left(\frac{2m_{n-1}}{N}-1\right)}\right),\text{as}\quad j\to+\infty,$$

where

$$\Phi_4:=\Phi_4(z_1,x,j):=1-x_{n+1}-x_nz_n2^{-\frac{j}{N}}-\sum_{l=1}^{n-1}x_l2^{-\frac{jm_l}{N}}z_n^{m_l}\omega(2^{-\frac{j}{N}}z_n)-$$
$$2^{-j}z_n^N\beta(2^{-\frac{j}{N}}z_n)+\sum_{l,k=1}^{n-1}B_{lk}(z_n,x,2^{-j}),$$

and B is a smooth function satisfying the condition $|B|\backsim 1$. Consequently, accounting the conditions (3.6) and the inequality $2m_{n-1}<N$, we establish the following lower bound:

$$\|M_ku_j\|_{L^{p'}(\mathbb{R}^3)}\geq 2^{j\left(2\left(n+1-\frac{\sum_{l=1}^{n-1}m_l+1}{N}\right)\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{n-1}{2}+\frac{\sum_{l=1}^{n-1}m_l}{N}-k\right)}c,$$

where $c>0$ is a constant which does not depend on j. Thus if

$$k<2\left(n+1-\frac{\sum_{l=1}^{n-1}m_l+1}{N}\right)\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{n-1}{2}+\frac{\sum_{l=1}^{n-1}m_l}{N}$$

then the operator M_k is not $L^p(\mathbb{R}^{n+1})\mapsto L^{p'}(\mathbb{R}^{n+1})$ bounded.

Analogical result holds true for the case $N=\infty$.

Thus, if $k<k_p(v)$ then the M_k is not $L^p-L^{p'}$ bounded operator. This completes a proof of the Theorem 3.2.

4. The model case

Now, we consider the case $n=3$ and function:

$$\phi(y_1,y_2,y_3)=(y_1-y_3^m)^2+y_2^2+y_3^N, \tag{4.1}$$

where $m\geq 2$ and $N\geq 2$.

Proposition 4.1. *Let ϕ be the phase function given by* (4.1).
(i) If $2m\geq N$ then

$$k_p(v)=2\left(3-\frac{1}{N}\right)\left(\frac{1}{p}-\frac{1}{2}\right); \tag{4.2}$$

(ii) If $m\geq 3$ and $2m<N$ then

$$k_p(v)=\max\left\{2\left(3-\frac{1}{2m}\right)\left(\frac{1}{p}-\frac{1}{2}\right),2\left(4-\frac{m+1}{N}\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1+\frac{m}{N}\right\}. \tag{4.3}$$

Proof. The case (i) of Proposition 4.1 follows from Theorem 2.2. The upper bound for the number $k_p(v)$ follows from results of the paper [7]. We show the lower bound for the number $k_p(v)$. Consider the sequence of test functions

$$u_j(x) := 2^{j\left(-\frac{4}{p'}+\frac{m+1}{Np'}\right)} F^{-1}(v_j(2^{-j}\cdot))(x),$$

where $v_j := v_j(\xi)$ is given by

$$v_j = \chi_0\left(\frac{2^{\frac{j}{N}}\xi_3}{\varphi(\xi)}\right)\chi_0\left(2^{\frac{jm}{N}}\left(\frac{\xi_1}{\varphi(\xi)} - \left(\frac{\xi_3}{\varphi(\xi)}\right)^m \omega\left(\frac{\xi_3}{\varphi(\xi)}\right)\right)\right)\chi_0\left(\frac{\xi_2}{\varphi(\xi)}\right)\frac{\chi_1(\varphi(\xi))|\xi|^k}{\varphi^3(\xi)G\left(\frac{\xi'}{\varphi(\xi)}\right)},$$

with $\xi' := (\xi_1, \xi_2, \xi_3)$. It can be proved that u_j is a bounded sequence in the space $L^p(\mathbb{R}^4)$.

We write $M_k u_j(x)$ after change of variables:

$$M_k u_j(x) = 2^{\left(\frac{4}{p}+\frac{m+1}{Np'}-k\right)j} 2^{-\frac{(m+1)j}{N}} \int_{\mathbb{R}^4} e^{i2^j\lambda\Phi_1(x,y)}\chi_0(y_1)\chi_0(y_2)\chi_0(y_3)\chi_1(\lambda)d\lambda dy,$$

where

$$\begin{aligned}\Phi_1(x,y) := 1 - x_4 - (x_1y_12^{-\frac{jm}{N}} + x_1y_3^m\omega(2^{-\frac{1}{N}}y_3)2^{-\frac{jm}{N}} +\\ x_2y_2 + x_3y_32^{-\frac{j}{N}} + x_4(1+\phi(y,2^{-j})),\end{aligned}$$

with

$$\phi(y, 2^{-j}) := 2^{-\frac{2m}{N}}y_1^2 + y_2^2 + 2^{-j}y_3^N.$$

Now, we assume that

$$|1 - x_4| << 2^{-j},\ |x_3| << 2^{\frac{(N-1)j}{N}},\ |x_2| << 2^{-j},\ |x_1| << 2^{\frac{(m-N)j}{N}}. \tag{4.4}$$

Then we use stationary phase method in y_1, y_2 variables and obtain:

$$M_k u_j(x) = 2^{\left(\frac{4}{p}+\frac{m+1}{Np'}-k\right)j} 2^{-\frac{j}{N}-j}c(x,j),$$

where $|c(x,j)| \gtrsim 1$ under the conditions (4.4). Consequently, we have

$$\|M_k u_j\|_{L^{p'}(\mathbb{R}^4)} \gtrsim 2^{j\left(2\left(4-\frac{m+1}{N}\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1+\frac{m}{N}-k\right)}$$

Consequently, if

$$k < 2\left(4 - \frac{m+1}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - 1 + \frac{m}{N}$$

then the operator M_k is not bounded from $L^p(\mathbb{R}^4)$ to $L^{p'}(\mathbb{R}^4)$. □

References

[1] Akramova D. I. and Ikromov I. A., Randol Maximal Functions and the Integrability of the Fourier Transform of Measures, Mathematical Notes,–2021.–109(5).–P.661-678.

[2] Arhipov, G. I., Karacuba, A. A., Čubarikov, V. N., Trigonometric integrals. *Izv. Akad. Nauk SSSR Ser. Mat.*, –1979.–43.–P.971-1003, 1197 (Russian); English translation in *Math. USSR-Izv.*, .,–1980.–15.–P.211-239.

[3] Arnol'd V. I., Gusein-zade S. M., and Varchenko A. N., "Singularities of differentiable mappings," in Classification of Critical Points of Caustics and Wavefronts, Vol. 1, –1985, Birkhäuser, Boston, Basel, Stuttgard.

[4] Brenner P. , On $L^p - L^{p'}$ estimates for the wave-equation, Math. Z.,–1975.–145.–P.251-254.

[5] L. Hörmander The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)]. Classics in Mathematics. Springer-Verlag, Berlin, –2003. x+440 pp. ISBN: 3-540-00662-1 35-02

[6] L. Hörmander The analysis of linear partial differential operators. III. Pseudo-Differential Operators. (1985) edition [Springer, Berlin; MR1065993 (91m:35001a)]. Classics in Mathematics. Springer-Verlag, Berlin, –1985.

[7] Ikromov A. I. and Ikromova D. I., On the Sharp Estimates for Convolution Operators with Oscillatory Kernel, ournal of Fourier Analysis and Applications,–2024.–30(29).–P.28-51.

[8] M. V. Fedoryuk, The Saddle-Point Method (Nauka, Moscow, –1977) [in Russian].

[9] Ruzhansky M., Sugimoto M., A local-to-global boundedness argument and Fourier integral operators, Journal of Mathematical Analysis and Applications,–2019.–473(2).–P.892-904.

[10] Sugimoto M., L^p-boundedness of Pseudo-differential operators satisfying Besov estimates I, J. Math. Soc. Japan,–1988.–40(1).–P.105-122.

[11] Sugimoto M., A priori estimates for higher order hyperbolic equations, Math. Z.,–1994.–215.–P.519-531.

[12] Sugimoto M., Estimates for Hyperbolic Equations of Space Dimension 3, Journal of Functional Analysis,–1998.–160.–P.382-407.

[13] Strichartz R. , Convolutions with kernels having singularities on a sphere, Trans. Amer. Math. Soc.,–1970.–148.–P.461-471.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.132-141
DOI: 10.29229/uzmj.2025-2-12

On a nonlocal problem of Bitsadze–Samarskii type for an elliptic equation with degeneration

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this paper, a nonlocal problem of the BitsadzeSamarskii type is studied for a degenerate elliptic equation in a vertical half-strip $\Omega = \{(x, y) : 0 < x < 1, y > 0\}$. The problem connects the value of the sought function on the right boundary of the domain with its value at an interior point of the same domain. Under certain conditions on the given functions, theorems on the existence and uniqueness of the solution are proved. The uniqueness of the solution is proved using the maximum principle, while the existence of a solution is established by methods of separation of variables and integral equations.

Keywords: degenerate equation, BitsadzeSamarskii type problem, spectral method, modified Bessel functions, eigenvalues, root functions, completeness, basis property.
MSC (2020): 34L10, 35J25, 35J70.

1. Introduction and Problem Statement

In [23], the Dirichlet problem and a Neumann-type problem (problem N) were studied for a degenerate elliptic equation with a power-type degeneration of the form

$$y^m u_{xx} + u_{yy} + \frac{\alpha}{y} u_y - b^2 y^m u = 0, 0 \leq x \leq 1,\ y > 0, \tag{1.1}$$

in case $m = 0$ and $b = 0$, in a vertical half-strip $\Omega = \{(x, y) : 0 < x < 1, y > 0\}$. The uniqueness of the solution was established using a combined method, which involves the maximum principle for elliptic equations and the "abc" method, while the existence of a solution was proven by employing separation of variables and integral transformation techniques.

In [14], for a mixed-type equation containing equation (1.1) in the case of $\alpha = 0$, $b = 0$ in a domain whose elliptic part is a half-strip Ω, the Tricomi problem was studied. The uniqueness of the solution was established using the "abc" method, while the existence was proven through the method of separation of variables and the theory of integral equations. Similar problems for mixed-type equations involving an equation of the form (1.1) were also considered in [10] and [16].

In [12], in a domain $\Omega = \{(x, y) : 0 < x < 1, y > 0\}$ for the case $\alpha = 0$, $b = 0$, a nonlocal Samarskii–Ionkin type problem was considered. By applying the spectral method, the existence and uniqueness of the solution were established, and an integral representation of the solution was obtained for the case $m \geq 0$.

In [11], for equation (1.1) in the same domain Ω and for the case $\alpha = 0$, $b = 0$, nonlocal problems with Samarskii–Ionkin type conditions were studied in a more general setting than in [10]. Using the maximum principle, theorems on the uniqueness of solutions were proven, and by means of spectral analysis, a theorem on the existence of solutions for the formulated problems was established.

In the work [15], for an elliptic equation of fractional order of the form $D^{2\alpha} u(x, y) - x^{2\beta} L u(x, y) = 0$ in the case where L is a self-adjoint positive operator with a discrete spectrum, the issues of solvability of various boundary value problems in infinite domains for

$\alpha \in (1/2, 1], \beta > -\alpha$, where $D^{2\alpha} = {}^C D^\alpha_{0+} \, {}^C D^\alpha_{0+}$, ${}^C D^\alpha_{0+}$ is the fractional Caputo derivative, are studied. It should be noted that this equation generalizes the well-known Tricomi, Gellerstedt, and Keldysh equations.

In [25], the first boundary value problem for a mixed-type equation with two degeneracy lines in a half-strip in the class of regular and bounded at infinity solutions is studied. Using spectral analysis methods, a criterion for the uniqueness of the problem is established, and the solution is constructed as a series in eigenfunctions of the corresponding one-dimensional problem. In [17], a nonlocal boundary value problem was formulated and studied for the degenerate equation (1.1) in a vertical half-strip. By applying the Hankel transform and the method of separation of variables, an explicit solution to the investigated problem was obtained. We also note [1], where a boundary value problem was investigated in an infinite half-strip for a generalized biaxially symmetric Helmholtz equation. Using the method of separation of variables and Fourier–Bessel series expansions, conditions for the solvability of the boundary value problem were established.

As for boundary value problems for degenerate equations in bounded domains, we note [19], where the first boundary value problem was studied for a mixed-type equation with a singular coefficient in a rectangular domain. Using spectral analysis methods, a criterion for the uniqueness of the solution was established, and the solution was constructed as a sum of a series in terms of the eigenfunctions of a one-dimensional spectral problem for eigenvalues. In [21] and [20], for a mixed-type equation of the second kind with a singular coefficient, uniqueness criteria for the Dirichlet problem were established using spectral expansions, and the solutions were constructed as sums of series.

A new type of nonlocal boundary value problem for a partial differential equation of elliptic type, arising in plasma theory, was formulated and studied by A.V. Bitsadze and A.A. Samarskii in [6]. In the scientific literature, this problem became known as the Bitsadze–Samarskii (BS) type problem. Problems of this type differ from other boundary value problems in that the boundary values of the sought solution or its derivatives are repeated at interior points of the domain, where the solution satisfies the differential equation. Similar problems were studied in the works [3], [2], [22], [8], [4], [5].

In this paper, for equation (1.1) in the case $\alpha = 0$, $b = 0$, posed in the vertical half-strip $\Omega = \{(x, y) : 0 < x < 1, y > 0\}$ a nonlocal problem is studied, where the boundary conditions with respect to the spatial variable include a nonlocal condition of the Bitsadze–Samarskii type.

For the equation

$$Lu \equiv y^m u_{xx} + u_{yy} = 0, m = const > 0 \tag{1.2}$$

in the vertical half-strip $\Omega = \{(x, y) : 0 < x < 1, y > 0\}$ we consider the following nonlocal problem.

Problem BS**.** Find a function $u(x, y)$, satisfying the following properties:

1) $u(x, t) \in C(\bar{\Omega}) \cap C^2(\Omega)$;

2) satisfies equation (1.2) in the domain Ω;

3) the following conditions are fulfilled:

$$\lim_{y \to +\infty} u(x, y) = 0 \text{ uniformly in } x \in [0, 1], \tag{1.3}$$

$$u_x(0, y) = \varphi_1(y), u(1, y) - u(x_0, y) = \varphi_2(y), y \geq 0, \tag{1.4}$$

$$u(x; 0) = \tau(x), 0 \leq x \leq 1, \tag{1.5}$$

where $\varphi_1(y), \varphi_2(y), \tau(x)$ are given functions.

2. Uniqueness of the Solution to Problem BS

We prove a theorem on the uniqueness of the solution to Problem BS. The following theorem holds:

Theorem 2.1. *Problem BS cannot have more than one solution.*

Proof. Suppose that the considered problem has two solutions $u_1(x,y)$ and $u_2(x,y)$. Then the difference of these solutions, i.e., the function $u(x,y)=u_1(x,y)-u_2(x,y)$ in the domain Ω satisfies equation (1.2), condition (1.3), and homogeneous boundary conditions of the form

$$u_x(0,y)=0, u(1,y)-u(x_0,y)=0, y\geq 0, u(x,0)=0, 0\leq x\leq 1. \tag{2.1}$$

Thus, it suffices to show that this homogeneous problem has only the trivial solution $u(x,y)\equiv 0$ in Ω.

Assume that $u(x,y)\neq 0$ in $\bar{\Omega}$. Suppose, for example, $u(c,h)\neq 0$, $h>0$. Then $u(x,y)\neq 0$ in $\bar{\Omega}_h$, where Ω_h is a rectangle with vertices at the points $O(0,0), A(1,0), O_h(0,h)$ and $A_h(1,h)$. We show that the maximum $\max\limits_{\bar{\Omega}_h}|u(x,y)|$ is attained only on the segment $\overline{O_hA_h}$. Without loss of generality, we can assume that $\max\limits_{\bar{\Omega}_h}|u(x,y)|=\max\limits_{\bar{\Omega}_h}u(x,y)>0$.

Indeed, by the well-known property of solutions to elliptic equations [22, p. 229, Theorem 4.2.4], the maximum cannot be attained inside Ω_h. Due to the first and third conditions in (2.1), $\max\limits_{\bar{\Omega}_h}u(x,y)$ cannot be attained on the segments $\overline{OO_h}$ and $\overline{OA}$. Let us show that $\max\limits_{\bar{\Omega}_h}u(x,y)$ cannot be attained on the segment $\overline{AA_h}$ either. Assume that the $\max\limits_{\bar{\Omega}_h}u(x,y)$ is attained at some interior point $(1,y_0)$ of the segment $\overline{AA_h}$. Then, by the second condition in (2.1), we find that the $\max\limits_{\bar{\Omega}_h}u(x,y)$ is also attained at some interior point (x_0,y_0) of the domain $\bar{\Omega}_h$, which contradicts the aforementioned property of elliptic equations. Therefore, $\max\limits_{\bar{\Omega}_h}u(x,y)$ is attained only on the segment $\overline{O_hA_h}$ and let $\max\limits_{\bar{\Omega}_h}|u(x,t)|=\varepsilon$.

Now, by condition (1.3), there exists $\tilde{h}>h$, such that for all $y\geq\tilde{h}$, we have $|u(x,y)|<\varepsilon/2$. This implies that $\max\limits_{\bar{\Omega}_{\tilde{h}}}|u(x,y)|\leq\varepsilon/2$, where $\Omega_{\tilde{h}}$ is a quadrilateral with vertices at points $O, A, O_h(0,\tilde{h})$ and $A_h(1,\tilde{h})$. On the other hand, since $\overline{\Omega}_h\subset\overline{\Omega}_{\tilde{h}}$, we must have $\max\limits_{\bar{\Omega}_{\tilde{h}}}|u(x,y)|\geq \max\limits_{\bar{\Omega}_h}|u(x,y)|=\varepsilon$, which contradicts the condition $\max\limits_{\bar{\Omega}_{\tilde{h}}}|u(x,y)|\leq\varepsilon/2$. Therefore, $u(x,y)\equiv 0$ in $\bar{\Omega}$, i.e., $u_1(x,y)=u_2(x,y)$. This completes the proof of Theorem 2.1. ☐

3. Existence of the Solution to Problem *BS*

Now we turn to proving the existence of a solution to Problem BS. The following theorem holds:

Theorem 3.1. *Let the functions $\tau(x),\varphi_i(y), i=1,2$ satisfy the following conditions:*

$$1)\varphi_i(y)\in C[0,\infty), y^{3m/4}\varphi_i(y)\in L_1(0,+\infty), \varphi_i(0)=0, \varphi_i(+\infty)=0, i=1,2,$$

$$2)\tau(x)\in C^2[0,1], \tau'''(x)\in L_2(0,1), \tau'(0)=0, \tau(1)=\tau(x_0), \tau''(1)=\tau''(x_0).$$

Then Problem BS has a solution.

Proof. We seek the solution of Problem BS in the domain Ω in the form of the sum of two functions:

$$u(x,y)=W(x,y)+V(x,y),$$

where $W(x,y)$ and $V(x,y)$ are solutions of equation (1.2) satisfying, respectively, the following conditions:

$$W_x(0,y)=\varphi_1(y), W(1,y)-W(x_0,y)=\varphi_2(y), y\geq 0, \tag{3.1}$$

$$\lim_{y\to+0} W(x,y)=0,\ \lim_{y\to+\infty} W(x,y)=0, 0\le x\le 1, \tag{3.2}$$

$$V_x(0,y)=0, V(1,y)-V(x_0,y)=0, y\ge 0, \tag{3.3}$$

$$\lim_{y\to+0} V(x,y)=\tau(x),\ \lim_{y\to+\infty} V(x,y)=0, 0\le x\le 1. \tag{3.4}$$

3.1. **Solution of Problem 2 (Problem** (1.2)**,** (3.1)**,** (3.2)**).** We seek the solution of Problem 2 in the form of an integral:

$$W(x,y)=\sqrt{y}\int_0^\infty \left(a_1(\lambda)e^{\lambda x}+a_2(\lambda)e^{-\lambda x}\right) J_{\frac{1}{m+2}}\left(\frac{2\lambda}{2+m}y^{\frac{m+2}{2}}\right)d\lambda, \tag{3.5}$$

where $a_1(\lambda), a_2(\lambda)$ are unknown functions, and $J_v(z)$ is the Bessel function of the first kind.

It should be noted that the function $W(x,y)$, defined by the integral (3.5), formally satisfies equation (1.2) and conditions (3.2).

From (3.5), taking into account the boundary conditions (3.1), we obtain the following system for the unknown functions $a_1(\lambda)$ and $a_2(\lambda)$

$$\begin{cases} \sqrt{y}\displaystyle\int_0^\infty \left(a_1(\lambda)-a_2(\lambda)\right) J_{\frac{1}{m+2}}\left(\frac{2\lambda}{2+m}y^{\frac{m+2}{2}}\right)d\lambda=\varphi_1(y),\\ \sqrt{y}\displaystyle\int_0^\infty \left(a_1(\lambda)(e^{\lambda}-e^{\lambda x_0})+a_2(\lambda)(e^{-\lambda}-e^{-\lambda x_0})\right) J_{\frac{1}{m+2}}\left(\frac{2\lambda}{2+m}y^{\frac{m+2}{2}}\right)d\lambda=\varphi_2(y). \end{cases} \tag{3.6}$$

Next, to find these functions, we use the Hankel transform:

$$F(p)=\int_0^\infty f(t)J_v(p\cdot t)tdt \Leftrightarrow f(t)=\int_0^\infty F(p)J_v(t\cdot p)pdp, \tag{3.7}$$

which holds for $\nu\ge -\frac{1}{2}$, provided that $\int_0^\infty \sqrt{t}f(t)dt<+\infty$.

Applying the Hankel transform to formulas (3.6) and using relation (3.7), while taking into account condition 1) of Theorem 3.1, we obtain the following system:

$$\begin{cases} a_1(\lambda)-a_2(\lambda)=\dfrac{1}{q}\displaystyle\int_0^\infty y^{2q-\frac{3}{2}}J_{\frac{1}{m+2}}\left(\frac{2\lambda}{2+m}y^{\frac{m+2}{2}}\right)\varphi_1(y)dy,\\ a_1(\lambda)(e^{\lambda}-e^{\lambda x_0})+a_2(\lambda)(e^{-\lambda}-e^{-\lambda x_0})=\dfrac{\lambda}{q}\displaystyle\int_0^\infty y^{2q-\frac{3}{2}}J_{\frac{1}{m+2}}\left(\frac{2\lambda}{2+m}y^{\frac{m+2}{2}}\right)\varphi_2(y)dy. \end{cases}$$

Thus, after simple transformations, we find:

$$a_1(\lambda)e^{\lambda x}+a_2(\lambda)e^{-\lambda x}=\frac{2}{(m+2)\left(\sin h\lambda-\sin h\lambda x_0\right)}\int_0^\infty t^{\frac{2m+1}{2}}J_{\frac{1}{m+2}}\left(\frac{2\lambda}{2+m}t^{\frac{m+2}{2}}\right)\times$$

$$\times\left(\left[\cos h\lambda(x_0-x)-\cos h\lambda\left(x-1\right)\right]\varphi_1(t)+\lambda\sin h\lambda x\varphi_2(t)\right)dt. \tag{3.8}$$

Thus, formulas (3.5) and (3.8) determine the solution to the considered problem. Furthermore, taking into account the conditions imposed on the functions $\varphi_i(y), i=1,2$, the properties of Bessel functions [13], and the properties of the Fourier transform [24], we conclude that the function $W(x,y)$, constructed in this way, satisfies equation (1.2) and conditions (3.1), (3.2).

3.2. **Solution of Problem 3 (Problem** (1.2)**,** (3.3)**,** (3.4)**).**

3.2.1. *Spectral Properties of Problem 3.* To solve Problem BS, we apply the spectral method, according to which seeks non-trivial solutions to equation (1.2) in the form of $V(x,y) = X(x)\cdot T(y)$. Substituting this expression into the original equation and satisfying the boundary conditions (3.3), with respect to the unknown function $X(x)$, we obtain the following spectral problem:

$$-X''(x) = \lambda X(x) = 0,\ 0 < x < 1, \tag{3.9}$$

$$X'(0) = 0,\ X(1) = X(x_0), \tag{3.10}$$

where λ is the separation parameter.

Alongside problem (3.9), (3.10), we also consider the problem adjoint to it. It is not difficult to determine that the adjoint problem has the following form

$$-Y''(x) = \lambda Y(x),\ x \in (0,x_0) \cup (x_0,1), \tag{3.11}$$

$$Y'(0) = 0,\ Y(1) = 0, \tag{3.12}$$

$$Y(x_0+0) = Y(x_0-0),\ Y'(1) = Y'(x_0+0) - Y'(x_0-0). \tag{3.13}$$

Note that problems (3.9), (3.10) and (3.11)-(3.13) are studied in the work [7]. According to this work, let us consider case when is a rational number x_0 in $(0,1)$. The following theorem holds [7]:

Theorem 3.2. *Let x_0 be any rational number from the interval $(0,1)$ such that $q-p=1$. Then the systems of root functions of problems (3.9), (3.10) and (3.11)-(3.13) form a Riesz basis in $L_2(0,1)$ (see Table 1).*

TABLE 1. Root functions and eigenvalues

No.	**Eigenvalues**	**Root functions of the main problem**	**Root functions of the adjoint problem**
1	$\lambda_0 = 0$	$X_0(x) = 1, \quad x \in [0,1]$	$Y_0(x) = \begin{cases} \dfrac{2}{1+x_0}, & x \in [0,x_0], \\ \dfrac{2(1-x)}{1-x_0^2}, & x \in [x_0,1] \end{cases}$
2	$\lambda_{1n} = \left(\frac{2q\pi n}{q+p}\right)^2,$ $n \neq k(q+p), \quad k,n \in \mathbb{N}$	$X_{1n}(x) = \cos(\sqrt{\lambda_{1n}}\,x), \quad x \in [0,1]$	$Y_{1n}(x) = \begin{cases} \dfrac{4\cos(\sqrt{\lambda_{1n}}\,x)}{1+x_0}, & x \in [0,x_0], \\ \dfrac{2\sin(\sqrt{\lambda_{1n}}(1-x))}{(1+x_0)\sin(\sqrt{\lambda_{1n}})}, & x \in [x_0,1] \end{cases}$
3	$\lambda_{2n} = (2qn\pi)^2, \quad n \in \mathbb{N}$	$X_{2n}(x) = \cos(\sqrt{\lambda_{2n}}\,x), \quad x \in [0,1]$	$\tilde{Y}_{2n}(x) = \begin{cases} \dfrac{4\cos(\sqrt{\lambda_{2n}}\,x)}{1+x_0}, & x \in [0,x_0], \\ \dfrac{4(1-x)\cos(\sqrt{\lambda_{2n}}\,x)}{1-x_0^2}, & x \in [x_0,1] \end{cases}$
4	$\lambda_{2n} = (2qn\pi)^2, \quad n \in \mathbb{N}$	$\tilde{X}_{2n}(x) = x\sin(\sqrt{\lambda_{2n}}\,x), \quad x \in [0,1]$	$Y_{2n}(x) = \begin{cases} 0, & x \in [0,x_0], \\ \dfrac{4\sin(\sqrt{\lambda_{2n}}\,x)}{1-x_0^2}, & x \in [x_0,1] \end{cases}$

3.2.2. *Existence of the Solution to Problem 3.* Let $V(x,y)$ be the solution to Problem 3. Consider the functions:

$$v_0(y) = (V(x,y), Y_0(x))_0, \tag{3.14}$$

$$v_{1n}(y) = (V(x,y), Y_{1n}(x))_0, n \neq k(q+p), k,n \in N, \tag{3.15}$$

$$v_{2n}(y) = (V(x,y), Y_{2n}(x))_0, n \in N, \tag{3.16}$$

$$\tilde{v}_{2n}(y) = (V(x,y), \tilde{Y}_{2n}(x))_0, n \in N, \tag{3.17}$$

where

$$(\varphi, \psi)_0 = (\varphi, \psi)_{L_2(a,b)} = \int_a^b \varphi(x)\psi(x)dx$$

is inner product in $L_2(a, b)$.

By differentiating (3.15) twice and taking into account equation (1.2) as well as the boundary conditions (1.4), we conclude that $v_{1n}(y)$ satisfies the differential equation

$$v''_{1n}(y) - \lambda_{1n} y^m v_{1n}(y) = 0 \tag{3.18}$$

with boundary conditions

$$\lim_{y\to+0} v_{1n}(y) = \tau_{1n}, \lim_{y\to+\infty} v_{1n}(y) = 0, \tag{3.19}$$

where

$$\tau_{1n} = (\tau(x), Y_{1n}(x))_0.$$

Now we find the general solution to equation (3.18). To do this, we perform the change of variables according to the formulas

$$v_{1n}(y) = \sqrt{y} Z_{1n}(t), t = \frac{\sqrt{\lambda_{1n}}}{q} y^q, q = \frac{m+2}{2}. \tag{3.20}$$

Then we have

$$t^2 Z''_{1n}(t) + t Z'_{1n}(t) - \left(t^2 + \frac{1}{(2q)^2}\right) Z_{1n}(t) = 0.$$

Substituting (3.20) to (3.18), the equation (3.18) is reduced to the modified Bessel equation, whose general solution is given by [23]:

$$Z_{1n}(t) = A_{1n} I_{\frac{1}{2q}}(t) + B_{1n} K_{\frac{1}{2q}}(t),$$

where A_{1n} and B_{1n} are arbitrary real constants, $I_v(t), K_v(t)$ are the modified Bessel functions [23]. Then, returning to the variable y and to the function $v_{1n}(y)$ using formulas (3.20), the general solution of equation (3.18) takes the form:

$$v_{1n}(y) = A_{in}\sqrt{y} I_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{1n}}}{q} y^q\right) + B_{1n}\sqrt{y} K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{1n}}}{q} y^q\right).$$

Now, to determine the unknown constants A_{1n} and B_{1n} we apply the boundary conditions (3.19). Taking into account the second condition in (3.19), as well as the asymptotic behavior of Bessel functions $I_\nu(z), K_\nu(z)$ for $z \to \infty$ [23,p.416]:

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z},$$

we obtain that $A_{1n} = 0$, that is

$$v_{1n}(y) = B_{1n}\sqrt{y} K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{1n}}}{q} y^q\right).$$

From here taking into account the first boundary condition (3.19), we have:

$$\lim_{y\to 0} v_{1n}(y) = B_{1n} \lim_{y\to+0} K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{1n}}}{q} y^q\right)\sqrt{y} = \tau_{1n}.$$

Next, using the behavior of the function $K_\nu(z)$ for $z \to 0$ [23,p.416]:

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}, \nu > 0 \tag{3.21}$$

we obtain

$$B_{1n} = \frac{2}{\Gamma\left(\frac{1}{2q}\right)}\left(\frac{\sqrt{\lambda_{1n}}}{2q}\right)^{1/2q}\tau_{1n}.$$

Thus, considering the found values of A_{1n} and B_{1n}, the solution of problem (3.18), (3.19) has the form:

$$v_{1n}(y) = \frac{2}{\Gamma\left(\frac{1}{2q}\right)}\left(\frac{\sqrt{\lambda_{1n}}}{2q}\right)^{1/2q}\tau_{1n}\sqrt{y}K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{1n}}}{q}y^q\right). \tag{3.22}$$

Similarly, we find $v_{2n}(y)$ from (3.16):

$$v_{2n}(y) = \frac{2}{\Gamma\left(\frac{1}{2q}\right)}\left(\frac{\sqrt{\lambda_{2n}}}{2q}\right)^{1/2q}\tau_{2n}\sqrt{y}K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{2n}}}{q}y^q\right). \tag{3.23}$$

Now, we find $v_0(y)$. Differentiating (3.14) twice and taking into account equation (1.2) as well as the boundary conditions (2.1), we find that the function $v_0(y)$ satisfies the equation and boundary conditions:

$$v_0''(y) = 0, v_0(0) = \tau_0, v(\infty) = 0, \tag{3.24}$$

where

$$\tau_0 = \int_0^1 \tau(x)Y_0(x)dx.$$

The problem (3.24) has a trivial solution under the condition

$$\tau_0 = 0 \Leftrightarrow \int_0^1 \tau(x)Y_0(x)dx = 0.$$

By similar calculations, to find the function $\tilde{v}_{2n}(y)$ from (3.17), we obtain the problem

$$\tilde{v}_{2n}''(y) - y^m\lambda_{2n}\tilde{v}_{2n}(y) = -2y^m\sqrt{\lambda_{2n}}v_{2n}(y),$$

$$\lim_{y\to+0}\tilde{v}_{2n}(y) = \tilde{\tau}_{2n}, \lim_{y\to+\infty}\tilde{v}_{2n}(y) = 0,$$

where the solution is determined by the formula

$$\tilde{v}_{2n}(y) = \frac{2}{\Gamma\left(\frac{1}{2q}\right)}\left(\frac{\sqrt{\lambda_{2n}}}{2q}\right)^{\frac{1}{2q}}\tau_{2n}\sqrt{y}\frac{y^q}{q}K_{\frac{1}{2q}-1}\left(\frac{\sqrt{\lambda_{2n}}}{q}y^q\right)+$$

$$+\frac{2}{\Gamma\left(\frac{1}{2q}\right)}\left(\frac{\sqrt{\lambda_{2n}}}{2q}\right)^{\frac{1}{2q}}\tilde{\tau}_{2n}\sqrt{y}K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{2n}}}{q}y^q\right), \tag{3.25}$$

where

$$\tilde{\tau}_{2n} = \int_0^1 \tau(x)\tilde{Y}_{2n}(x)dx.$$

From formulas (3.22), (3.23), and (3.25), the uniqueness of the solution to Problem 3 follows, since $\tau(x)=0$ on $(0,1)$, then $\tilde{v}_{2n}(y)=0$ for $n=0,1,2,...$ and $v_{2n}(y)=0$ for $n=1,2,...$ on $(0,\infty)$. Thus, due to the completeness of the system $\{\tilde{Y}_{2n}(x)\}_{n=1}^{\infty}$, $Y_0(x)$, $\{Y_{2n}(x)\}_{n=1}^{\infty}$, $\{Y_{1n}(x)\}_{n=1}^{\infty}$ in the space $L_2(0,1)$, it follows that $V(x,y)=0$ for all $x\in[0,1]$, $y\in(0,\infty)$. Thus, the uniqueness of the solution to problems (1.2), (3.3), (3.4) is proven.

Now, let us address the question of the existence of the solution to Problem 3.

Theorem 3.3. *If $\tau(x)\in C^2[0,1]$, $\tau'''(x)\in L_2(0,1)$, $\tau'(0)=0, \tau(1)=\tau(x_0)$, $\tau''(1)=\tau''(x_0)$ and the condition $(\tau(x),Y_0(x))_0=0$ is satisfied, then a solution to Problem 3 exists.*

Proof. Function $V(x,y)$ can be represented as a biorthogonal series

$$V(x,y)=v_0(y)+\sum_{n=1}^{\infty *}v_{1n}(y)X_{1n}(x)+\sum_{n=1}^{\infty}(v_{2n}(y)\tilde{X}_{2n}(x)+\tilde{v}_{2n}(y)X_{2n}(x)), \tag{3.26}$$

that converges in $L_2(0,1)$ for each $y\in(0,\infty)$, where "*" means that the sum is taken over $n\in N$, different from $k(q+p)$, $k\in N$. Using the formulas for $v_{1n}, v_{2n}, \tilde{v}_{2n}, v_0$ the series (3.26) can be rewritten in the form

$$\begin{aligned} V\left(x,y\right)=&\textstyle\sum_{n=1}^{\infty *}\frac{2}{\Gamma\left(\frac{1}{2q}\right)}\left(\frac{\sqrt{\lambda_{1n}}}{2q}\right)^{\frac{1}{2q}}\tau_{1n}\sqrt{y}K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{1n}}}{q}y^q\right)\cos\sqrt{\lambda_{1n}}x+\\ &+\textstyle\sum_{n=1}^{\infty}\frac{2}{\Gamma\left(\frac{1}{2q}\right)}\left(\frac{\sqrt{\lambda_{2n}}}{2q}\right)^{\frac{1}{2q}}\sqrt{y}K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{2n}}}{q}y^q\right)\left(\tau_{2n}x\sin\sqrt{\lambda_{2n}}x+\tilde{\tau}_{2n}\cos\sqrt{\lambda_{2n}}x\right)+\\ &+\textstyle\sum_{n=1}^{\infty}\frac{2}{\Gamma\left(\frac{1}{2q}\right)}\left(\frac{\sqrt{\lambda_{2n}}}{2q}\right)^{\frac{1}{2q}}\tau_{2n}y^q\frac{\sqrt{y}}{q}K_{\frac{1}{2q}-1}\left(\frac{\sqrt{\lambda_{2n}}}{q}y^q\right)\cos\sqrt{\lambda_{2n}}x. \end{aligned} \tag{3.27}$$

It can be seen that for $y\ge\delta>0$, where δ is sufficiently small, the series (3.27), together with all its derivatives of any order, converges uniformly due to the exponential decay of the function $K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{in}}}{q}y^q\right), i=1,2$ at $n\to\infty$. Therefore, the series (3.27) satisfies equation (1.2) for $y>0$. This series also satisfies the boundary conditions (1.3)-(1.5), (2.1), except the condition $V(x,0)=\tau(x)$. In order to verify the last condition, we need to establish uniform convergence of the series (3.27) for $y\ge0$, which would provide continuity of the function (3.27) over the domain $x\in[0,1]$, $y\in[0,\infty)$, and justify the possibility of term-by-term differentiation of the series (3.27) with respect to $y>0$. Integrating by parts the expressions for the coefficients $\tau_{in}=(\tau(x),Y_{in}(x))_0, i=1,2$, $\tilde{\tau}_{2n}=(\tau(x),\tilde{Y}_{2n}(x))_0$, taking into account the conditions of Theorem 3.3, we obtain that the following representation holds

$$\tau_{in}=\frac{1}{\sqrt{\lambda_{in}^3}}\tau_{in}^{(3)}, \tau_{in}^{(3)}=\int_0^1\tau'''(x)Z_{in}(x)dx, i=1,2, \tag{3.28}$$

$$\tilde{\tau}_{2n}=\frac{1}{\sqrt{\lambda_{2n}^3}}\tilde{\tau}_{2n}^{(3)}-\frac{3}{\lambda_{2n}^2}\tau_{2n}^{(3)}, \tilde{\tau}_{2n}^{(3)}=\left(\tau^{(3)}(x),\tilde{Z}_{2n}(x)\right)_0, \tau_{2n}^{(3)}=\left(\tau^{(3)}(x),Z_{2n}(x)\right)_0, \tag{3.29}$$

where

$$Z_{1n}(x)=\left\{\begin{array}{l}\dfrac{4\sin\sqrt{\lambda_{1n}}x}{1+x_0}, x\in[0,x_0)\\ \dfrac{2\cos\sqrt{\lambda_{1n}}(1-x)}{(1+x_0)\sin\sqrt{\lambda_{1n}}}, x\in(x_0,1]\end{array}\right., \quad Z_{2n}(x)=\left\{\begin{array}{l}0, x\in[0,x_0)\\ -\dfrac{4\cos\sqrt{\lambda_{2n}}x}{(1-x_0^2)}, x\in(x_0,1]\end{array}\right.,$$

$$\tilde{Z}_{2n}(x)=\begin{cases}\dfrac{4\sin\sqrt{\lambda_{2n}}x}{1+x_0}, x\in[0,x_0)\\ \dfrac{4(1-x)\sin\sqrt{\lambda_{2n}}x}{(1-x_0^2)\sin\sqrt{\lambda_{2n}}}, x\in(x_0,1]\end{cases}.$$

Now, using the asymptotic formula (3.21), we estimate the following expressions for fixed $n\in N$ and small $y\geq 0$:

$$\left|\left(\frac{\sqrt{\lambda_{1n}}}{2q}\right)^{\frac{1}{2q}}\sqrt{y}K_{\frac{1}{2q}}\left(\frac{\sqrt{\lambda_{1n}}}{q}y^q\right)\right|\leq\left|\left(\frac{\sqrt{\lambda_{1n}}}{2q}\right)^{\frac{1}{2q}}\frac{\Gamma\left(\frac{1}{2q}\right)}{2}\sqrt{y}\left(\frac{\sqrt{\lambda_{1n}}}{q}y^q\right)^{-\frac{1}{2q}}\right|\leq C_1, \tag{3.30}$$

$$\left|\left(\frac{\sqrt{\lambda_{2n}}}{2q}\right)^{\frac{1}{2q}}y^q\sqrt{y}K_{\frac{1}{2q}-1}\left(\frac{\sqrt{\lambda_{2n}}}{q}y^q\right)\right|\leq\left|\left(\frac{\sqrt{\lambda_{2n}}}{2q}\right)^{\frac{1}{2q}}\frac{\Gamma\left(\left|\frac{1}{2q}-1\right|\right)}{2}\sqrt{y}\frac{y^q}{q}\left(\frac{\sqrt{\lambda_{2n}}}{q}y^q\right)^{-\left|\frac{1}{2q}-1\right|}\right|\leq$$

$$\leq\left|\left(\frac{\sqrt{\lambda_{2n}}}{2q}\right)^{\frac{1}{2q}}\frac{\Gamma\left(1-\frac{1}{2q}\right)}{2}\sqrt{y}\frac{y^q}{q}\left(\frac{\sqrt{\lambda_{2n}}}{q}y^q\right)^{\frac{1}{2q}-1}\right|\leq\frac{C_2}{n^{1-\frac{1}{q}}}. \tag{3.31}$$

Formulas (3.28)–(3.31) guarantee the uniform convergence of the series (3.27) in the domain $x\in[0,1]$, $y\in[0,\infty)$, as well as the fulfillment of conditions (3.3) and (3.4) for the function $V(x,y)$. Theorem 3.3 is proven. □

Hence, the solution to the Problem BS exists, is unique, and is determined as the sum of the solutions to Problems 2 and 3. Thus, the proof of Theorem 3.1 is complete.

FUNDING

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488086).

References

[1] Abashkin A.A.; On a problem for the generalized biaxially symmetric Helmholtz equation in an infinite half-strip. Vestn. Sam. state tech. univ. Ser. Phys.-math. sci., –2012. –1. –26. –P.39–45.

[2] Ashurov R., Kadirkulov B., Ergashev O.; Inverse problem of BitsadzeSamarskii type for a two-dimensional parabolic equation of fractional order. Journal of Mathematical Sciences, –2023. –274. –2. –P.172–185.

[3] Ashurov R., Kadirkulov B., Jalilov M.; On an inverse problem of the Bitsadze-Samarskii type for a parabolic equation of fractional order. Boletin de la Sociedad Matematica Mexicana., –2023. –29. –70. –P.104–115.

[4] Baderko E.A., Cherepova M.; Solution of the BitsadzeSamarskii problem for a parabolic system in a semibounded domain. Differ. Equ., –2017. –53. –10. –P. 1294–1302.

[5] Baderko E.A., Cherepova M.; BitsadzeSamarskii problem for parabolic systems with Dini continuous coefficients. Complex Var. Elliptic Equ., –2018. –64. –5. –P.753–765.

[6] Bitsadze A.V., Samarskii A.A.; Some elementary generalizations of linear elliptic boundary value problems. Russ. Sov. Math. Dokl., –1969. –10. –P.398–400.

[7] Kadirkulov B., Ergashev O.; On a problem of the Bitsadze-Samarskii type for a second-order ordinary differential equation. Bull. Inst. Math., –2023. –6. –4. –P.113-121. (In Russian).

[8] Karachik V.V., Turmetov B.Kh.; On solvability of some nonlocal boundary value problems for biharmonic equation. Math. Slovaca., –2020. –70. –2. –P.329–342.

[9] Khubiev K.U.; The BitsadzeSamarskii problem for some characteristically loaded hyperbolic-parabolic equation. Journal of Samara State Technical University, Ser. physical and mathematical sciences., –2019. –23. –4. –P.789–796.

[10] Lerner M.E., Repin O.A.; On a problem with two nonlocal boundary conditions for an equation of mixed type. Siberian Math. J., –1999. –40. –6. –P.1064–1078

[11] Lerner M.E., Repin O.A.; On Frankl'-type problems for some elliptic equations with degeneration of various types. Differ. Equ., –1999. –35. –8. –P.1098–1104.

[12] Moiseev E.I.; On the solution of a nonlocal boundary value problem by the spectral method. Differ. Equ., –1999. –35. –8. –P.1105–1112.

[13] Olver F.; Asymptotics and special functions: Trans. from English/Ed. by A. P. Prudnikov., Moscow. Nauka. Chief editor of physical and mathematical literature., –1990. –P.528. (In Russian).

[14] Repin O.A.; The Tricomi problem for an equation of mixed type in a domain whose elliptic part is a half-strip. Differ. Equ., –1996. –32. –4. –P.571–574

[15] Ruzhansky M., Torebek B.T., Turmetov B.; Well-posedness of Tricomi-Gellerstedt-Keldysh-type fractional elliptic problems. J. Int. Eq. Appl., –2022. –34. –3. –P.373–387.

[16] Ruziev M.Kh.; Boundary value problem for a mixed-type equation with a singular coefficient in a domain whose elliptic part is a half-strip. Vestn. Sam. state tech. univ. Ser. Phys.-math. sciences., –2009. –1. –18. –P.33–40. (In Russian).

[17] Ruziev M., Kazakbaeva K.; On a non-local boundary value problem for the Holmgren equation with a singular coefficient. Bull. Inst. Math., –2024. –7.–5. –P.117–124.

[18] Sabitov K.B.; Towards a theory of mixed-type equations. Moscow., FIZMATLIT., –2014. –P.304. (In Russian).

[19] Sabitov K.B., Safina R.M.; The first boundary-value problem for an equation of mixed type with a singular coefficient. Izv. Math., –2018. –82. –2. –P.318–350.

[20] Sabitova Yu.K.; The Dirichlet Problem for a Mixed-Type Equation with Characteristic Degeneration. Vestn. Sam. State Tech. Univ. Ser. Phys.-Math. Sci., –2019. –23. –4. –P.622–645. (In Russian).

[21] Safina R.M.; Dirichlet problem for mixed type equation with characteristic degeneration and singular coefficient. LMJ., –2020. –41. –P.80–88.

[22] Shamsi H.Al., Kadirkulov B.J., Kerbal S.; The BitsadzeSamarskii type problem for mixed type equation in the domain with the deviation from the characteristics. Lobachevskii J. Math., –2020. –41. – 6. –P.1021–1030.

[23] Shimkovich E.V.; On weighted boundary value problems for a degenerate elliptic equation in a half-strip. Lithuanian mathematical collection. –1990. –30. –P.185–196. (in Russian).

[24] Sneddon I.; Fourier Transformation. Foreign Literature Publishing House., Moscow. –1955. –P.670.

[25] Vagapov V.Z.; Dirichlet problem for the mixed type equation with two degeneration lines in a half-strip. Vestn. Sam. State Tech. Univ. Ser. Phys.-Math. Sci., –2019. –23. –1. –P.7–19. (In Russian).

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.142-152
DOI: 10.29229/uzmj.2025-2-13

Spatially Nonlocal Problems with Integral Conditions for Ultraparabolic Equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. The article is devoted to investigating the solvability of new nonlocal boundary value problems for linear ultraparabolic equations with two time variables and one spatial variable. The main peculiarities of the problems is first that nonlocal conditions of integral form are given with respect to the spatial variable and second, that the coefficient at one of the time derivatives may degenerate. The article aims to prove existence and uniqueness theorems for regular solutions, i.e., or solutions having all weak derivatives in the sense of S. L. Sobolev occurring in the equation.

Keywords: ultraparabolic equations, nonlocal problems, integral conditions, existence, uniqueness.

MSC (2020):35K70 35K65

Introduction

Ultraparabolic equations are a subclass of the class of elliptic-parabolic differential equations (also referred to as differential equations with nonnegative characteristic form); see [4, 18]. The theory of local boundary value problems for ultraparabolic equations, i.e., of problems in which certain conditions at the boundary points are given seems to be rather well developed (see [4, 18, 5, 27, 23, 7]). We observe that, alongside an independent meaning as a part of the general theory of boundary value problems for differential equations, the theory of boundary value problems for ultraparabolic equations also makes sense for the development of other areas of science, for instance, mathematical biology and probability theory [2].

The problems for ultraparabolic equations studied in the article are nonlocal problems, i.e., problems in which, instead of point (local) conditions, some conditions are given that connect the value of the solution or (and) its derivatives at the boundary points with the values of the solution or (and) its derivatives at the points of boundary or inner manifolds.

Nonlocal problems for ultraparabolic equations were studied earlier (see [8, 11, 6]) but in general the theory of nonlocal boundary value problems for ultraparabolic equations seems insufficiently developed.

In the present article, we partially fill this gap (certainly, to a very small extent). More exactly, in the present article, we will study the question of the solvability of nonlocal problems for ultraparabolic equations with integral conditions with respect to the spatial variables.

Problems with integral conditions are sufficiently well studied for classical second-order differential equations of hyperbolic, parabolic, and elliptic types, for some degenerating equations, some high-order equations including equations of Sobolev and mixed types (see [6, 1, 14, 21, 28, 19, 29, 30, 3, 26, 10, 9, 13, 20, 12]). As for ultraparabolic equations, spatially nonlocal equations were studied for them earlier only in [16]. And it is the paper [16] that served a source for the present article. More exactly, in [16], conditions were obtained for the solvability of the corresponding nonlocal problems related to the injectivity of some integral operators constructed from the data of the problem. In the present article, we substantially weaken the conditions of [16].

We will consider nonlocal problems for ultraparabolic equations in some model situations. More general problems and the discussion of possible generalizations will be presented at the end of the article.

The whole construction and the argument in the article are based on the properties of functions of the Lebesgue spaces L_p and the Sobolev spaces W_p^l. The necessary definitions and a description of the properties of functions of these spaces can be found in [22, 15, 24, 25].

1. Statements of the Problems

Let Ω be the interval (0,1) of the axis OX, $y \in (0,1)$, $Q = \Omega \times (0,1) \times (0,T)$. Suppose next that $a(t)$, $c(x,y,t)$, $f(x,y,t), N_1(y), N_2(y)$ are given functions defined for $x \in \bar{\Omega},\ y \in [0,1]\ , t \in [0,T]$; L is the differential operator whose action at a given function $v(x,y,t)$ is defined by the equality

$$Lv = v_t + a(t)v_x - v_{yy} + c(x,y,t)v.$$

Boundary Value Problem I: Find a function $u(x,y,t)$ that is a solution in Q to the equation

$$Lu = f(x,y,t) \tag{1.1}$$

and satisfies the conditions

$$u(x,y,0) = 0, \qquad x \in \Omega, y \in (0,1), \tag{1.2}$$

$$u(0,y,t) = 0, \quad y \in (0,1), t \in (0,T), \tag{1.3}$$

$$u(x,0,t) = \int\limits_0^1 N_1(y)u(x,y,t)dy, \quad (x,t) \in \Omega \times (0,T). \tag{1.4}$$

$$u(x,1,t) = \int\limits_0^1 N_2(y)u(x,y,t)dy, \quad (x,t) \in \Omega \times (0,T). \tag{1.5}$$

Boundary Value Problem II: Find a function $u(x,y,t)$ that is a solution in Q to equation (1.1) and satisfies conditions (1.2),(1.3) and also the conditions

$$u_y(x,0,t) = \int\limits_0^1 N_1(y)u(x,y,t)dy, \quad (x,t) \in \Omega \times (0,T). \tag{1.6}$$

$$u_y(x,1,t) = \int\limits_0^1 N_2(y)u(x,y,t)dy, \quad (x,t) \in \Omega \times (0,T). \tag{1.7}$$

Boundary Value Problem III: Find a function $u(x,y,t)$ that is a solution in Q to equation (1.1) and satisfies conditions (1.2),(1.3) and also the conditions

$$u(x,0,t) = \int\limits_0^1 N_1(y)u(x,y,t)dy, \quad (x,t) \in \Omega \times (0,T). \tag{1.8}$$

$$u_y(x,1,t) = \int\limits_0^1 N_2(y)u(x,y,t)dy, \quad (x,t) \in \Omega \times (0,T). \tag{1.9}$$

Boundary Value Problems I, II, and III can be called the integral analogs of the first, second, and mixed problems for ultraparabolic equations (1.1). In [16], the solvability of Problems I and II was studied while Problem III has not been studied before.

Denote by V_0 the linear space

$$V_0 = \{v(x,y,t) : \int\limits_Q (v^2 + v_x^2 + v_t^2 + v_{yy}^2)dxdydt < +\infty\}$$

(here the derivatives are understood as weak derivatives in the sense of S L. Sobolev). Endow V_0 with the norm

$$\|v\|_{V_0} = \left(\int\limits_Q (v^2 + v_x^2 + v_t^2 + v_{yy}^2)dxdydt \right)^{\frac{1}{2}}.$$

The main goal of the article is to prove the solvability of Boundary Value Problems I–III in this space.

2. Solvability of Boundary Value Problem II

All the arguments and calculations concerning Boundary Value Problem II, on the one hand, look easy and transparent, and on the other hand, demonstrate the essence of the methods applied. Therefore, we start studying the solvability of the problems from Problem II.

We put

$$R_0(y,z) = \frac{y^2}{2}[N_2(z) - N_1(z)] + yN_1(z),$$

$$R_1(x,y,z,t) = R_{0yy}(y,z) + R_0(y,z)[c(x,z,t) - c(x,y,t)],$$

$$M_0 = (1 - \frac{1}{2}\int\limits_0^1 y^2[N_2(y) - N_1(y)]dy)(1 - \int\limits_0^1 yN_1(y)dy) - \frac{1}{2}\int\limits_0^1 y^2 N_1(y)dy \int\limits_0^1 y[N_2(y) - N_1(y)]dy.$$

Next, let B the Fredholm operator whose action at a given function $w(y)$ is defined by the equality

$$(Bw)(y) = w(y) - \int\limits_0^1 R_0(y,z)w(z)dz$$

This operator is an integral operator with degenerate kernel, and if $M_0 \neq 0$ then it is continuously invertible as an operator from $L_2([0,1])$ into $L_2([0,1])$. The condition of the invertibility of B was the main condition for the solvability of Boundary Value Problem II in [16]. Below the solvability of Boundary Value Problem II will be proved without the condition $M_0 \neq 0$.

Theorem 2.1. *Suppose the fulfillment of the condition*

$$a(t) \in C([0,T]), \qquad a(t) \geq 0 \quad \textit{for} \quad t \in [0,T];$$

$$c(x,y,t) \in C(\bar{Q}), \qquad c_x(x,y,t) \in C(\bar{Q});$$

$$N_1(y) \in L_2([0,1]), \qquad N_2(y) \in L_2([0,1]).$$

Then for every function $f(x,t)$ such that $f(x,y,t) \in L_2(Q)$, $f_x(x,y,t) \in L_2(Q)$, $f(0,y,t) = 0$ for $y \in [0,1]$, $t \in [0,T]$, Boundary Value Problem II has a solution $u(x,y,t)$ belonging to V_0, and such a solution is unique.

Proof. If $M_0 \neq 0$ then the desired assertion is proved in [16]. Consider the case of $M_0 = 0$.

Let ε be a positive number. Consider the boundary value problem: Find a function $u(x, y, t)$ that is a solution in Q to the equation

$$L_\varepsilon u \equiv u_t + a(t)u_x - \varepsilon u_{xx} - u_{yy} + c(x, y, t)u = f(x, y, t) \tag{2.1}$$

an satisfies conditions (1.2) and (1.3) and also the conditions

$$u_x(1, y, t) = 0, \quad y \in (0, 1), \quad t \in (0, T), \tag{2.2}$$

$$(1 + \varepsilon)u_y(x, 0, t) = \int\limits_0^1 N_1(y)u(x, y, t)dy, \quad (x, t) \in \Omega \times (0, T), \tag{2.3}$$

$$(1 + \varepsilon)u_y(x, 1, t) = \int\limits_0^1 N_2(y)u(x, y, t)dy, \quad (x, t) \in \Omega \times (0, T), \tag{2.4}$$

For fixed (sufficiently small) positive ε and under the conditions of the theorem, this problem has a solution $u(x, y, t)$ (at first, we omit the index "ε" for the solution) such that $u(x, y, t) \in V_0$, $u_{xx}(x, y, t) \in L_2(Q)$ (see [12] and [16]). Let us show that this solution admits a priori estimates uniform in ε, which will enable us in the sequel to organize passage to the limit.

Consider the equality

$$\int\limits_0^t \int\limits_0^1 \int\limits_\Omega L_\varepsilon u(x, y, \tau)u(x, y, \tau)dxdyd\tau = \int\limits_0^t \int\limits_0^1 \int\limits_\Omega f(x, y, \tau)u(x, y, \tau)dxdyd\tau \tag{2.5}$$

where $t \in (0, T]$. Integrating by parts, using conditions (1.2), (1.3), and (2.2)–(2.4), we conclude that (2.5) implies the equality

$$\frac{1}{2}\int\limits_0^1 \int\limits_\Omega u^2(x, y, t)dxdy + \frac{1}{2}\int\limits_0^t \int\limits_0^1 a(\tau)u^2(1, y, \tau)dyd\tau + \int\limits_0^t \int\limits_0^1 \int\limits_\Omega u_y^2 dxdyd\tau$$

$$+\varepsilon \int\limits_0^t \int\limits_0^1 \int\limits_\Omega u_x^2 dxdyd\tau = \int\limits_0^t \int\limits_0^1 \int\limits_\Omega fudxdyd\tau - \int\limits_0^t \int\limits_0^1 \int\limits_\Omega cu^2 dxdyd\tau$$

$$-\frac{1}{1+\varepsilon}\int\limits_0^t \int\limits_\Omega u(x, 0, \tau)\left(\int\limits_0^1 N_1(y)u(x, y, \tau)dy\right)dxd\tau$$

$$+\frac{1}{1+\varepsilon}\int\limits_0^t \int\limits_\Omega u(x, 1, \tau)\left(\int\limits_0^1 N_2(y)u(x, y, \tau)dy\right)dxd\tau. \tag{2.6}$$

Estimate the last two summands on the right hand-side of [21] using Hölder's inequality and also the inequality

$$u^2(x, y^*, \tau) \leqslant \delta \int\limits_0^1 u_y^2(x, y, \tau)dy + c(\delta)\int\limits_0^1 u^2(x, y, \tau)dy, \tag{2.7}$$

in which y^* is one of the numbers 0 or 1, δ is an arbitrary positive number (see [15]). Now, estimating the first summand on the right-hand side of (2.6) with the use of Young's inequality, choosing δ small, and utilizing Gronwall's lemma, we infer that a solution $u(x,y,t)$ to the boundary value problem (2.1), (1.2), (1.3), (2.2)−(2.4) satisfies the first a priori estimate

$$\int_0^1\int_\Omega u^2(x,y,t)dxdy+\int_0^t\int_0^1 a(\tau)u^2(1,y,\tau)dyd\tau+\int_0^t\int_0^1\int_\Omega u_y^2dxdyd\tau$$

$$+\,\varepsilon\int_0^t\int_0^1\int_\Omega u_x^2dxdyd\tau\leqslant K_1\int_Q f^2dxdydt, \tag{2.8}$$

in which $t\in[0,T]$, K_1 is a constant defined by the functions $c(x,y,t)$, $N_1(y)$, $N_2(y)$ and also by the number T.

At the next step, consider the equality

$$-\int_0^t\int_0^1\int_\Omega L_\varepsilon u(x,y,\tau)u_{xx}(x,y,\tau)dxdyd\tau=-\int_0^t\int_0^1\int_\Omega f(x,y,\tau)u_{xx}(x,y,\tau)dxdyd\tau, \tag{2.9}$$

where again $t\in(0,T]$. It is not hard to transform this equality to the form

$$\frac{1}{2}\int_0^1\int_\Omega u_x^2(x,y,t)dxdy+\frac{1}{2}\int_0^t\int_0^1 a(\tau)u_x^2(0,y,\tau)dyd\tau+\int_0^t\int_0^1\int_\Omega u_{xy}^2dxdyd\tau$$

$$+\varepsilon\int_0^t\int_0^1\int_\Omega u_{xx}^2dxdyd\tau=\int_0^t\int_0^1\int_\Omega fu_{xx}dxdyd\tau-\int_0^t\int_0^1\int_\Omega cu_x^2dxdyd\tau$$

$$-\int_0^t\int_0^1\int_\Omega c_xu_xudxdyd\tau-\frac{1}{1+\varepsilon}\int_0^t\int_\Omega u_x(x,0,\tau)\left(\int_0^1 N_1(y)u_x(x,y,\tau)dy\right)dxd\tau$$

$$+\frac{1}{1+\varepsilon}\int_0^t\int_\Omega u_x(x,1,\tau)\left(\int_0^1 N_2(y)u_x(x,y,\tau)dy\right)dxd\tau$$

Using Hölder's inequality once again and inequality (2.7) but for the function $u_x(x,y^*,\tau)$, applying Young's inequality and Gronwall's lemma, finally, reckoning with estimate (2.8), it is not hard to show that equality (2.9) implies the estimate

$$\int_0^1\int_\Omega u_x^2(x,y,t)dxdy+\int_0^t\int_0^1 a(\tau)u_x^2(0,y,\tau)dyd\tau+\int_0^t\int_0^1\int_\Omega u_{xy}^2dxdyd\tau$$

$$+\,\varepsilon\int_0^t\int_0^1\int_\Omega u_{xx}^2dxdyd\tau\leqslant K_2\int_Q (f^2+f_x^2)dxdydt, \tag{2.10}$$

where the constant K_2 is defined only by the functions $c(x,y,t)$, $N_1(y)$, $N_2(y)$ and also by the number T.

For obtaining the following estimate uniform in ε, consider the equality

$$\int_0^t \int_0^1 \int_\Omega L_\varepsilon u(x,y,\tau)u_\tau(x,y,\tau)dxdyd\tau = \int_0^t \int_0^1 \int_\Omega f(x,y,\tau)u_\tau(x,y,\tau)dxdyd\tau,$$

Transform this equality to the form

$$\int_0^t \int_0^1 \int_\Omega u_\tau^2 dxdyd\tau + \frac{1}{2}\int_0^1 \int_\Omega u_y^2(x,y,t)dxdy = \int_0^t \int_0^1 \int_\Omega f u_\tau dxdyd\tau$$

$$-\int_0^t \int_0^1 \int_\Omega cuu_\tau dxdyd\tau - \frac{1}{1+\varepsilon}\int_0^t \int_\Omega u(x,0,\tau)\left(\int_0^1 N_1(y)u_\tau(x,y,\tau)dy\right)dxd\tau$$

$$+\frac{1}{1+\varepsilon}\int_\Omega u(x,0,t)\left(\int_0^1 N_1(y)u(x,y,t)dy\right)dx$$

$$+\frac{1}{1+\varepsilon}\int_0^t \int_\Omega u(x,1,\tau)\left(\int_0^1 N_2(y)u_\tau(x,y,\tau)dy\right)dxd\tau$$

$$-\frac{1}{1+\varepsilon}\int_\Omega u(x,1,t)\left(\int_0^1 N_1(y)u(x,y,t)dy\right)dx. \tag{2.11}$$

Estimating each summand on the right-hand side of (2.11) using Hölder's and Young's inequalities and (2.7), applying Gronwall's lemma, and reckoning with estimates (2.8) and (2.10), we see that equality (2.11) implies the a priori estimate

$$\int_0^t \int_0^1 \int_\Omega u_\tau^2 dxdyd\tau + \frac{1}{2}\int_0^1 \int_\Omega u_y^2(x,y,t)dxdy + \varepsilon\int_0^1 \int_\Omega u_x^2(x,y,t)dxdy \leqslant K_3 \int_Q (f^2+f_x^2)dxdydt, \tag{2.12}$$

where K_3 is a constant defined only by the functions $c(x,y,t)$, $N_1(y)$, $N_2(y)$ and also by the number T.

The last estimate uniform in ε, namely, the estimate of the function $u_{yy}(x,y,t)$ in $L_2(Q)$, obviously follows from estimates (2.8), (2.10), and (2.12).

Let $\{\varepsilon_m\}_{m=1}^\infty$ be a sequence of positive numbers such that $\varepsilon_m \to 0$ as $m \to \infty$. The above-obtained estimates, the reflexivity of a Hilbert space, and the embedding theorems [22, 15, 24] mean that there exist a sequence $\{m_k\}_{k=1}^\infty$ of naturals and a function $u(x,y,t)$ for which the following convergences hold as $k \to \infty$:

$u_{m_k}(x,y,t) \to u(x,y,t)$ weakly in V_0 and almost everywhere in $\bar{Q}$,

$\varepsilon_{m_k} u_{m_k\lambda x}(x,y,t) \to 0$ weakly in $L_2(Q)$,

$u_{m_k y}(x,0,t) \to u_y(x,0,t)$ weakly in $L_2(\Omega\times(0,T))$,

$u_{m_k y}(x,1,t) \to u_y(x,1,t)$ weakly in $L_2(\Omega\times(0,T))$,

$\varepsilon_{m_k} u_{m_k y}(x,0,t) \to 0$ weakly in $L_2(\Omega\times(0,T))$,

$\varepsilon_{m_k} u_{m_k y}(x,1,t) \to 0$ weakly in $L_2(\Omega\times(0,T))$.

These convergences imply that the limit function $u(x, y, t)$ is a desired solution to Boundary Value Problem II.

The uniqueness of solutions to Boundary Value Problem II in V_0 obviously follows from estimate [28], which is valid also for $\varepsilon = 0$.

The theorem is completely proved. □

3. Solvability of Boundary Value Problems I and III

The idea of proving the solvability of Boundary Value Problem I is in general quite analogous to the idea of proving the solvability of Boundary Value Problem II.

We put

$$\tilde{R}_0(y, z) = y[N_2(z) - N_1(z)] + N_1(z),$$

$$\tilde{M}_0 = (1 - \int_0^1 y[N_2(y) - N_1(y)]dy)(1 - \int_0^1 N_1(y)dy) - \int_0^1 yN_1(y)dy \int_0^1 [N_2(y) - N_1(y)]dy.$$

Further, from the given function $v(y)$, define the function $w(y)$:

$$w(y) = v(y) - \int_0^1 \tilde{R}_0(y, z)dz$$

If the condition $\tilde{M}_0 \neq 0$ is fulfilled then it is not hard to express $v(y)$ through $w(y)$, and this is the case considered in [16]. Below we will examine the case admitting the equality $M_0 = 0$.

In what follows, we will study the simplified situation $c(x, y, t) \equiv c(x, t)$; the situation with a general $c(x, y, t)$ will be commented on at the end of the article.

Theorem 3.1. *Suppose the fulfillment of the condition*

$$a(t) \in C^1([0, T]), \qquad a(t) \geq a_0 > 0 \quad for \quad t \in [0, T];$$

$$c(x, y, t) \equiv c(x, t), \quad c(x, t) \in C(\bar{\Omega} \times [0, T]), \quad c_t(x, t) \in C(\bar{\Omega} \times [0, T]);$$

$$N_1(y) \in W_2^1([0, 1]), \qquad N_2(y) \in W_2^1([0, 1]).$$

Then for any function $f(x, t)$ such that $f(x, y, t) \in L_2(Q)$, $f_t(x, y, t) \in L_2(Q)$, $f(x, y, 0) = 0$ for $x \in \bar{\Omega}$, $y \in [0, 1]$, Boundary Value Problem I has a solution $u(x, y, t)$ in the space V_0, and this solution is unique.

Proof. We once again use the regularization method.

Given a positive number ε, consider the problem: Find a function $u(x, y, t)$ that is a solution in Q to equation (2.1) and satisfies conditions (1.2), (1.3), (2.2), and also the conditions

$$(1 + \varepsilon)u(x, 0, t) = \int_0^1 N_1(y)u(x, y, t)dy, \quad (x, t) \in \Omega \times (0, T), \tag{3.1}$$

$$(1 + \varepsilon)u(x, 1, t) = \int_0^1 N_2(y)u(x, y, t)dy, \quad (x, t) \in \Omega \times (0, T), \tag{3.2}$$

For fixed (sufficiently small) ε, if $f(x,y,t) \in L_2(Q)$ and the hypotheses of the theorem are fulfilled then this problem has a solution $u(x,y,t)$ such that $u(x,y,t) \in V_0$, $u_{xx}(x,y,t) \in L_2(Q)$ (see [12] and [16]). Moreover, since $f_x(x,y,t) \in L_2(Q)$ and $f(x,y,0) = 0$ for $(x,y) \in \bar{\Omega} \times [0,1]$, we can pass to the problem formally differentiated with respect to t and infer that the solution $u(x,y,t)$ satisfies the memberships $u_t(x,y,t) \in V_0$ and $u_{xxt}(x,y,t) \in L_2(Q)$. We show that the functions $u(x,y,t)$, i.e., the solutions to problem (2.1), (1.2), (1.3), (2.2), (3.1), (3.2), satisfy the desired a priori estimates uniform in ε.

We put

$$w(x,y,t) = u(x,y,t) - \frac{1}{1+\varepsilon}\int_0^1 \tilde{R}_0(y,z)u(x,z,t)dz,$$

$$g(x,y,t) = f(x,y,t) - \frac{1}{1+\varepsilon}\int_0^1 \tilde{R}_0(y,z)f(x,z,t)dz.$$

The function $w(x,y,t)$ satisfies the condition

$$w_t + a(t)w_x - \varepsilon w_{xx} - w_{yy} + cw = g - \frac{1}{1+\varepsilon}\int_0^1 \tilde{R}_0(y,z)w_{zz}(x,z,t)dz \tag{3.3}$$

in the cylinder Q, conditions (1.2), (1.3), (2.2), and also the condition

$$w(x,0,t) = w(x,1,t) = 0, \quad (x,t) \in \Omega \times (0,T). \tag{3.4}$$

Transform the lass summand in (3.3) by integration by parts:

$$\frac{1}{1+\varepsilon}\int_0^1 \tilde{R}(y,z)w_{zz}(x,z,t)dz = -\frac{1}{1+\varepsilon}\int_0^1 \tilde{R}_{0z}(y,z)w_z(x,z,t)dz$$

$$+\frac{1}{1+\varepsilon}\left[\tilde{R}(y,1)w_z(x,1,t) - \tilde{R}(y,0)w_z(x,0,t)\right]. \tag{3.5}$$

Multiply equation (3.3) written down in the variables x, y, τ by the function $w(x,y,\tau) - w_{yy}(x,y,\tau)$ and then integrate the result over the domain $\Omega \times (0,1) \times (0,t)$. After using conditions (1.2), (1.3), (2.2)), and (3.4), taking into account representation (3.5), applying (2.7) and Gronwall's lemma, it is not hard to obtain the estimate

$$\int_0^1\int_\Omega [w^2(x,y,t) + w_y^2(x,y,t)]dxdy + \varepsilon\int_0^t\int_0^1\int_\Omega w_{xy}^2 dxdyd\tau$$

$$+\int_0^t\int_0^1\int_\Omega w_{yy}^2 dxdyd\tau \leqslant K_4\int_Q f^2 dxdydt, \tag{3.6}$$

where K_4 is a constant defined only by the functions $c(x,y,t)$, $N_1(y)$, $N_2(y)$ and also by the number T.

Due to the equality $u_{yy}(x,y,t) = w_{yy}(x,y,t)$, from (3.6) we have the following estimate for $u(x,y,t)$:

$$\int_0^t\int_0^1\int_\Omega u_{yy}^2 dxdyd\tau \leqslant K_4\int_Q f^2 dxdydt, \tag{3.7}$$

Let us return to equation (2.1). After multiplying this equation by the function $u(x,y,\tau)$ and integrating, reckoning with (3.7), we obtain the second a priori estimate

$$\int_0^1\int_\Omega u^2(x,y,t)dxdy+\varepsilon\int_0^t\int_0^1\int_\Omega u_x^2dxdyd\tau+\int_0^t\int_0^1\int_\Omega u_y^2dxdyd\tau\leqslant K_5\int_Q f^2dxdydt, \tag{3.8}$$

where K_5 is a constant defined only by the functions $c(x,y,t)$, $N_1(y)$, $N_2(y)$ and also by the number T.

Now, passing to problem (2.1), (1.2), (1.3), (2.2), (3.1), (3.2) differentiated with respect to t, repeating the above actions, we obtain the estimates

$$\int_0^1\int_\Omega [w_t^2(x,y,t)+w_{yt}^2(x,y,t)]dxdy+\varepsilon\int_0^t\int_0^1\int_\Omega w_{xy\tau}^2dxdyd\tau$$

$$+\int_0^t\int_0^1\int_\Omega w_{yy\tau}^2dxdyd\tau\leqslant K_6\int_Q(f^2+f_t^2)dxdydt, \tag{3.9}$$

$$\int_0^t\int_0^1\int_\Omega w_{yy\tau}^2dxdyd\tau\leqslant K_6\int_Q(f^2+f_t^2)dxdydt, \tag{3.10}$$

$$\int_0^1\int_\Omega u_t^2(x,y,t)dxdy+\varepsilon\int_0^t\int_0^1\int_\Omega u_{x\tau}^2dxdyd\tau+\int_0^t\int_0^1\int_\Omega u_{y\tau}^2dxdyd\tau\leqslant K_7\int_Q(f^2+f_t^2)dxdyd\tau, \tag{3.11}$$

where K_6 and K_7 are constants defined only by the functions $c(x,y,t)$, $N_1(y)$, $N_2(y)$ and also by the number T.

The last estimates — estimates for the derivatives of $u(x,y,t)$ with respect to x — are easy to obtain after multiplying (2.1) by the function $u_x-\varepsilon u_{xx}$ and integrating (with the use of the condition $a(t)\geq a_0>0$).

The above-proven estimates are sufficient for proving the solvability of Boundary Value Problem I. Again using the reflexivity of a Hilbert space and the embedding theorems, it is not hard to demonstrate (see the proof of Theorem 4.1) that there exist sequences $\{\varepsilon_{m_k}\}_{k=1}^\infty$ of positive numbers and $\{u_{m_k}(x,y,t)\}_{k=1}^\infty$ of solutions to the boundary value problem (2.1), (1.2), (1.3), (2.2), (3.1), (3.2) with $\varepsilon=\varepsilon_{m_k}$ such that $\varepsilon_{m_k}\to 0$, $L_{\varepsilon_{m_k}}u\to Lu$, and the limit function $u(x,y,t)$ is a desired solution to Boundary Value Problem I.

The uniqueness of solutions to Boundary Value Problem I in V_0 follows from inequality (3.8), which holds also for $\varepsilon=0$.

The theorem is proved. $\square$

Theorem 3.2. *Suppose the fulfillment of all the hypotheses of Theorem 2.3. Then for any function $f(x,y,t)$ such that $f(x,y,t)\in L_2(Q)$, $f_t(x,y,t)\in L_2(Q)$, $f(x,y,0)=0$ for $x\in\bar\Omega$, $y\in[0,1]$, Boundary Value Problem III has a solution $u(x,y,t)$ in V_0, and this solution is unique.*

The proof of this theorem is carried out by analogy with the proof of Theorem 2.3 with the only difference that the function $\tilde{\tilde{R}}_0(y,z)=yN_2(z)+N_1(z)$ is used instead of the function $\tilde{R}_0(y,z)$.

4. Comments and Appendices

The results obtained in the article mean, on the one hand, that conditions [16], which give the solvability of problems with integral conditions for ultraparabolic equations, can be substantially weakened in some cases, and on the other hand, that the conditions of the present article may also contain some additional smoothness requirements on the input data.

Some of the obtained results can obviously be strengthened. For example, in Boundary Value Problems II and III, conditions (1.6), (1.7) and (1.9) can be replaced by the conditions of Boundary Value Problem III; namely, by the corresponding conditions

$$u_y(x,0,t) + \sigma_1(x,t)u(x,0,t) = \int_0^1 N_1(y)u(x,y,t)dy,$$

$$u_y(x,1,t) + \sigma_2(x,t)u(x,1,t) = \int_0^1 N_2(y)u(x,y,t)dy.$$

The essence of the results on the existence and uniqueness of regular solutions will not change. The function a in (1.1) may also depend on other variables; equation (1.1) may contain also the summand $b(x,y,t)u_y$.

Other generalizations are also possible.

5. Funding

The work was carried out in the framework of the State Task to the Sobolev Institute of Mathematics (Project FWNF–2022–0008).

References

[1] Abdrakhmanov A. M.; Solvability of a boundary-value problem with an integral boundary condition of the second kind for equations of odd order. 2010. URL: `https://link.springer.com/article/10.1134/S000143461007014X`

[2] Bogachev V. I., Krylov N. V., Rckner M., Shaposhnikov S. V.; Uravneniya FokkeraPlankaKolmogorova. M.: Izhevsk, Institut kompyuternykh issledovaniy, 2013. 592 s.

[3] Bogatov A. V., Gilev A. V., Pulkina L. S.; A problem with a non-local condition for a fourth-order equation with multiple characteristics. *Russian Universities Reports. Mathematics*, –2022. –27. –(139). –P. 214230. URL: `https://doi.org/10.20310/2686-9667-2022-27-139-214-230`

[4] Fichera G.; On a unified theory of boundary value problems for elliptic-parabolic equations of second order. *Boundary problems. Differential equations, Univ. Wisconsin Press, Madison, WI*, –1960. –P. 97120

[5] Il'in A. M.; On a class of ultraparabolic equations. *Dokl. Akad. Nauk SSSR*, –1964. –159. –(6). –P. 12141217

[6] Ivanchov N. I.; Boundary value problems for a parabolic equation with integral conditions. *Differ. Uravn.*, –2004. –40. –(4). –P. 547564

[7] Kozhanov A. I., Tsyrempilova I.; Ultraparabolic equations with degeneration. *J. Math. Sci.*, –2024. URL: `https://doi.org/10.1007/s10958-024-07415-3`

[8] Kozhanov A. I.; On the solvability of boundary value problems for quasilinear ultraparabolic equations in some mathematical models of the dynamics of biological systems. 2009. URL: `https://link.springer.com/content/pdf/10.1134/S199047891004006X.pdf`

[9] Kozhanov A. I., Kenzhebay Kh.; Boundary value problems with an integro-differential non-local condition for composite type differential equations of the fourth order. *Chelyab. Fiz.-Mat. Zh.*, –2023. –8. –(4). –P. 516527

[10] Kozhanov A. I., Khromchenko D. S.; Nonlocal problems with an integrally-disturbed A. A. Samarskii condition for third order quasi-parabolic equations. *Math. Notes of NEFU*, –2023. –30. –(4). –P. 1223

[11] Kozhanov A. I., Kosheleva Yu. A.; Linear inverse problems for ultraparabolic equations: the case of an unknown coefficient of space type. *Sib. J. Pure and Appl. Math.*, –2016. –16. –(3). –P. 2739

[12] Kozhanov A. I.; Nelokalnye zadachi i zadachi s integralnymi usloviyami dlya differentsialnykh uravneniy v chastnykh proizvodnykh: svodka rezultatov, nereshenniye zadachi. Novosibirsk: Nauka, 2024. 96 s.

[13] Kozhanov A. I., Abdrakhmanov A. M.; Spatially nonlocal boundary value problems with generalized SamarskiiIonkin condition for quasi-parabolic equations. *Sib. Electron. Math. Rep.*, –2022. –20. –P. 110123

[14] Kozhanov A. I., Popov N. S.; O razreshimosti nekotorykh zadach so smesheniem dlya psevdoparabolicheskikh uravneniy. *Vestn. NGU. Ser. Mat., Mekh., Inform.*, –2010. –10. –(3). –P. 4662

[15] Ladyzhenskaya O. A., Uraltseva N. N.; Linear and nonlinear equations of elliptic type. Moscow: Nauka, 1973

[16] Lukina G. A.; Prostranstvenno-nelokalnye kraevye zadachi dlya ultraparabolicheskikh uravneniy s integralnym granichnym usloviem. *Matematicheskie zametki SVFU*, –2012. –(1). –P. 3443

[17] Mamedov I. G.; Three-dimensional integro-multipoint boundary value problem for loaded Volterra-hyperbolic integro-differential equations of Bianchi type. *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki*, –2012. –1. –P. 820

[18] Oleynik O. A., Radkevich E. V.; Uravneniya s neotritsatelnoy xarakteristikoy formoy. M.: MGU, 2010

[19] Popov N. S.; O razreshimosti kraevykh zadach dlya mnogomernykh parabolicheskikh uravneniy chetvertogo poryadka s nelokalnymi granichnymi usloviyami integralnogo vida. *Matematicheskie SVFU*, –2016. –23. –(1). –P. 7986

[20] Pulkina L. S.; A problem with nonlocal integral first kind conditions for fourth-order partial differential equations. *Vestnik SamU. Estestvenno-Nauchnaya Ser.*, –2024. –30. –(2). –P. 3044

[21] Pulkina L. S.; Problems with nonclassical conditions for hyperbolic equations. Samara: Izdat. Samar. Univ., 2012

[22] Sobolev S. L.; Some applications of functional analysis in mathematical physics. Moscow: Nauka, 1988

[23] Tersenov A. S.; Basic boundary value problems for one ultraparabolic equation. *Sib. Mat. Zh.*, –2001. –42. –(6). –P. 14131430

[24] Triebel H.; Interpolation theory: functional spaces, differential operators. Berlin: VEB Deutscher Verlag Wissenschaften, 1978

[25] Trenogin V. A.; Functional analysis. Moscow: Nauka, 1980

[26] Varlamova G. A., Kozhanov A. I.; Nonlocal problems with integral conditions for hyperbolic equations with two time variables. *Math. Notes of NEFU*, –2023. –30. –(3). –P. 1226

[27] Vladimirov V. S., Drozhzhinov Yu. N.; Generalized Cauchy problem for an ultraparabolic equation. *Math. USSR-Izvestiya*, –1967. –1. –(6). –P. 12851303

[28] Yusubov Sh. Sh., Mamedova J. J.; On the solvability of the equation with dominant mixed derivative with integral boundary conditions. *Bull. Baku State Univ., Ser. Math.*, –2012. –(3). –P. 5762

[29] Zaitseva N. V.; Mixed problems with integral conditions for hyperbolic equations with the Bessel operator. URL: `https://link.springer.com/content/pdf/10.1134/S00122661230130013.pdf`

[30] Zikirov O. S.; Kraevye zadachi teorii uravneniy giperbolicheskogo i sostavnogo tipov tretego poryadka. Tashkent: Universitet, 2022. 200 s.

[31] Zikirov O. S., Sagdullayeva M. M.; Solvability of nonlocal problem with integral condition for third-order equation. *J. Math. Sci.*, –2024. –284. –(2).

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.153-164
DOI: 10.29229/uzmj.2025-2-14

Weighted (m,ψ)-capacity $C_m(K,D,\psi)$ of a condenser (K,D)

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this paper, we introduce the concept of the capacity of a condenser (K,D), with a weighted function $\psi(z)\in C(K)$ for a compact set $K\subset D$, where $D\subset\mathbb{C}^n$ is a domain, in the class of m- subharmonic functions.

Keywords: m-subharmonic function, m-subharmonic measure, weighted (m,ψ)-subharmonic measure, weighted (m,ψ)- capacity.

MSC (2020): 32U05, 32U20.

1. Introduction

Capacity of condensers is one of the important concepts of the potential theory and it has been extensively studied by many researchers. In the works of A. Sadullaev and B. Abdullaev [7], [1], Z. Blocki [2], S. Dinev, S. Kolodziej [3] the concept of condenser capacity were introduced, in the class of m-subharmonic functions, where extensive research was conducted, leading to significant results.

Recall that m- subharmonic functions in the domain $D\subset\mathbb{C}^n$ is defined using the operators

$$(dd^cu)^k\wedge\beta^{n-k},\qquad 1\le k\le n,\tag{1.1}$$

where $d=\partial+\bar\partial$, $d^c=\frac{\partial-\bar\partial}{4i}$ and $\beta=dd^c|z|^2=\frac{i}{2}\sum\limits_{i=1}^{n}dz_i\wedge d\bar z_i$ is standard canonical (1,1) form in $\mathbb{C}^n$. Then $\frac{1}{n!}\beta^n=dV_n$ is volume form in $\mathbb{C}^n$. Operator (1.1) gives the Laplace operator for $k=1$ and the Monge-Ampere operator for $k=n$. The operator (1.1) is called the complex Hessians operator, as it can be shown for $u\in C^2(D)$ that

$$(dd^cu)^k\wedge\beta^{n-k}=k!(n-k)!H_k(u)\beta^n,$$

where

$$H_k(u)=\sum_{1\le j_1<j_2<...<j_k\le n}\lambda_{j_1}\cdot\lambda_{j_2}\cdot...\cdot\lambda_{j_k}$$

is the Hessian of dimension k of the eigenvalue vector $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)$ of the matrix

$$(u_{j,\bar l}),\ \ u_{j,\bar l}=\frac{\partial^2u}{\partial z_j\partial\bar z_l},\ j,l=1,2,...,n.$$

Below, we outline the definition of subharmonic function and highlight its key concepts.

Definition 1.1. Twice smooth function $u(z)\in C^2(D)$ is called m-subharmonic, $u(z)\in sh_m(D)$, if the conditions

$$(dd^cu)^k\wedge\beta^{n-k}\ge 0\,,\ \forall k=1,\,2,...,\,n-m+1$$

holds at each point $z\in D$.

Z. Blocki proved that for all twice differentiable m-subharmonic functions $u,\ v_1,\ v_2,\ ...,\ v_{n-m}$ it is true

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0. \qquad (1.2)$$

Moreover, if a twice differentiable function u satisfies (1.2) for all twice differentiable m-subharmonic functions $v_1,\ v_2,\ \dots,\ v_{n-m}$ then u is necessarily m-subharmonic function [2]. Using this, we can define m-subharmonic functions in the class of the upper semicontinuous functions.

Definition 1.2. A function $u(z) \in L^1_{loc}(D)$ is called m-subharmonic in the domain $D \subset \mathbb{C}^n$, if it is upper semicontinuous and for any twice differentiable m-subharmonic functions $v_1,\ v_2,\ \dots,\ v_{n-m}$ the current

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}$$

defined as

$$[dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}](\omega) =$$
$$= \int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c\omega, \quad \omega \in F^{0,0}(D)$$

is positive, i.e.

$$\int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c\omega \geq 0, \quad \forall\omega \in F^{0,0}(D),\ \ \omega \geq 0.$$

Class of m-subharmonic functions we denote as $sh_m(D)$. It is clear, that

$$psh = sh_1 \subset sh_2 \subset sh_m \subset ... \subset sh_n = sh. \qquad (1.3)$$

Definition 1.3. A set $E \subset D$ is called m-polar in $D \subset \mathbb{C}^n$ if there exist a function $u(z) \in sh_m(D)$, $u(z) \not\equiv -\infty$, such that $u|_E = -\infty$.

(1.3) follows , that a m-polar set is polar in the sense of the classical potential theory, so that for m-polar set $E \subset D$ the Hausdorf measure $H_{2n-2+0}(E) = 0$.

Definition 1.4. A domain $D \subset \mathbb{C}^n$ is called m-regular if there exists a m-subharmonic function $\rho \in sh_m(D)$ such that $\rho|_D < 0$, $\lim\limits_{z\to\partial D}\rho(z) = 0$, i.e. $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$. It is called strongly m-regular if the function $\rho \in sh_m(D^+)\bigcap C^2(D^+)$ and is strongly m-subharmonic in D^+, where D^+ is a neighborhood of the closure $\bar{D}$.

The well-known comparison principle is one of the important inequalities between m-subharmonic functions and the integrals of their Hessians [7]. Since we have made extensive use of the comparison principle in our work, we present it.

Theorem 1.5. *If $u, v \in sh_m(D)\bigcap L^\infty_{loc}(D)$ and the set $F = \{z \in D : u(z) < v(z)\} \subset\subset D$, then*

$$\int\limits_F (dd^c u)^k \wedge \beta^{n-k} \geq \int\limits_F (dd^c v)^k \wedge \beta^{n-k},\ \ 1 \leq k \leq n - m + 1. \qquad (1.4)$$

2. m-subharmonic measure and the capacity of a condenser (K, D).

The m-subharmonic measure is defined as an extremal function in the class of m-subharmonic (sh_m) functions. Let $E \subset D$ be a subset of the domain $D \subset \mathbb{C}^n$. For the sake of simplicity, we assume that D is a bounded and strongly m-regular domain, We denote by $\mathcal{U}(E, D)$ the class of all functions $u \in sh_m(D)$, such that $u|_E \leq -1,\ \ u|_D < 0$ and we define

$$\omega(z, E, D) = \sup\{u(z) :\ u(z) \in \mathcal{U}(E, D)\}.$$

Definition 2.1. The regularization $\omega^*(z,E,D)=\overline{\lim\limits_{w\to z}}\,\omega(w,\,E,D)$ is called the m-subharmonic measure ($\mathcal{P}_m-$ measure) of E with respect to D (see [7], [1]).

Let $D\subset\mathbb{C}^n$ be a domain and $K\subset D$ a compact.

Definition 2.2. A point $z^0\in K$ is said to be globally m-regular if $\omega^*(z^0,K,D)=-1$. It is said to be locally m-regular if for any neighborhood $B\neq\varnothing$, $z^0\in B\subset\mathbb{C}^n$, the intersection $K\bigcap\bar{B}$ is globally m-regular at the point z^0, i.e. $\omega^*(z^0,K\bigcap\overline{B},D)=-1$. If all points of a compact set K are globally (or locally) m-regular, then the compact set K is called to be globally (or locally) m-regular compact.

m-capacity of the condenser (K,D) is introduced as follows, using the operator

$$(dd^cu)^{n-m+1}\wedge\beta^{m-1}.$$

Definition 2.3. Let K be a compact subset of $D\subset\mathbb{C}^n$. The following quantity

$$C_m(K,D)=\inf\left\{\int\limits_D(dd^cu)^{n-m+1}\wedge\beta^{m-1}:u\in sh_m(D)\bigcap C(D),\ u|_K\le-1,\ \varliminf_{z\to\partial D}u(z)\ge0\right\}$$

is called the m-capacity of the condenser (K,D).

Note that, for m-regular compact $K\subset D$ the m-capacity of the condenser is equal

$$C_m(K,D)=\int\limits_K(dd^c\omega^*(z,K,D))^{n-m+1}\wedge\beta^{m-1}=\int\limits_D(dd^c\omega^*(z,K,D))^{n-m+1}\wedge\beta^{m-1}.$$

Moreover, external capacity $C_m^*(E,D)=0$ if and only if E is m-polar set in D.

3. Weighted m-subharmonic measure.

(m,ψ)-subharmonic measure and its properties. Let $D\subset\mathbb{C}^n$ be a strongly m-regular domain, $E\subset D$ be any fixed set and $\psi(z)$ be a bounded and negative function in E. We denote by $\mathcal{U}(E,D,\psi)$ the class of all functions $u(z)\in sh_m(D)$, such that

$$u|_E\le\psi|_E,\ \ u|_D<0.$$

Using this family of functions, we define

$$\omega(z,E,D,\psi)=\sup\{u(z):u(z)\in\mathcal{U}(E,D,\psi)\}.$$

Definition 3.1. The regularization

$$\omega^*\left(z,\,E,\,D,\psi\right)=\overline{\lim_{w\to z}}\,\omega\left(w,\,E,\,D,\psi\right)$$

is called weighted $(m,\psi)-$subharmonic measure ($\mathcal{P}_{(m,\psi)}-$measure) of the set E with respect to D.

Note that $\omega^*(z,E,D,-1)$ $(\psi\equiv-1)$, coincides with the m-subharmonic measure of the potential theory in the class of $u(z)\in sh_m(D)$, i.e. $\omega^*(z,E,D,-1)=\omega^*(z,E,D)$.

As one can see from the definition 3.1, the function $\omega^*(z,\,E,\,D,\psi)$ is m-subharmonic in D. If $0\le\inf\limits_{z\in E}\psi(z)$, then $\omega^*(z,\,E,\,D,\psi)=0$, $\forall z\in D$. Therefore, in this paper, we will consider the special case, where $\sup\limits_{z\in E}\psi(z) < 0$ is satisfied. The weighted (m,ψ)-subharmonic measure satisfies the properties m-subharmonic measure and the inequality

$$-\inf_{z\in E}\psi(z)\cdot\omega^*(z,E,D) \leq \omega^*(z,E,D,\psi) \leq -\sup_{z\in E}\psi(z)\cdot\omega^*(z,E,D) \tag{3.1}$$

holds for any set $E\subset D$ and for all $z\in D$ (see [5]).

$(m,\ \psi)-$ **regularity of compacts.** Let the function $\psi(z)$ be extended to the domain D as a function from the class $\mathcal{U}(E,D,\psi)$ i.e. if there is a function

$$\tilde{\psi}\in sh_m(D), \tilde{\psi}|_E=\psi|_E,\ \tilde{\psi}|_D<0, \tag{3.2}$$

then it is obvious $\omega(z,E,\ D,\psi)\geq\tilde{\psi}(z)$, $\forall z\in D$ and

$$\omega(z,E,\ D,\psi)=\psi(z),\ \forall z\in E. \tag{3.3}$$

However, if condition (3.2) is not satisfied, then, in general, equality (3.3) does not hold, a priori, it may happen that $\omega(z,E,\ D,\ \psi)<\psi(z)$ at some points $z\in E$ (see [5]).

Below we assume that the condition (3.3) satisfied. We also assume, that $D\subset\mathbb{C}^n$ is a strongly m-regular domain and $K\subset D$ is a compact.

Definition 3.2. A point $z^0\in K$ is said to be globally (m,ψ)-regular if $\omega^*(z^0,\ K,D,\psi)=\psi(z^0)$. It is said to be locally (m,ψ)-regular if for any neighborhood B, $z^0\in B\subset\mathbb{C}^n$, the intersection $K\bigcap\bar{B}$ is globally (m,ψ)-regular at the point z^0, i.e. $\omega^*(z^0,\ K\bigcap\overline{B},\ D,\ \psi)=\psi(z^0)$. If all points of a compact set K are globally (or locally) $(m,\psi)-$regular, then the compact set K is called a globally (or locally) $(m,\psi)-$regular compact.

Now we present some important theorems on the (m,ψ)-regularity of a compact K.

Theorem 3.3. *Let K be (m,ψ)-regular compact set and $\psi(z)$ be continuous in the compact K. Then*

$$\omega^*(z,K,D,\psi)\equiv\omega(z,K,D,\psi)\in C(\bar{D})$$

for any $z\in D$.

Theorem 3.4. *Let $\psi\in C(K)$ and condition (3.3) be satisfied, i.e.*

$$\omega(z,K,\ D,\psi)=\psi(z),\forall z\in K.$$

A fixed point $z^0\in K\subset\mathbb{C}^n$ is locally (m,ψ)-regular if and only if it is locally m-regular, $\omega^*(z^0,\ K\bigcap\overline{B},D)=-1\ \ \forall B=B(z^0,r),\ \ r>0.$

Theorem 3.5. *Let the function $\psi(z)$ be continuous in the compact K and extended to $\mathcal{U}(K,D,\psi)$ as a strictly m-subharmonic function in some neighbourhood $D^+\supset\bar{D}$ of the closure $\bar{D}$, i.e, there exists a function $\tilde{\psi}$ such that it is strictly m-subharmonic in the domain D^+ and $\tilde{\psi}|_K=\psi|_K,\tilde{\psi}|_D<0$. Then a fixed point $z^0\in K\subset D$ is locally (m,ψ)-regular if and only if it is globally (m,ψ)-regular.*

Theorem 3.3, Theorem 3.4 and Theorem 3.5 mentioned above were proven in our previous works (see [5], [6]).

From the Theorem 3.4 and the Theorem 3.5, we obtain several important corollaries.

Corollary 3.6. *If the compact set $K\subset D$ is globally (m,ψ)- regular, where the function $\psi(z)$ is extended to $\mathcal{U}(K,\ D,\psi)$ as a strictly m-subharmonic function in some neighbourhood $D^+\supset\bar{D}$ of closure $\bar{D}$, then K is locally m-regular.*

Corollary 3.7. *If ψ_1 and ψ_2 are continuous in the compact K and extended to $\mathcal{U}(K,D,\psi_1)$ and $\mathcal{U}(K,D,\psi_2)$ as strictly m-subharmonic functions in some neighbourhood $D^+ \supset \bar{D}$ of closure $\bar{D}$, respectively, then the point $z^0 \in K \subset D$ is $(m,\psi_1)-$regular if and only if it is $(m,\psi_2)-$regular.*

Corollary 3.8. *If the compact set $K \subset D$ is globally (m,ψ)-regular, where $\psi(z)$ is continuous in the compact K and extended to $\mathcal{U}(K,D,\psi)$ as a strictly m-subharmonic function in some neighbourhood $D^+ \supset \bar{D}$ of closure $\bar{D}$, then K is not m-polar at each of its point. It means that for any $z^0 \in K$ and for any neighborhood $B \subset D$, $z^0 \in B$ the intersection $E = B \bigcap K$ is not m-polar.*

Maximality of (m,ψ)-subharmonic measures. Maximal functions are one of the important concepts of the potential theory and they are analog of harmonic functions in the class of m-subharmonic functions. We remember,

Definition 3.9. A function $u \in sh_m(D)$ is called maximal in the domain $D \subset \mathbb{C}^n$ if it satisfies the dominance principle within the class of m-subharmonic functions, i.e., if $\forall v \in sh_m(D)$: $\varliminf\limits_{z\to\partial D}(u(z)-v(z)) \geq 0$, then $u(z) \geq v(z)$, $\forall z \in D$ (see [7]).

Let $B(0,1) \subset \mathbb{C}^n$ be a ball and $\varphi(\xi)$ be a continuous function defined on the boundary ∂B. Construct the function

$$w(z) = \sup\{u(z):\ u \in \mathcal{U}(\varphi, B)\}$$

using the class of $\mathcal{U}(\varphi,B) = \{u \in sh_m(D)\bigcap C(\bar{D}) : u|_{\partial B} \leq \varphi\}$. Z. Blocki in [2] proved that the function $w(z)$ is continuous and maximal, $w \in sh_m(B)\bigcap C(\bar{B})$, $w|_{\partial B} = \varphi$. Moreover, the operator $(dd^c w)^{n-m+1} \wedge \beta^{m-1} = 0$. We will prove an analog of this theorem on the maximality of the (m,ψ)-subharmonic measure.

Theorem 3.10. *Let $K \subset D$ be (m,ψ)-regular compact set and $\psi(z)$ be a continuous function in the compact set K. Then, the $\mathcal{P}_{(m,\psi)}-$measure is maximal in the open set $D\backslash K$, i.e.,*

$$(dd^c\omega^*(z,K,D,\psi))^{n-m+1} \wedge \beta^{m-1} = 0.$$

Proof. According to Theorem 3.3, $\omega(z,K,D,\psi) \equiv \omega^*(z,K,D,\psi) \in C(\bar{D})$ for any $z \in D$. We fix a ball $B \subset\subset D\backslash K$ and construct the following function

$$v(z) = \sup\{u(z) : u \in sh_m(B)\bigcap C(\bar{B}),\ u|_{\partial B} \leq \omega^*(K,D,\psi)|_{\partial B}\}.$$

Then $v \in sh_m(B)\bigcap C(\bar{B})$ and it is maximal in the ball B, i.e., $(dd^c v)^{n-m+1} \wedge \beta^{m-1} = 0$. Since $v(z) = \omega^*(z,K,D,\psi),\ \forall z \in \partial B$ and it is maximal in B, then

$$v(z) \geq \omega^*(z,K,D,\psi),\ \forall z \in B.$$

Let us define the following function

$$w(z) = \begin{cases} \omega^*(z,K,D,\psi), & z \in D\backslash B \\ v(z), & z \in B. \end{cases}$$

It can be easily seen that according to the above definition we have $w(z) \in sh_m(D)\bigcap C(D)$ and $w(z) \in \mathcal{U}(K,D,\psi)$. As a result,

$$w(z) \leq \omega^*(z,K,D,\psi),\ \forall z \in D.$$

Consequently, we have $v(z) = \omega^*(z,K,D,\psi),\ \forall z \in B$ and $(dd^c\omega^*(z,K,D,\psi))^{n-m+1} \wedge \beta^{m-1} = 0$ in B. From the arbitrariness of the ball $B \subset D\backslash K$, we can conclude that

$$(dd^c\omega^*(z,K,D,\psi))^{n-m+1} \wedge \beta^{m-1} = 0$$

in the open set $D\backslash K$. $\square$

4. The weighted capacity $\mathcal{P}_m(E,D,\psi)$ and (m,ψ)-capacity $C_m(K,D,\psi)$ of a condenser (K,D).

We introduce the weighted capacity value $\mathcal{P}_m(E,D,\psi)$ and (m,ψ)-capacity of the condenser (K,D) using the weighted $\mathcal{P}_{m,\psi)}-$measure.

Let $E\subset D$ be a set and $\psi(z)$ be a bounded, negative and continuous function on E. As mentioned above, we consider the case $\sup\limits_{z\in E}\psi(z)<0$ in constructing the function $\omega^*(z,E,D,\psi)$.

Definition 4.1. The quantity

$$\mathcal{P}_m(E,D,\psi)=-\int\limits_D \omega^*(z,E,D,\psi)dV$$

is called the $\mathcal{P}_{(m,\psi)}-$capacity of the set E with respect to D.

Similarly, $\mathcal{P}_m(E,D)$ capacity (case $\psi(z)=-1$) the function $\mathcal{P}_m(E,D,\psi)\geq 0$ and it satisfies monotonicity, countable subadditivity. Moreover, $\mathcal{P}_m(E,D,\psi)=0$ if and only if E is a m-polar set.

Definition 4.2. Let K be a compact subset of a strongly m-regular domain $D\subset\mathbb{C}^n$. Then the following quantity

$$C_m(K,D,\psi)=\inf\left\{\int\limits_D (dd^cu)^{n-m+1}\wedge\beta^{m-1} : u\in sh_m(D)\bigcap C(D), u|_K\leq\psi|_K,\ \varliminf_{z\to\partial D}u(z)\geq 0\right\}$$

is called (m,ψ)-capacity of the condenser (K,D).

Note that, if $\psi\equiv -1$, then the weighted (m,ψ)-capacity $C_m(K,D,\psi)$ coincides with $C_m(K,D)$, $\ C_m(K,D,-1)=C_m(K,D)$.

In the study of weighted (m,ψ)-capacity $C_m(K,D,\psi)$ for simplicity, we assume that the weight function ψ is continuous, $\psi(z)\in C(E)$, although many of the properties proved below remain valid for the general case of $\psi(z)$.

The (m,ψ)-capacity $C_m(K,D,\psi)$ has the following properties.

Proposition 4.3. *a) if $K_1\subset K_2\subset D$, then $C_m(K_1,D,\psi)\leq C_m(K_2,D,\psi)$.*
b) if $\psi_1\leq\psi_2$, then $C_m(K,D,\psi_1)\geq C_m(K,D,\psi_2)$.

The proof of the monotonicity properties of the condenser $C_m(K,D,\psi)$ follows easily from its definition.

Proposition 4.4. *If $K\subset D$ is a (m,ψ)-regular compact, then*

$$C_m(K,D,\psi)=\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}.$$

Proof. $\omega^*(z,K,D,\psi)\in sh_m(D)\bigcap C(D)$ and according to Theorem 3.10,

$$\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\geq C_m(K,D,\psi).$$

Conversely, $\forall\varepsilon$, $0<2\varepsilon<-\max\limits_{z\in K}\psi(z)$ and for any function

$$u\in sh_m(D)\bigcap C(D),\ u|_K\leq\psi|_K,\ \varliminf_{z\to\partial D}u(z)\geq 0,$$

the set $F=\{z\in D: u(z)<(1+\frac{2\varepsilon}{\max\limits_{z\in K}\psi(z)})\cdot\omega^*(z,K,D,\psi)-\varepsilon\}\subset\subset D$ is open. Therefore, according to the comparison principle (Theorem 1.5),

$$\int\limits_F (dd^c u)^{n-m+1}\wedge\beta^{m-1}\geq (1+\frac{2\varepsilon}{\max\limits_{z\in K}\psi(z)})^{n-m+1}\cdot\int\limits_F (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}.$$

Since $K\subset F$ and $(dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}=0$ in $D\backslash K$, it follows that

$$(1+\frac{2\varepsilon}{\max\limits_{z\in K}\psi(z)})^{n-m+1}\cdot\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}=$$

$$=(1+\frac{2\varepsilon}{\max\limits_{z\in K}\psi(z)})^{n-m+1}\cdot\int\limits_F (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\leq$$

$$\leq\int\limits_F (dd^c u)^{n-m+1}\wedge\beta^{m-1}\leq\int\limits_D (dd^c u)^{n-m+1}\wedge\beta^{m-1}.$$

The arbitrariness of $\varepsilon>0$ implies that

$$\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\leq\int\limits_D (dd^c u)^{n-m+1}\wedge\beta^{m-1}$$

and

$$\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\leq C_m(K,D,\psi).$$

□

Proposition 4.5. *For any compact $K\subset D$,*

$$C_m(K,D,\psi)=\inf\{C_m(E,D,\tilde\psi):\ E\supset K\},$$

where $\tilde\psi\in C(E)$, $\tilde\psi|_K=\psi|_K$ and E is $(m,\tilde\psi)-$regular compact in the domain D.

Proof. For any $\varepsilon>0$, there exists a function $u(z)$ with $u\in sh_m(D)\bigcap C(D), u|_K\leq\psi|_K$, $\varliminf\limits_{z\to\partial D} u(z)\geq 0$ such that the following inequality

$$\int\limits_D (dd^c u)^{n-m+1}\wedge\beta^{m-1}-C_m(K,D,\psi)<\varepsilon$$

holds. Since the function $\psi(z)$ is continuous on the compact set K, according to Whitney's theorem [8], there exists some continuous function $\tilde\psi(z)$ in D such that $\tilde\psi|_K=\psi|_K$. Then, $U=\{z\in D:\ u(z)<\tilde\psi(z)+\varepsilon\}$ is open, $U\supset K$. We take a $(m,\tilde\psi)-$regular compact E such that $E:\quad K\subset E\subset\subset U$ and consider the open set

$$F=\{z\in D:\ u(z)<(1+\frac{2\varepsilon}{\max\limits_{z\in E}\tilde\psi(z)})\cdot\omega^*(z,E,D,\tilde\psi)-\varepsilon\}$$

where $0 < 2\varepsilon < -\max\limits_{z\in E}\tilde{\psi}(z)$. It is not difficult to check that $E \subset F \subset\subset D$. Applying the comparison principle again we have

$$C_m(E, D, \tilde{\psi}) = \int\limits_E \left(dd^c\omega^*(z, E, D, \tilde{\psi})\right)^{n-m+1} \wedge \beta^{m-1} =$$

$$= \int\limits_F \left(dd^c\omega^*(z, E, D, \tilde{\psi})\right)^{n-m+1} \wedge \beta^{m-1} \leq$$

$$\leq \frac{1}{\left(1 + \frac{2\varepsilon}{\max\limits_{z\in E}\tilde{\psi}(z)}\right)^{n-m+1}} \cdot \int\limits_F (dd^c u)^{n-m+1} \wedge \beta^{m-1} \leq$$

$$\leq \frac{1}{\left(1 + \frac{2\varepsilon}{\max\limits_{z\in E}\tilde{\psi}(z)}\right)^{n-m+1}} \int\limits_D (dd^c u)^{n-m+1} \wedge \beta^{m-1} <$$

$$< \frac{1}{\left(1 + \frac{2\varepsilon}{\max\limits_{z\in E}\tilde{\psi}(z)}\right)^{n-m+1}} (C_m(K, D, \psi) + \varepsilon).$$

Thus,

$$C_m(K, D, \psi) \leq C_m(E, D, \tilde{\psi}) < \frac{1}{\left(1 + \frac{2\varepsilon}{\max\limits_{z\in E}\tilde{\psi}(z)}\right)^{n-m+1}} \cdot (C_m(K, D, \psi) + \varepsilon).$$

The arbitrariness of $\varepsilon > 0$ implies that

$$C_m(K, D, \psi) = \inf\{C_m(E, D, \tilde{\psi}) : \ E \supset K\}.$$

□

Proposition 4.6. *If K is a (m, ψ)-regular compact, then*

$$C_m(K, D, \psi) = \sup\left\{\int\limits_K (dd^c u)^{n-m+1} \wedge \beta^{m-1} : u \in sh_m(D) \bigcap C(D),\ \psi|_K \leq u|_K,\ u|_D < 0\right\}.$$

Proof. Since $C_m(K, D, \psi) = \int\limits_K (dd^c\omega^*(z, K, D, \psi))^{n-m+1} \wedge \beta^{m-1}$ and

$$\psi|_K = \omega^*(z, K, D, \psi)|_K,\ \ \omega^*(z, K, D, \psi)|_D < 0,$$

it follows that

$$C_m(K, D, \psi) \leq \sup\left\{\int\limits_K (dd^c u)^{n-m+1} \wedge \beta^{m-1} : u \in sh_m(D) \bigcap C(D),\ \psi|_K \leq u|_K,\ u|_D < 0\right\}.$$

On the other hand, for any function

$$u \in sh_m(D) \bigcap C(D),\ \psi|_K \leq u|_K,\ u|_D < 0, \forall z \in D,$$

we will construct a function v such that

$$v(z) = \max\{(1 + \varepsilon)\omega^*(z, K, D, \psi),\ u(z)\},\ \varepsilon > 0.$$

Then we have

$$v\in sh_m(D)\bigcap C(D),\quad \psi|_K\le v|_K,\quad v|_D<0,\quad v|_K=u|_K,\quad \lim_{z\to\partial D}v(z)=0.$$

Let us now consider the following open set

$$F=\left\{z\in D:\ (1+\varepsilon)\,\omega^*(z,K,D,\psi)+\varepsilon^2<v(z)\right\},$$

where $0<\varepsilon<-\max\limits_{z\in K}\psi(z)$. It is easy to check that $K\subset F\subset\subset D$.

Thus, according to the comparison principle and Theorem 3.10,

$$(1+\varepsilon)^{n-m+1}\int\limits_K[dd^c\omega^*(z,K,D,\psi)]^{n-m+1}\wedge\beta^{m-1}=$$

$$=(1+\varepsilon)^{n-m+1}\int\limits_F[dd^c\omega^*(z,K,D,\psi)]^{n-m+1}\wedge\beta^{m-1}\ge$$

$$\ge\int\limits_F(dd^cv)^{n-m+1}\wedge\beta^{m-1}\ge\int\limits_K(dd^cv)^{n-m+1}\wedge\beta^{m-1}=\int\limits_K(dd^cu)^{n-m+1}\wedge\beta^{m-1}.$$

The arbitrariness of $\varepsilon>0$ implies that

$$\int\limits_K(dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\ge\int\limits_K(dd^cu)^{n-m+1}\wedge\beta^{m-1}.$$

$\square$

Definition 4.7. Let U be an open subset of D. The quantity

$$C_m(U,D,\psi)=\sup\{C_m(K,D,\psi):K\subset U\}$$

is called (m,ψ)- capacity of the open set U.

It follows from the definition of the $(m,\psi)-$ capacity of the open set U and from proposition 4.5 that $C_m(U,D,\psi)=\sup\{C_m(K,D,\psi):\ K\subset U\}$, where K is a $(m,\psi)-$regular compact. The monotonicity properties mentioned above for the compact $K\subset D$ also hold for an open set U. The proof methods used for the corresponding properties in [7] can be applied to prove propositions below.

Proposition 4.8. *If $U\subset D$ is an open set, then*

$$C_m(U,D,\psi)=\sup\left\{\int\limits_U(dd^cu)^{n-m+1}\wedge\beta^{m-1}:u\in sh_m(D)\bigcap C(D),\ \psi|_U\le u|_U,\ u|_D<0\right\}=$$

$$=\sup\left\{\int\limits_U(dd^cu)^{n-m+1}\wedge\beta^{m-1}:u\in sh_m(D)\bigcap C^\infty(D),\ \psi|_U\le u|_U,\ u|_D<0\right\}.$$

Proposition 4.9. *For any increasing sequence of open sets $U_j\subset U_{j+1}\subset D$, $j\in\mathbb{N}$, we have*

$$C_m\left(\bigcup_j U_j,D,\psi\right)=\lim_{j\to\infty}C_m(U_j,D,\psi).$$

Now, we will define the (m,ψ) −external capacity of an arbitrary set $E \subset D$ by using the capacity of open sets.

Definition 4.10. The quantity

$$C_m^*(E,D,\psi) = \inf\{C_m(U,D,\psi):\ U \supset E\}$$

is called the (m,ψ)-external capacity of the set $E \subset D$, where $U \subset D$ is an open set in D.

Proposition 4.11. *(m,ψ) −external capacity is countably subadditive, i.e.*

$$C_m^*\left(\bigcup_j E_j, D, \psi\right) = \sum_j C_m^*\left(E_j, D, \psi\right),\ \ E_j \subset D,\ \forall j \in \mathbb{N}.$$

In [4], it is shown that for $\psi \in \digamma$ and $E \subset\subset D$, there exist positive constants c_1 and c_2 such that the relation

$$c_1 C_m^*\left(E,D,\psi\right) \le \mathcal{P}_m\left(E,D,\psi\right) \le c_2 (C_m^*\left(E,D,\psi\right))^{\frac{1}{n-m+1}}$$

holds. Where

$$\digamma = \digamma\left(D\right) = \Big\{\varphi \in sh_m\left(D\right): \forall z^0 \in D,\ \exists \text{ a neighbourhood } U \ni z^0,\ \ \exists \varphi_j \in sh_m\left(D\right) \bigcap L^\infty\left(D\right),$$

$$\lim_{z\to\partial D} \varphi_j\left(z\right) = 0,\ \forall j \in \mathbb{N},\ \ \varphi_j \searrow \varphi \text{ on } U,\ \sup_j \int_D \left(dd^c\varphi_j\right)^{n-m+1} \wedge \beta^{m-1} < +\infty \Big\}.$$

From this, for $\psi \in \digamma\left(D\right)$ and $E \subset\subset D$, we obtain the important result: $C_m^*\left(E,D,\psi\right) = 0$ if and only if E is m-polar.

The last fact, that the weighted external capacity $C_m^*\left(E,D,\psi\right) = 0$ if and only if E is m-polar, also follows from the following inequality.

Theorem 4.12. *The following inequality holds:*

$$\left(-\inf_{z\in E} \psi\left(z\right)\right)^{n-m+1} C_m^*\left(E,D\right) \ge C_m^*\left(E,D,\psi\right) \ge \left(-\sup_{z\in E} \psi\left(z\right)\right)^{n-m+1} C_m^*\left(E,D\right), \forall E \subset D. \tag{4.1}$$

Proof. From the definition of the (m,ψ) −external capacity, it suffices to prove the inequality (4.1) for (m,ψ) −regular compact sets.

Since $D \subset \mathbb{C}^n$ is a strongly m-regular domain, there exists a function $\rho \in sh_m\left(D\right) \bigcap C^2\left(D\right)$ such that $\rho|_D < 0$ and $\lim\limits_{z\to\partial D} \rho\left(z\right) = 0$. According to inequality (3.1), for any $\varepsilon > 0$ the following holds:

$$-\inf_{z\in E} \psi\left(z\right)\cdot\omega^*\left(z,E,D\right) + 2\varepsilon\rho\left(z\right) < \omega^*\left(z,E,D,\psi\right) + \varepsilon\rho\left(z\right) < -\sup_{z\in E} \psi\left(z\right)\cdot\omega^*\left(z,E,D\right),\ \forall z \in D.$$

The functions $\omega^*\left(z,E,D\right)$, $\rho\left(z\right)$ and $\omega^*\left(z,E,D,\psi\right)$ can be extended to a domain $G \supset\supset D$ such that the following inequality holds for the extended functions:

$$-\inf_{z\in E} \psi\left(z\right)\cdot\tilde{\omega}^*\left(z,E,D\right) + 2\varepsilon\tilde{\rho}\left(z\right) \ge \tilde{\omega}^*\left(z,E,D,\psi\right) + \varepsilon\tilde{\rho}\left(z\right) \ge -\sup_{z\in E} \psi\left(z\right)\cdot\tilde{\omega}^*\left(z,E,D\right),\ \ \forall z \in G\backslash D,$$

where, $\tilde{\omega}^*(z,E,D)$, $\tilde{\rho}(z)$ and $\tilde{\omega}^*(z,E,D,\psi)$ are, respectively, the m-subharmonic and continuous extensions of the functions $\omega^*(z,E,D)$, $\rho(z)$ and $\omega^*(z,E,D,\psi)$ to the domain G. Applying the comparison principe, we obtain:

$$\int\limits_D \left[dd^c\left(-\inf_{z\in E}\psi(z)\,\omega^*(z,E,D)+2\varepsilon\rho(z)\right)\right]^{n-m+1}\wedge\beta^{m-1}\geq$$

$$\geq\int\limits_D \left(dd^c\left(\omega^*(z,E,D,\psi)+\varepsilon\rho(z)\right)\right)^{n-m+1}\wedge\beta^{m-1}$$

$$\geq\int\limits_D \left[dd^c\left(-\sup_{z\in E}\psi(z)\,\omega^*(z,E,D)\right)\right]^{n-m+1}\wedge\beta^{m-1}.$$

It is known that the integrals

$$\int\limits_D \left[dd^c\left(-\inf_{z\in E}\psi(z)\,\omega^*(z,E,D)+2\varepsilon\rho(z)\right)\right]^{n-m+1}\wedge\beta^{m-1}$$

and

$$\int\limits_D \left[dd^c\left(\omega^*(z,E,D,\psi)+\varepsilon\rho(z)\right)\right]^{n-m+1}\wedge\beta^{m-1}$$

can be expressed as,

$$\left(-\inf_{z\in E}\psi(z)\right)^{n-m+1}\int\limits_D \left[dd^c\omega^*(z,E,D)\right]^{n-m+1}\wedge\beta^{m-1}+\varepsilon C_1$$

and

$$\int\limits_D \left[dd^c\omega^*(z,E,D,\psi)\right]^{n-m+1}\wedge\beta^{m-1}+\varepsilon C_2,$$

where C_1 and C_2 are positive constants. According to Theorem 3.10,

$$\left(-\inf_{z\in E}\psi(z)\right)^{n-m+1}\int\limits_E \left[dd^c\omega^*(z,E,D)\right]^{n-m+1}\wedge\beta^{m-1}+\varepsilon C_1\geq$$

$$\geq\int\limits_E \left(dd^c\omega^*(z,E,D,\psi)\right)^{n-m+1}\wedge\beta^{m-1}+\varepsilon C_2\geq$$

$$\geq\left(-\sup_{z\in E}\psi(z)\right)^{n-m+1}\int\limits_E \left[dd^c\omega^*(z,E,D)\right]^{n-m+1}\wedge\beta^{m-1}.$$

Since $\varepsilon>0$ is arbitrary, we conclude that

$$\left(-\inf_{z\in E}\psi(z)\right)^{n-m+1}C_m^*(E,D)\geq C_m^*(E,D,\psi)\geq\left(-\sup_{z\in E}\psi(z)\right)^{n-m+1}C_m^*(E,D),\forall E\subset D.$$

□

Corollary 4.13. *$C_m^*(E,D,\psi)=0$ if and only if E is m-polar.*

Indeed, it is known that the external capacity $C_m^*(E,D)=0$ if and only if E is m-polar set in D. From inequality (4.1), it follows that Corollary 4.13 holds.

References

[1] Abdullayev B., Sadullayev A., Sharipov R.; Currents and theory of m-subharmonic functions. Tashkent. Adast poligraf, –2023.–P.112 (in Russian).

[2] Blocki Z., Weak solutions to the complex Hessian equation. Ann.Inst.Fourier, Grenoble,–2005.–5.–55.–P.1735–1756.

[3] Dinew S., Kolodziej S., A priori estimates for the complex Hessian equation. Analysis and PDE,–2014.–7.–5.–P.227–244.

[4] El Aini H., Elkhadhra F., Complex Hessian operator associated to an m-positive closed current and weighted m-capacity. Complex Analysis and Operator Theory,–2021.–15.–56.–P.1–38.

[5] Kuldoshev K., Narzillayev N., Weighted m-subharmonic measure and $(m,\psi)-$ regularity of compacts. Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences,–2023.–6.–2.–P.76–86.

[6] Kuldoshev K., Narzillayev N., (m,ψ,δ) $-$regularity of compacts in $\mathbb{C}^n$. Acta NUUz,–2024.–2.–2.–1.–P.65–76.

[7] Sadullaev A., Abdullaev B.; Potential theory in the class of m-subharmonic functions. Trudy Mat. Inst. Steklova,–2012.–279.–P.166–192 (in Russian).

[8] Whitney H., Analytic extensions of differentiable functions defined in closed sets. Trans. American Mathematical Society,–1934.–36.–P.63-89.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.165-172
DOI: 10.29229/uzmj.2025-2-15

Generalized Dirichlet-Neumann problem for a fourth-order elliptic equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In a bounded domain, a fourth-order equation with generalized Dirichlet and Neumann conditions is considered. For this problem, the unique solvability of a regular solution is proved.

Keywords: Generalized Dirichlet and Neumann problem, fourth-order elliptic equation, unique solvability of the problem.

MSC (2020): 35J40

1. Statement of the problem and the criterion for its Fredholm property. Let the domain D be bounded by a simple smooth contour Γ and $n(t) = n_1(t) + in_2(t)$ be the inward normal to Γ at the point t. It is assumed that the contour Γ belongs to the class $C^{3,\nu}$ with some $0 < \nu < 1$. The latter means that the function $n(t)$ with respect to the arc length parameter on Γ belongs to the class $C^{2,\nu}$.

Consider in this domain the fourth-order elliptic equation

$$\frac{\partial^4 u}{\partial y^4} - a_1\frac{\partial^4 u}{\partial x^4} - a_2\frac{\partial^4 u}{\partial x^3\partial y} - a_3\frac{\partial^4 u}{\partial x^2\partial y^2} - a_4\frac{\partial^4 u}{\partial x\partial y^3} + \sum_{i+j\le 3} a_{ij}\frac{\partial^3 u}{\partial x^i\partial y^j} = f^0, \tag{1.1}$$

with coefficients $a_i \in \mathbb{R}$ and $a_{ij} \in C^{\mu}(\overline{D})$, $0 < \mu < \nu$, and the right-hand side $f^0 \in C^{\mu}(\overline{D})$. Its solution is sought in the class of functions $u \in C^4(D) \cap C^{3,\mu}(\overline{D})$, in particular, the main part of the differential expression

$$\frac{\partial^4 u}{\partial y^4} - a_1\frac{\partial^4 u}{\partial x^4} - a_2\frac{\partial^4 u}{\partial x^3\partial y} - a_3\frac{\partial^4 u}{\partial x^2\partial y^2} - a_4\frac{\partial^4 u}{\partial x\partial y^3} = f^0 - \sum_{i+j\le 3} a_{ij}\frac{\partial^3 u}{\partial x^i\partial y^j}$$

belongs to the class $C^{\mu}(\overline{D})$. The ellipticity condition is that the characteristic polynomial

$$z^4 - (a_1 + a_2 z + a_3 z^2 + a_4 z^3) = (z-\nu_1)(z-\nu_2)(z-\overline{\nu}_1)(z-\overline{\nu}_2) \tag{1.2}$$

has no real roots. In particular, two cases are possible (i) and (ii), when in the upper half-plane, respectively, there are two different roots ν_1, ν_2 of this polynomial and when there is one multiple root ν.

For equation (1.1) the boundary value problem is posed

$$\left(\frac{\partial^r u}{\partial n^r}\right)^+ = f_1, \quad \left(\frac{\partial^s u}{\partial n^s}\right)^+ = f_2, \tag{1.3}$$

where $0 \le r < s \le 3$, the symbol $+$ means the boundary value on Γ and the normal derivative of the $r-th$ order is understood as the expression

$$\left(n_1\frac{\partial}{\partial x} + n_2\frac{\partial}{\partial y}\right)^r u.$$

For $r=0$ it is natural to call it the generalized Dirichlet problem, and for $r>0$ – the generalized Neumann problem. All six possible boundary conditions can be divided into two groups, the first of which is determined by the $s-r\le 2$ condition, and the second group consists of one problem

$$u^+=f_1,\quad \left(\frac{\partial^3 u}{\partial n^3}\right)^+=f_2. \tag{1.4}$$

In the general case of $2l-$ order equations of the problem type (1.3) from the Fredholm property point of view were studied in [3]. The calculation of the index of problem (1.4) for the case $l=2$ of the fourth-order equation was carried out in [4]. In this paper, we consider similar questions for problem (1.3), including the special case of problem (1.4), where, compared to [4], the calculation of the index of the latter problem is carried out in more detail and in a slightly different way.

Let the contour Γ be oriented so that when moving in the positive direction, the region D remains on the left. Let $e(t)=e_1(t)+ie_2(t)$ be the unit tangent vector to Γ at the point t, directed in the positive direction. In particular, it is related to the normal vector n by the relations $n_1=-e_2,\ n_2=e_1$. When the variable t changes on the contour Γ, the vectors $e(t)$ and $n(t)=ie(t)$ run through the unit circle on the complex plane, which we denote by $\mathbb{T}$.

As shown in [3], problem (1.1), (1.3) is Fredholm equivalent to the problem defined by the boundary condition

$$\left(\frac{\partial^3 u}{\partial e^{3-r}\partial n^r}\right)^+=g_1,\quad \left(\frac{\partial^3 u}{\partial e^{3-s}\partial n^s}\right)^+=g_2,$$

i.e. they have the Fredholm property simultaneously and their indices coincide. In accordance with this, the functions

$$h_1(e,z)=(e_1+e_2z)^{3-r}(-e_2+e_1z)^r,\quad h_2(e,z)=(e_1+e_2z)^{3-s}(-e_2+e_1z)^s \tag{1.5}$$

of the variables $z\in\mathbb{C}$ and $e\in\mathbb{T}$. Following [3], using these functions, respectively for the two cases (i) and (ii) of roots of the characteristic polynomial (1.2), we compose matrices - functions

$$(i)\ G_0(e)=\begin{pmatrix} h_1(e,\nu_1) & h_1(e,\nu_2)\\ h_2(e,\nu_1) & h_2(e,\nu_2)\end{pmatrix},\quad (ii)\ G_0(e)=\begin{pmatrix} h_1(e,\nu) & h_1'(e,\nu)\\ h_2(e,\nu) & h_2'(e,\nu)\end{pmatrix}, \tag{1.6}$$

where differentiation is taken with respect to the variable z. In terms of these matrices, the main result of [3] regarding the Fredholm property of the problem and its index can be formulated as follows.

Theorem 1. *Problem (1.1), (1.3) is Fredholm in the class $C^{3,\mu}(\overline{D})\cap C^4(D)$ if and only if*

$$\det G_0(e)\ne 0,\quad e\in\mathbb{T}, \tag{1.7}$$

and if this condition is satisfied, e index æ is given by the formula

$$æ=-2\mathrm{Ind}\,_\Gamma G+8, \tag{1.8}$$

where $G(t)=G[e(t)],\ t\in\Gamma$ and $\mathrm{Ind}\,G$ *is the Cauchy index – the increment of the continuous branch* $\arg\det G$ *on the contour Γ (in the positive direction) divided by 2π.*

The following lemma about the Cauchy index, intuitively obvious, was used in [3, 4] without proof.

Lemma 1. *Let a complex-valued function $a_0(e)$ be invertible on $\mathbb{T}$ and $a(t)=a_0[e(t)]$. Then*

$$[\arg a]_\Gamma=[\arg a_0]_\mathbb{T}, \tag{1.9}$$

where the increment on the unit circle is taken in accordance with its counterclockwise traversal.

Proof. Consider the conformal mapping α of the unit disk D_0 onto the domain D, subject to the condition

$$\alpha'(0) = -i. \tag{1.10}$$

Under the accepted assumptions regarding the smoothness of the contour Γ, we can use Kellogg's theorem [2], according to which the derivative α' is continuous in the closed disk $\overline{D}_0$ and is everywhere different from zero. The function $t = \alpha(u)$ implements a homeomorphism of $\mathbb{T} \to \Gamma$ with preservation of orientation, so that

$$[\arg a]_\Gamma = [\arg b]_\mathbb{T}, \quad b = a \circ \alpha. \tag{1.11}$$

Under this mapping, the unit tangent vector $e(t),\ t \in \Gamma$, goes to

$$e[\alpha(u)] = iu\frac{\alpha'(u)}{|\alpha'(u)|}, \quad u \in \mathbb{T}.$$

In particular,

$$b(u) = a[\alpha(u)] = a_0[(e \circ \alpha)(u)] = a_0\left[iu\frac{\alpha'(u)}{|\alpha'(u)|}\right].$$

We set

$$c_r(u) = a_0\left[iu\frac{\alpha'(ru)}{|\alpha'(ru)|}\right], \quad u \in \mathbb{T},\ 0 \le r \le 1.$$

Obviously, the function c_r depends continuously on r and, therefore,

$$[\arg c_0]_\mathbb{T} = [\arg c_1]_\mathbb{T}.$$

By (1.10), the function c_0 coincides with a_0 and by definition $c_1 = b$, so that, taking into account (1.11), the previous equality becomes (1.9).

Using Lemma 1, Theorem 1 can be given the following more explicit form.

Theorem 2. *In the case* (i) *for* $s - r \le 2$*, problem (1.1), (1.3) is Fredholm and its index is zero. For* $s - r = 3$*, this problem, i.e. problem (1.4) is Fredholm if and only if* $\nu_1^3 \ne \nu_2^3$ *and the polynomial*

$$P(z) = (q\nu_2 - \nu_1)z^2 + (1 - q)(\nu_1\nu_2 - 1)z + \nu_2 - q\nu_1, \quad q = e^{2\pi i/3}, \tag{1.12}$$

has no real roots and its index

$$æ = 8(2 - k), \tag{1.13}$$

where k *is the number of roots of the polynomial* P *in the upper half-plane.*

In case (ii)*, problem (1.1), (1.3) is always Fredholm and its index is zero.*

Proof for two possible cases of roots of the characteristic polynomial (1.2) we will carry out separately.

(i) According to (1.6)

$$\det G_0(e) = h_1(e, \nu_1)h_1(e, \nu_2)\left[\left(\frac{h_2}{h_1}\right)(e, \nu_1) - \left(\frac{h_2}{h_1}\right)(e, \nu_2)\right].$$

In the notation

$$\omega(e, z) = \frac{-e_2 + e_1 z}{e_1 + e_2 z} \tag{1.14}$$

we can write functions (1.5) as

$$h_1(e,\nu) = (e_1+\nu e_2)^3\omega^r(e,\nu), \quad h_2(e,\nu) = (e_1+\nu e_2)^3\omega^s(e,\nu),$$

so that

$$\det G_0(e) = (e_1+\nu_1 e_2)^3(e_1+\nu_2 e_2)^3\omega^r(e,\nu_1)\omega^r(e,\nu_2)[\omega^{s-r}(e,\nu_2)-\omega^{s-r}(e,\nu_1)]. \tag{1.15}$$

Obviously,

$$\omega^n(e,\nu_2)-\omega^n(e,\nu_1) = \left\{\begin{array}{ll} (\nu_2-\nu_1)[(e_1+\nu_1 e_2)(e_1+\nu_2 e_2)]^{-1}, & n=1, \\ (\nu_2-\nu_1)h_0(e)[(e_1+\nu_1 e_2)(e_1+\nu_2 e_2)]^{-2}, & n=2, \end{array}\right. \tag{1.16}$$

with function $h_0(e) = [(-e_2+\nu_2 e_1)(e_1+\nu_1 e_2)+(-e_2+\nu_1 e_1)(e_1+\nu_2 e_2)]$. Explicitly

$$h_0(e) = (\nu_1+\nu_2)(2e_1e_2\zeta+e_1^2-e_2^2), \quad \zeta = \frac{\nu_1\nu_2-1}{\nu_1+\nu_2}, \tag{1.17}$$

and $\operatorname{Im}\zeta > 0$. Therefore, for $r-s \le 2$, condition (1.7) is satisfied.

As for the case $r=0$, $s=3$, corresponding to problem (1.4), then in accordance with the identity $a^3-b^3 = (a-b)(a-qb)(a-q^{-1}b)$, where $q = e^{2\pi i)/3}$, we have equality

$$\omega^3(e,\nu_2)-\omega^3(e,\nu_1) = (\nu_2-\nu_1)h_+(e)h_-(e)[(e_1+\nu_1 e_2)(e_1+\nu_2 e_2)]^{-3} \tag{1.18}$$

with functions $h_\pm(e) = [(-e_2+\nu_2 e_1)(e_1+\nu_1 e_2)-q^{\pm 1}(-e_2+\nu_1 e_1)(e_1+\nu_2 e_2)]$. Since $\omega^3(e,\nu) = \nu^3$ for $e=1$, the condition $\nu_1^3 \ne \nu_2^3$ is necessary for the invertibility of function (1.15) on the unit circle.

It is easy to verify that

$$h_-(e) = q^{-1}h_+(ie), \quad h_+(e) = e_1^2 P(e_2/e_1) \tag{1.19}$$

with the polynomial P from (1.12). Consequently, (1.7) reduces to the condition $h_+(e) \ne 0,\ e \in \mathbb{T}$, which in turn is equivalent to the fact that the polynomial P has no zeros on the real axis. Note that since $h_+(e) \ne 0$ for $e=1$ and $e=i$, both coefficients $q\nu_2-\nu_1$ and $\nu_2-q\nu_1$ of the polynomial P are nonzero.

Turning to the index formula, we first note that the Cauchy index

$$\operatorname{Ind}(e_1+\nu e_2) = 1. \tag{1.20}$$

Indeed, the left-hand side of this equality depends continuously on ν in the upper half-plane, remaining an integer. Therefore it does not depend on ν and is equal to 1 for $\nu = i$.

It is further asserted that

$$\operatorname{Ind}\omega(e,\nu) = 0, \quad \operatorname{Ind} h_0 = 2. \tag{1.21}$$

Indeed, since $\operatorname{Im}\omega(e,\nu) = |e_1+\nu e_2|^{-2}\operatorname{Im}\nu$, the function ω takes its values in the upper half-plane, where a continuous branch of the argument can be chosen. As a result, we obtain the first equality. By virtue of (1.17) in the second equality h_0 can be replaced by the function $\widetilde{h}_0(e) = 2e_1e_2\zeta+e_1^2-e_2^2$, which is everywhere non-zero on $\mathbb{T}$. In addition, $\operatorname{Ind}\widetilde{h}_0$ depends continuously on ν, remaining an integer. Since $\widetilde{h}_0(e) = (e_1+ie_2)^2$ for $\nu_1 = \nu_2 = i$, the second equality in (1.21) follows from this.

From (1.15), (1.16) and (1.20), (1.21) it follows immediately that for $s-r \le 2$ the Cauchy index $\operatorname{Ind} G_0 = 4$. Together with Lemma 1 and (1.8), we conclude that the index of the problem æ $= 0$. Similarly, for $s - r = 3$, from (1.15) and (1.18), taking into account the first relation in (1.19), we deduce that

$$\operatorname{Ind} G_0 = \operatorname{Ind} h_+ + \operatorname{Ind} h_- = 2\operatorname{Ind} h_+. \tag{1.21}$$

The function h^+ is even, so that

$$\operatorname{Ind}_{\mathbb{T}} h_+ = 2\operatorname{Ind}_{\mathbb{T}^0} h_+, \tag{1.22}$$

where $\mathbb{T}^0$ is a right semicircle. The transformation $e \to e_2/e_1$ implements a homeomorphism of this semicircle onto the entire line $\mathbb{R}$, and going around it counterclockwise corresponds to the positive direction on the line. Since $t^2 + 1 = (e_2/e_1)^2 + 1 = e_1^{-2}$, we can rewrite the second equality in (1.19) in the form $h_+(e) = R(t)$ with the rational function $R(t) = (t^2 + 1)^{-1}P(t)$. By the argument principle [1] from here

$$\operatorname{Ind}_{\mathbb{T}^0} h_+ = \operatorname{Ind}_{\mathbb{R}} R = k - 1,$$

where k is the number of zeros of the polynomial $P(z)$ in the half-plane $\operatorname{Im} z > 0$. Taking into account (1.21), (1.22) from here $\operatorname{Ind} G_0 = 4(k - 1)$, which together with Lemma 1 and (1.8) leads to the index formula (1.13) of problem (1.4).

(*ii*) According to (1.6), in this case

$$\det G_0(e) = h_1^2(e, \nu)\left(\frac{h_2}{h_1}\right)'(e, \nu).$$

Substituting here the expressions

$$h_1(e, \nu) = (e_1 + \nu e_2)^3 \omega^r(e, \nu), \quad h_2/h_1 = \omega^{s-r},$$

taking into account the derivative $\omega'(e, z) = (e_1 + e_2 z)^{-2}$ of function (1.14), we obtain the equality

$$\det G_0(e) = (s - r)(e_1 + \nu e_2)^4 \omega^{s+r-1}(e, \nu).$$

It shows that condition (1.7) is satisfied and $\operatorname{Ind} G_0 = 4$, which, taking into account Lemma 1 and (1.8), leads to the index æ $= 0$ of problem (1.1), (1.3).

Note that due to the condition $\nu_1^3 \neq \nu_2^3$ the coefficients $q\nu_2 - \nu_1$ and $\nu_2 - q\nu_1$ of the polynomial P are nonzero and its roots are given by the formulas

$$z^{\pm} = \frac{(q-1)(\nu_1\nu_2 - 1) \mp i\sqrt{q}\sqrt{3(\nu_1\nu_2 - 1)^2 + 4(\nu_1^2 + \nu_1\nu_2 + \nu_2^2)}}{2(q\nu_2 - \nu_1)},$$

where it is taken into account that $q^2 + q + 1 = 0$. Since $\sqrt{q} = e^{\pi i/3}$, $q - 1 = i\sqrt{3q}$, hence

$$z^{\pm} = \frac{\sqrt{3}(1 - \nu_1\nu_2) \pm \sqrt{3(\nu_1\nu_2 - 1)^2 + 4(\nu_1^2 + \nu_1\nu_2 + \nu_2^2)}}{2i(e^{\pi i/3}\nu_2 - e^{-\pi i/3}\nu_1)}. \tag{1.23}$$

Let $P = P_1$ and, accordingly, denote the roots of (1.23) by $z_1^{\pm}$. Let the polynomial P_2 be constructed similarly with respect to h_-, i.e. $h_-(e) = e_1^2 P_2(e_2/e_1)$. This polynomial is obtained from (1.12) by replacing q with q^{-1}. It is easy to see that it is related to P_1 by the relations

$$z^2 P_1(-1/z) = qP_2(z) = -\widetilde{P}_1(z),$$

where $\widetilde{P}_1$ is obtained from (1.12) by permuting the roots ν_1, ν_2. These relations show that the roots $z_2^{\pm}$ of the polynomial P_2 are obtained from (1.23) by this permutation, i.e.

$$z_2^{\pm} = \frac{\sqrt{3}(1 - \nu_1\nu_2) \pm \sqrt{3(\nu_1\nu_2 - 1)^2 + 4(\nu_1^2 + \nu_1\nu_2 + \nu_2^2)}}{2i(e^{\pi i/3}\nu_1 - e^{-\pi i/3}\nu_2)},$$

and go to $z_1^\pm$ under the transformation $z \to -1/z$, which takes the upper half-plane onto itself. It is easy to see that in the notation of Lemma 1 in [4] the sets $\{z_1^\pm,\, z_2^\pm\}$ and $\{a_1^\pm,\, a_2^\pm\}$ coincide, more precisely, $z_1^\pm = a_2^\pm$, $z_2^\pm = a_1^p m$. Therefore, the number $2(2-k)$ in Theorem 2 coincides with the number n of Lemma 1 in [4]. Accordingly, Theorem 2 agrees with Theorem 1 in [4].

In addition, Theorem 2 in [4] for some special pairs ν_1, ν_2 gives an explicit description of the Fredholm property of problem (1.4) and its index.

Theorem 3 [4]. *Let in the notation (1.2) the pair ν_1, ν_2 belong to one of the following types:*

$$|\nu_1| = |\nu_2| = 1, \tag{i_1}$$

$$\nu_1 = i\rho_1, \quad \nu_2 = i\varrho_2, \tag{i_2}$$

$$\nu_1 = \nu, \quad \nu_2 = -1/\nu, \tag{i_3}$$

and accordingly

$$\delta = \begin{cases} |\nu_1 - \nu_2| - \sqrt{3}, & (i)_1, \\ |\rho_1 - \rho_2| - \sqrt{3}(1 + \rho_1\rho_2), & (i)_2, \\ 4\mathrm{Im}\,\nu - |\nu|^2 - 1, & (i)_3. \end{cases} \tag{1.24}$$

Then the Fredholm property of problem (1.1), (1.4) is equivalent to the condition $\delta \neq 0$ and its index æ $= 0$ for $\delta < 0$ and æ $= 8$ for $\delta > 0$.

As shown by the following lemma, in this theorem with respect to (i_3) one should replace δ in (1.24) with $-\delta$. For pairs (i_1) and (i_2) the proof of this theorem in [4] is established by direct calculations. In the case of (i_3) it is based on Theorem 3 from [5], which is the reason for the error mentioned.

Lemma 2. *Let $\nu_1 = \nu$, $\quad \nu_2 = -1/\nu$, where $\mathrm{Im}\,\nu > 0$ and $\nu \neq i$.*

Then the Fredholm property of problem (1.1), (1.4) is equivalent to the condition

$$\delta = 1 + |\nu|^2 - 4\mathrm{Im}\nu \neq 0 \tag{1.25}$$

and its index is given by the formula

$$æ = \begin{cases} 0, & \delta < 0, \\ 8, & \delta > 0. \end{cases} \tag{1.26}$$

Proof. It is easy to verify that function (1.14) has the property

$$\omega(e, -1/\nu) = -1/\omega(e, \nu).$$

Therefore, in the case under consideration $s - r = 3$, equality (1.5) becomes

$$\det G_0(e) = \frac{(e_1 + e_2\nu)^3(-e_2 + e_2\nu)^3}{\nu^3\omega^3(e,\nu)}[\omega^6(e,\nu) + 1]. \tag{1.27}$$

We denote by Γ_ν the image of the circle $\mathbb{T}$ under the mapping $e \to \omega(e,\nu)$. Since $\omega(e,\nu)$ coincides with

$$\gamma(t) = \frac{\nu - t}{1 + \nu t}, \tag{1.28}$$

for $t = e_2/e_1$, the curve Γ_ν coincides with the image of the line $\mathbb{R}$ under the fractional-linear transformation γ and is a circle located in the upper half-plane.

With respect to the roots $\zeta_s = e^{(\pi i + 2\pi i s)/6}$, $0 \leq s \leq 5$, of the sixth power from -1 we can write

$$\omega^6(e,\nu) + 1 = \prod\nolimits_{s=0}^{5}[\omega(e,\nu) - \zeta_s].$$

In particular, the condition $\zeta_s \notin \Gamma_\nu,\ 0 \le s \le 5$, is necessary and sufficient for the invertibility of the matrix G_0.

Transformation (1.28) is involutive in the sense that the equality $\zeta = \gamma(z)$ is equivalent to $z = \gamma(\zeta)$. Therefore, the point ζ_s does not lie on the circle Γ_ν if and only if

$$\operatorname{Im}\gamma(\zeta_s) = \frac{2\operatorname{Im}\nu - (1+|\nu|^2)\operatorname{Im}\zeta_s}{|1+\nu\zeta_s|^2} \ne 0. \tag{1.29}$$

For three points from the set

$$\zeta_s = \pm i,\ e^{\pm\pi i/6},\ e^{\pm 5\pi i/6},$$

lying in the lower half-plane, and for $\zeta = i$ this condition is certainly satisfied. For points $\zeta = e^{\pi i/6},\ e^{5\pi i/6}$ it is equivalent to the inequality $4\operatorname{Im}\nu - (1+|\nu|^2) \ne 0$, i.e. condition (1.25).

Let this condition be satisfied. From (1.27) taking into account (1.20), (1.21) we have:

$$\operatorname{Ind} G_0 = 6 + \sum_{s=0}^{5} \frac{1}{2\pi}[\arg(\omega(e,\nu) - \zeta_s)]_{\mathbb{T}}. \tag{1.30}$$

The fractional-linear transformation (1.28) takes the upper half-plane to the exterior of the circle Γ_ν, since

$$\operatorname{Im}\gamma(i) = \frac{2\operatorname{Im}\nu - (|\nu|^2+1)}{|1+i\nu|^2} < 0.$$

Therefore, when going around the unit circle $\mathbb{T}$ counterclockwise, the point $\omega(e,\nu)$ goes around Γ_ν twice clockwise. Consequently, the increment

$$[\arg(\omega(e,\nu) - \zeta_s)]_{\mathbb{T}} = [\arg(\gamma(t) - \zeta_s)]_{\mathbb{R}} = -2[\arg(z-\zeta_s)]_{\Gamma_\nu}.$$

Note further that the clockwise increment

$$[\arg(z-\zeta_s)]_{\Gamma_\nu} = \begin{cases} -2\pi, & \text{if } \zeta_s \text{ inside } \Gamma_\nu, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting these expressions into (1.30), we arrive at the equality $\operatorname{Ind} G_0 = 6 - 2r$, where r is the number of points ζ_s lying inside the circle Γ_ν. Accordingly, the index of the problem $æ = 4(r-1)$.

Expression (1.29) for the imaginary part of $\gamma(\zeta_s)$ shows that $r = 1$ for $4\operatorname{Im}\nu > 1 + |\nu|^2$ and $r = 3$ for $4\operatorname{Im}\nu < 1 + |\nu|^2$). Therefore, the equality $æ = 4(r-1)$ goes over to the index formula (1.26).

Comparing the expressions for the index in Theorem 2 and Lemma 2, we come to the conclusion that for $|\nu - 2| < \sqrt{3}$ both roots of (1.24) lie in the upper half-plane, and for $|\nu - 2| > \sqrt{3}$ they lie on both sides of the real line.

Note that for $\nu_1 = \nu, \quad \nu_2 = -1/\nu$, formulas (1.23) can be rewritten as

$$z^\pm = \pm i e^{\pi i/3} \frac{\nu^2 \pm \nu\sqrt{3} + 1}{\nu^2 + e^{2\pi i/3}}. \tag{1.31}$$

Assuming $u_1 = ie^{\pi i/3}$, $u_2 = -ie^{2\pi i/3}$, we obtain

$$z^\pm = u_1 \frac{\nu^2 \pm \nu\sqrt{3} + 1}{(\nu - u_1)(\nu + u_1)} = u_1 \frac{(\nu \pm u_2)(\nu \pm \bar{u}_2)}{(\nu - u_1)(\nu + u_1)}.$$

Obviously, the sets $\pm u_1$, $\pm u_2$, and $\pm\bar{u}_2$ form all six roots of the equation $z^6 = -1$. As in the general situation (1.23), a direct check of the signs of the imaginary part of the numbers (1.31) is difficult.

According to Theorem 3, for the three types of pairs (ν_1, ν_2) specified in it, the index of the problem takes the values 0.8. On the other hand, according to Theorem 2, the value æ $= 16$ is also possible, when both roots of the polynomial P lie in the lower half-plane. It is also realized, for example, for $\nu_1 = i,\ \nu_2 = i + 2$. In other words, for the equation

$$\Delta\left(\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial x^2}\right) = f^0$$

problem (1.4) is Fredholm and its index is æ $= 16$.

References

[1] Bitsadze A.V.; Fundamentals of the theory of analytic functions. Moscow: Nauka. 1984. [in russian]

[2] Goluzin G.M.; Geometric theory of functions of a complex variable. Moscow, Nauka, 1972. [in russian]

[3] Koshanov B., Soldatov A.P.; Boundary value problem with normal derivatives for an elliptic equation on the plane// Differential equations, 2016, Vol. 52, No. 12, pp. 1666-1681 [in russian]

[4] Soldatov A.P.; On a boundary value problem for a fourth-order elliptic equation on the plane// Computational Mathematics and Mathematical Physics. 2022, Vol. 62, No. 4, pp. 616-624 [in russian]

[5] Soldatov A.P.; On the Fredholm property and the index of the generalized Neumann problem, Differential equations, 2020, Vol. 56, 2, P. 217-225 [in russian]

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.173-184
DOI: 10.29229/uzmj.2025-2-16

A nonlocal boundary value problem for a mixed type equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In the present paper, in a certain finite mixed domain for the equation $(signy)|y|^m u_{xx} + u_{yy} + \frac{\alpha_0}{|y|^{1-\frac{m}{2}}} u_x + \frac{\beta_0}{y} u_y = 0$ we study the problem with a nonlocal condition pointwise connecting the value of the unknown function on the boundary and parallel to it internal characteristic, using the fractional order differentiation operator in the sense of Riemann-Liouville. On the segment of the degeneration line, a general conjugation condition and an analogue of the Frankl condition type are specified. The uniqueness of the solution to the problem is proved using the extremum principle of A.V. Bitsadze. The existence of a solution to the problem is proved using the theory of singular integral equations, Wiener-Hopf equations and Fredholm equations of the second kind.

Keywords: Gellerstedt equation, Bitsadze-Samarskii condition, Wiener-Hopf equation, and Fredholm equation, singular coefficients.

MSC (2020): 35M10, 35M12.

1. Statement of the BSF problem (Bitsadze-Samarsky, Frankl).

Let $D = D^+ \bigcup D^- \bigcup I$ -be an unbounded mixed domain on the complex plane $C = \{z = x + iy\}$, where: D^+ -is the half-plane $y > 0$, D^- -is a finite domain of the half-plane $y < 0$, bounded by the characteristics of the equation

$$(signy)|y|^m u_{xx} + u_{yy} + \frac{\alpha_0}{|y|^{1-\frac{m}{2}}} u_x + \frac{\beta_0}{y} u_y = 0, \tag{1.1}$$

emanating from the points $A(-1, 0)$, $B(1, 0)$ and the segment AB of the straight line $y = 0$. By C_0 and C_1 respectively, we denote the intersection points of the characteristics AC and BC with the characteristics emanating from the point $E(c, 0)$, where $c \in I = (-1, 1)-$ is the interval on the axis $y = 0$.

In equation (1.1) it is assumed that m, α_0 and β_0 are some real numbers satisfying the conditions $m > 0$, $|\alpha_0| < (m + 2)/2$, $-m/2 < \beta_0 < 1$.

Note that the constructive, functional and differential properties of solutions of equation (1.1) depend significantly on the numerical parameters α_0 and β_0 at the lower terms of (1.1). On the plane of parameters α_0 and β_0 the triangle is considered $A_0^* B_0^* C_0^*$ bounded by straght lines [8]

$$A_0^* C_0^* : \beta_0 + \alpha_0 = -m/2, \; B_0^* C_0^* : \beta_0 - \alpha_0 = -m/2, \; A_0^* B_0^* : \beta_0 = 1,$$

and depending on the location of the point $P(\alpha_0, \beta_0)$ in this triangle, problems for equation (1.1) are formulated and studied.

Let us consider the case when $P(\alpha_0, \beta_0) \in \Delta E_0^* C_0^* B_0^* \bigcup E_0^* C_0^*$, where $E_0^* = E_0^*(0, 1)$.

In [9] the problem with Bitsadze-Samarskii condition [1] on the boundary characteristic AC and parallel to it internal characteristic EC_0 was investigated in a finite domain. In the present paper the problem is investigated in an unbounded domain, where Bitsadze-Samarskii condition is specified on the part AC_0 of the boundary characteristic AC and parallel to it internal characteristic EC_1, i.e. the part C0C of the boundary characteristic AC is freed from Bitsadze-Samarskii condition and this missing nonlocal condition is replaced by an analogue of Frankl condition [3],[4], [7], [10], [11] on the degeneration segment AB [9]. On the segment AB of the parabolic degeneration line the general conjugation condition of V. I. Zhegalov [22], G. Karatoprakliev [6] is specified.

In the problem with the Bitsadze-Samarskii condition on the boundary characteristic AC and parallel to it internal characteristic EC_0 was investigeted in a finite domain. In the present paper the the problem is investigated in an unbounded domain, where Bitsadze-Samarskii condition is specified on the part AC_0 of the boundary characteristic AC and parallel internal characteristic EC_1 This means that the segment C_0C of the boundary characteristic AC is excluded from the Bitsadze-Samarskii condition, and the missing non-local condition is replaced by an analogue of the Frankl condition [3], [4], [7], [10], [11] on the degeneration segment AB [9]. On the segmentAB of the parabolic degeneration line, a general conjugation condition by V.I. Zhegalov [22], and G. Karatoprakliev [6] is imposed.

Let D_R^+ -be a bounded region obtained by cutting off a portion of the domain D^+ with an arc of a normal curve σ_R having endpoints at the points $A_R = A_R(-R, 0),\ B_R = B_R(R, 0)$. The arc σ_R is defined as follows:

$$\sigma_R : x^2 + 4(m+2)^{-2} y^{m+2} = R^2,\ -R \le x \le R,\ 0 \le y \le ((m+2)R/2)^{2/(m+2)}.$$

This construction forms a finite subdomain bounded by the arc σ_R providing a controlled and restricted portion of the original unbounded domain D^+.

We introduce the following notations: $I = \{(x, y) : -1 < x < 1,\ y = 0\}$ the open interval on the line $y = 0$, $\overline{I}_1 = \{(x, y) : -\infty < x \le -1,\ y = 0\}$ the semi-infinite interval extending to the left on the line $y = 0$, $\overline{I}_2 = \{(x, y) : 1 \le x < +\infty,\ y = 0\}$ the semi-infinite interval extending to the right on the line $y = 0$, Define the domain D_R as follows $D_R = D_R^+ \bigcup D^- \bigcup I,\ D_R$. Here, D_R is a subdomain of the unbounded region D. It combines the bounded domain D_R^+, the region D^-, and the interval I, forming a comprehensive subdomain for further analysis.

We introduce the linear functions $p(x) = ax - b$ and $q(x) = a - bx$, which map the interval $[-1, 1]$ onto the segments $[-1, c]$ and $[c, 1]$ respectively. These functions satisfy the conditions: $p(-1) = -1,\ p(1) = c,\ q(-1) = 1,\ q(1) = c$, The coefficients a and b are determined as follows: $a = (1 + c)/2,\ b = (1 - c)/2$ This mapping approach is based on the works [10, 11] and is used to construct appropriate transformations within the defined intervals.

BSF Problem. Find a function $u(x, y)$ satisfying the following properties:

1) The function $u(x, y)$ is continuous in any subdomain D_R of the unbounded domain D,except on the segment $AB = \{(x, 0) : -1 \le x \le 1\}$ where it may have finite one-sided limits;

2)The function $u(x, y)$ belongs to the space $C^2(D^+)$ and satisfies equation (1.1) in this region;

3) The function $u(x, y)$ is a generalized solution of class R_1 [20], [21] in the domain D^-.

4) On the degeneration interval I, the following general conjugation condition holds [6], [22].

$$u(x, -0) = a_1(x)u(x, +0) + a_2(x),\ x \in [-1, 1], \tag{1.2}$$

$$\lim_{y\to -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} = b_1(x) \lim_{y\to +0} y^{\beta_0} \frac{\partial u}{\partial y} + b_2(x),\ x \in I, \tag{1.3}$$

Moreover, the limits in ((1.3)) at $x = \pm 1$ may have singularities of an order lower than $1 - \alpha - \beta$, where $\alpha = \frac{m+2(\beta_0+\alpha_0)}{2(m+2)}, \quad \beta = \frac{m+2(\beta_0-\alpha_0)}{2(m+2)}, \quad \alpha > 0, \quad \beta > 0, \quad \alpha + \beta < 1$.

5) Equality

$$\lim_{R\to\infty} u(x, y) = 0, \tag{1.4}$$

holds, where $R^2 = x^2 + 4(m+2)^{-2}y^{m+2}$;

6) The function $u(x,y)$ satisfies the following boundary conditions:

$$u(x,y)\big|_{y=0} = \varphi_i(x), \quad \forall x \in \overline{I}_i, \quad i = 1,2; \tag{1.5}$$

$$\mu_0(1+x)^{\alpha} D^{1-\beta}_{-1,x} u\left[\theta\left(p(x)\right)\right] = \mu_1(1-x)^{\alpha} D^{1-\beta}_{x,1} u\left[\theta^*\left(q(x)\right)\right] + \psi(x), \quad x \in [-1,1]; \tag{1.6}$$

$$u\left(p(x), -0\right) - \nu\left(q(x), +0\right) = f(x), \quad x \in [-1,1]; \tag{1.7}$$

where $\mu_0,\ \mu_1$ are some constants such that $\mu_0^2 + \mu_1^2 \neq 0$; $D^{1-\beta}_{-1,x}$ and $D^{1-\beta}_{x,1}$ are fractional-order differential operators [22],

$$\theta(x_0) = \frac{x_0 - 1}{2} - i\left[\frac{(m+2)}{4}(1+x_0)\right]^{2/(m+2)}, \quad x_0 \in [-1,c],$$

is the affix of the intersection point of the characteristic $AC_0 \subset AC$ with the characteristic emerging from the point $M_0(x_0,0)$, $x_0 \in [-1,c]$.

$$\theta^*(x_0) = \frac{x_0 + c}{2} - i\left[\frac{(m+2)}{4}(x_0 - c)\right]^{2/(m+2)}, \quad x_0 \in [c,1],$$

is the affix of the intersection point of the characteristic EC_1 with the characteristic emerging from the point $M_0(x_0,0)$, $x_0 \in [c,1]$.

The functions $\varphi_1(x)$, $\varphi_2(x)$, $\psi(x)$, and $f(x)$ are given and satisfy the following conditions: $\varphi_1(-1) = 0$, $\varphi_2(1) = 0$, $\varphi_1(-\infty) = 0$, $\varphi_2(+\infty) = 0$, $f(1) = 0$; $\psi(x) \in C\left[-1,1\right] \cap C^1(-1,1)$, $\quad f(x) \in C\left[-1,1\right] \cap C^1(-1,1)$

The functions $\varphi_i(x)$ are continuously differentiable on any intervals $[-N,-1]$ and $[1,N]$ and satisfy the inequality for sufficiently large $|x|$:

$$|\varphi_i(x)| \leq M|x|^{-\delta_0}, \delta_0 = const > 0$$

where δ_0 is a positive constant.

The Bitsadze-Samarskii condition (1.6) is set on the part AC_0 (where $\theta\left(p(x)\right) \in AC_0$), the boundary characteristic AC, and the internal characteristic EC_1 (where $\theta^*\left(q(x)\right) \in EC_1$).

The condition (1.7) (where $-1 \leq p(x) \leq c,\ c \leq q(x) \leq 1$) is an analog of the Frankl condition on the segments $[-1,c]$ and $[c,1]$ of the degeneration line AB.

Let's introduce the following notations:

$$\tau^-(x) = u(x,-0), \quad \nu^-(x) = \lim_{y\to-0}(-y)^{\beta_0}\frac{\partial u}{\partial y}, \tag{1.8}$$

$$\tau(x) = u(x,+0), \quad \nu(x) = \lim_{y\to+0} y^{\beta_0}\frac{\partial u}{\partial y}, \tag{1.9}$$

According to notations (1.10) and (1.11), the gluing conditions (1.2), (1.3), and Frankl's condition take the following forms:

$$\tau^-(x) = a_1(x)\tau(x) + a_2(x), \quad x \in [-1,1], \tag{1.10}$$

$$\nu^-(x) = b_1(x)\nu(x) + b_2(x), \quad x \in (-1,1)/\{c\}, \tag{1.11}$$

$$\tau^-\left(p(x)\right) - \tau^-\left(q(x)\right) = f(x),\ x \in [-1,1]. \tag{1.12}$$

2. Uniqueness of the solution to the BSF problem

The solution of equation ((1.1)) in the region D^- satisfying the modified Cauchy initial data:

$$u(x,-0)=\tau^-(x), x\in \overline{I};\ \lim_{y\to -0}(-y)^{\beta_0}\frac{\partial u}{\partial y}=\nu^-(x),\ x\in I,$$

is given by the Darboux formula [20]:

$$u(x,y)=\gamma_1\int_{-1}^{1}\tau^-\left[x+\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1-t)^{\alpha-1}(1+t)^{\beta-1}dt+$$

$$+\gamma_2(-y)^{1-\beta_0}\int_{-1}^{1}\nu^-\left[x+\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1-t)^{-\beta}(1+t)^{-\alpha}dt,$$

where $\gamma_1=\frac{\Gamma(\alpha+\beta)2^{1-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)},\ \gamma_2=-\frac{\Gamma(2-\alpha-\beta)2^{\alpha+\beta-1}}{(1-\beta_0)\Gamma(1-\alpha)\Gamma(1-\beta)}$.

By virtue of the formula Darboux and boundry condition ((1.7)), we obtain the following form

$$\mu_0a^{1-\alpha-\beta}\nu^-(p(x))-\mu_1b^{1-\alpha-\beta}\nu^-(q(x))=$$
$$=\gamma[\mu_0D^{1-\alpha-\beta}_{-1,x}\tau^-(p(x))-\mu_1D^{1-\alpha-\beta}_{x,1}\tau^-(q(x))]+\Psi_0(x),\quad x\in I, \tag{2.1}$$

where $\gamma=\left(\frac{m+2}{4}\right)^{\alpha+\beta-1}\frac{\Gamma(\alpha+\beta)\Gamma(1-\alpha)}{\Gamma(2-\alpha-\beta)},\quad \Psi_0(x)=\left(\frac{m+2}{4}\right)^{\alpha+\beta-1}\frac{(\beta_0-1)\Gamma(1-\alpha)\psi(x)}{\Gamma(2-\alpha-\beta)}$.

Based on conditions (1.10) and (1.11), the relation (2.1) can be written as

$$\mu_0a^{1-\alpha-\beta}b_1(p(x))\nu\left(p(x)\right)-\mu_1b^{1-\alpha-\beta}b_1(q(x))\nu\left(q(x)\right)=$$

$$=\gamma\left[\mu_0D^{1-\alpha-\beta}_{-1,x}a_1(p(x))\tau\left(p(x)\right)-\mu_1D^{1-\alpha-\beta}_{x,1}a_1(q(x))\tau\left(q(x)\right)\right]+\Psi_1(x),\ x\in(-1,1), \tag{2.2}$$

where

$$\Psi_1(x)=-\mu_0a^{1-\alpha-\beta}b_2(p(x))+\mu_1b^{1-\alpha-\beta}b_2(p(x))+\gamma[\mu_0D^{1-\alpha-\beta}_{-1,x}a_2(p(x))-\mu_1D^{1-\alpha-\beta}_{x,1}a_2(q(x))]+\Psi_0(x).$$

The relation (2.2) is the first functional relationship between the unknown functions $\tau(x)$ and $\nu(x)$ introduced on the interval $(-1,1)$ of the axis $y=0$ from the region D^-.

Uniqueness of the solution to the BSF problem.

Theorem 2.1. *Let $\varphi_1(x)\equiv 0,\ \varphi_2(x)\equiv 0,\ \psi(x)\equiv 0,\ f(x)\equiv 0$*

$$\mu_0>0,\ \mu_1<0,\ a_1(x)>0,\ b_1(x)>0, \tag{2.3}$$

then the solution $u(x,y)$ of the BSF problem attains its greatest positive value (GPV) and least negative value (LNV) in the region $\overline{D}^+_R$ on the curve $\bar{\sigma}_R$.

Proof. By virtue of the Hopf principle [2], the solution $u(x,y)$ does not attain its GPV (greatest positive value) or LNV (least negative value) at interior points (x_0,y_0) of the region D^+_R. Suppose that the function $u(x,y)$ attains its GPV in the region $\bar{D}^+_R$ at some interior point $(x_0,0)$ of the segment AB.

Here, we separately consider two possible cases for the location of the point x_0.

1. Let $x_0 \in (-1, c]$, $x_0 = p(\xi_0)$. Then, by virtue of the corresponding homogeneous condition (1.8) ($c\ f(x) \equiv 0$), the solution $u(x, y)$ attains its greatest positive value (GPV) at two points $(p(\xi_0), 0)$ and $(q(\xi_0), 0)$. Consequently, at these points $\nu\,(p(\xi_0)) < 0$, $\nu\,(q(\xi_0)) < 0$ [20].

From this, in view of (2.3), it follows that

$$\mu_0 a^{1-\alpha-\beta} b_1(p(\xi_0))\nu\,(p(\xi_0)) - \mu_1 b^{1-\alpha-\beta} b_1(q(\xi_0))\nu\,(q(\xi_0)) < 0, \ \xi_0 \in (-1, 1), \tag{2.4}$$

On the other hand, it is well known that at the point of the positive maximum of the function $\tau(x)$, for fractional differentiation operators, the inequalities $D^{1-\alpha-\beta}_{-1,x} a_1(p(x))\tau\,(p(x))\,|_{x=x_0} > 0$, $D^{1-\alpha-\beta}_{x,1} a_1(q(x))\tau\,(q(x))\,|_{x=x_0} > 0$ hold. Then, by virtue of (2.3)

$$\mu_0 D^{1-\alpha-\beta}_{-1,x} a_1(p(x))\tau\,(p(x))\,|_{x=\xi_0} - \mu_1 D^{1-\alpha-\beta}_{x,1} a_1(q(x))\tau\,(q(x))\,|_{x=\xi_0} > 0.$$

From this, we conclude that the left-hand side of the corresponding homogeneous relation (2.2) ($c\ \Psi_1(x) \equiv 0$) is strictly positive, which contradicts inequality (2.4). Therefore, $x_0 = p(\xi_0) \notin (-1, c]$.

2. Let $x_0 \in [c, 1)$, $x_0 = q(\eta_0)$. By arguments similar to those in the case of $x_0 \in (-1, c]$, we conclude that $x_0 = q(\eta_0) \notin [c, 1)$.

Thus, the solution $u(x, y)$, satisfying the conditions of Theorem 2.1, does not attain its greatest positive value (GPV) at interior points of the interval $(-1, 1)$ along the axis $y = 0$.

By virtue of the corresponding homogeneous boundary conditions (1.5) ($c\ \varphi_1(x) \equiv 0$, $\varphi_2(x) \equiv 0$), the function $u(x, y)$ does not attain its GPV at points of the segments $[-R, -1] \cup [1, R]$ either. Consequently, from the previous reasoning, it follows that $(x_0, y_0) \in \overline{\sigma}_R$.

Similarly, as above, it can also be shown that the point (x_0, y_0), where the solution $u(x, y)$ attains its least negative value (LNV) in the region D^+_R, belongs to $\overline{\sigma}_R$, i.e., $(x_0, y_0) \in \overline{\sigma}_R$. Theorem 2.1 is proved. □

From Theorem 2.1, it follows:

Corollary 2.1 *The solution $u(x, y)$, satisfying the conditions of Theorem 2.1, is identically zero in the region $\overline{D}^+$.*

Proof. The solution to the BSF problem, under the conditions of Theorem 2.1, attains its GPV and LNV in the region $\overline{D}^+_R$ at points of the normal curve $\overline{\sigma}_R$. By virtue of (1.4), for any $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon)$ such that for $R > R_0(\varepsilon)$, the inequality $|u(x, y)| < \varepsilon$, $(x, y) \in \sigma_R$ holds. Consequently, by Theorem 2.1,

$$|u(x, y)| < \varepsilon, \ \forall (x, y) \in \overline{D}^+_R. \tag{2.5}$$

From this, due to the arbitrariness of ε, as $R \to +\infty$, we conclude that $u(x, y) \equiv 0$ in the region $D^+ \cup I_1 \cup I \cup I_2$. Corollary 2.1 is proved. □

Corollary 2.2. *The BSF problem, under the conditions of Theorem 2.1, has at most one solution.*

Proof. By virtue of Corollary 2.1, taking into account the conjugation condition (1.2), (1.3), we have

$$\lim_{y\to -0} u(x, y) \equiv 0, \ \forall\ x \in \bar{I}; \ \lim_{y\to -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} \equiv 0, \ x \in I. \tag{2.6}$$

Now, in the region $\overline{D}^-$, reconstructing the solution $u(x, y)$ using the Darboux formula with zero data (2.6), we obtain that $u(x, y) \equiv 0$ also in the region $\overline{D}^-$. Corollary 2.2 is proved. □

Thus, the uniqueness of the solution to the BSF problem is established.

3. Existence of the solution BSF problem.

Theorem 3.1. *The BSF problem, under the conditions* (2.3) *and*

$$\frac{-\lambda\pi^2\sqrt{b}}{\lambda_0\sqrt{a}\sin(\delta\pi)}\left[\mu_0\frac{a^{2-4a_0}}{b^{1+\delta}e^{b_0\pi}}-\mu_1\frac{b^{1-4a_0}}{a^{\delta}e^{-b_0\pi}}\right]<1, \tag{3.1}$$

where

$$\lambda_0=\mu_0\left[\pi e^{-b_0\pi}\cot(2a_0\pi)-e^{b_0\pi}\Gamma(2a_0)\Gamma(1-2a_0)-\gamma_0\Gamma(1-2a_0)\right]+$$

$$+\mu_1\left[\pi e^{b_0\pi}\cot(2a_0\pi)-e^{-b_0\pi}\Gamma(2a_0)\Gamma(1-2a_0)+\gamma_0\Gamma(1-2a_0)\cos(2a_0\pi)\right]\neq 0,$$

$\lambda=-\left(\mu_0e^{-b_0\pi}-\mu_1e^{b_0\pi}+\mu_1\gamma_0\left(\Gamma(2a_0)\right)^{-1}\right)/\lambda_0,$ $a_1(x)=a_1=const$ *and* $b_1(x)=b_1=const,$ *is uniquely solvable.*

Let us show that the set of numerical parameters of the BSF problem satisfying inequality (3.1) is nonempty. Indeed, assuming in (3.1) that $\mu_1=-a^{2-4a_0+\delta}$, we obtain

$$-\frac{\lambda\pi^2\sqrt{b}a^{2-4a_0-0.5}}{\lambda_0\sin(\delta\pi)}\left[\mu_0\frac{e^{-b_0\pi}}{b^{1+\delta}}+b^{1-4a_0}e^{b_0\pi}\right]<1, \tag{3.2}$$

if $2-4a_0-0.5>0$ (i.e., $\beta_0<\frac{6-m}{8}$), then for sufficiently close values of the numerical parameter c to minus one, the factor $a^{2-4a_0-0.5}=\left(\frac{1+c}{2}\right)^{2-4a_0-0.5}$ in (3.2) will be sufficiently small, and the inequality (3.2), i.e., (3.1), will hold for such values of the parameter c.

3.1. Derivation of a singular integral equation with non-Fredholm operators in the non-characteristic part of the equation with respect to the unknown function $\tau_1(x)$.

The solution to the Dirichlet problem in the half-plane $y\geq 0$, satisfying the condition

$$u(x,+0)=\tau(x),\ x\in(-\infty,+\infty), \tag{3.3}$$

is given by the formula [15]

$$u(x,y)=k_2(1-\beta_0)y^{1-\beta_0}\int\limits_{-\infty}^{+\infty}\tau(t)(r_0^2)^{a_0-1}\times$$

$$\times\exp\left(-2b_0arcsin\frac{t-x}{r_0}\right)dt,\ -\infty<x<+\infty,\ y\geq 0, \tag{3.4}$$

where $k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{1-2a_0}\frac{\Gamma(1-\delta)\Gamma(1-\bar{\delta})}{\Gamma(2-\delta-\bar{\delta})},\quad r_0^2=(x-t)^2+\frac{4y^{m+2}}{(m+2)^2}, 2a_0=\alpha+\beta,\ \delta=a_0+b_0i,\ b_0=-\frac{\alpha_0}{m+2}.$

We can write the following relationship between the unknown functions $\tau(x)$ and $\nu(x)$

$$\nu(x)=-k_2(1-\beta_0)\frac{m+2}{2}\int\limits_{-1}^{1}\tau'(t)\left[\frac{x-t}{|x-t|^{2-2a_0}}\cdot\exp\left(-2b_0\arcsin\frac{t-x}{|t-x|}\right)\right]dt+$$

$$+\Phi(x),\ x\in(-1,1), \tag{3.5}$$

where

$$\Phi(x) = -k_2(1-\beta_0)\frac{m+2}{2}\left(e^{b_0\pi}\int\limits_{-\infty}^{-1}\frac{\tau_1'(t)dt}{(x-t)^{1-2a_0}} - e^{-b_0\pi}\int\limits_{1}^{+\infty}\frac{\tau_2'(t)dt}{(t-x)^{1-2a_0}}\right).$$

Note that relation (3.5) holds for the entire interval $I=(-1,1)$.

By virtue of (3.5), eliminating $\nu\left(p(x)\right)$ and $\nu\left(q(x)\right)$ from relation (2.2)), we obtain

$$-\mu_0 k_2(1-\beta_0)\frac{m+2}{2}a^{1-2a_0}b_1\int_{-1}^{1}\tau'(t)\left[\frac{p(x)-t}{|p(x)-t|^{2-2a_0}}\exp\left(-2b_0\arcsin\frac{t-p(x)}{|t-p(x)|}\right)\right]dt+$$

$$+\mu_1 k_2(1-\beta_0)\frac{m+2}{2}b^{1-2a_0}b_1\int_{-1}^{1}\tau'(t)\left[\frac{q(x)-t}{|q(x)-t|^{2-2a_0}}\exp\left(-2b_0\arcsin\frac{t-q(x)}{|t-q(x)|}\right)\right]dt=$$

$$=\gamma\left[\mu_0 D_{-1,x}^{1-2a_0}a_1\tau\left(p(x)\right)-\mu_1 D_{x,1}^{1-2a_0}a_1\tau\left(q(x)\right)\right]+\Psi_2(x),\ x\in(-1,1), \tag{3.6}$$

where

$$\Psi_2(x)=\Psi_1(x)-\gamma\left[\mu_0 D_{-1,x}^{1-2a_0}a_2(p(x))-\mu_1 D_{x,1}^{1-2a_0}a_2(q(x))\right]-\mu_0 a^{1-2a_0}b_1\Phi\left(p(x)\right)+$$

$$+\mu_1 b^{1-2a_0}b_1\Phi\left(q(x)\right)-\mu_0 a^{1-2a_0}b_2(p(x))+\mu_1 b^{1-2a_0}b_2(q(x)).$$

Taking into account the inequalities $-1\le p(x)\le c,\ c\le q(x)\le 1$, the first and second integrals of the integro-differential equation (3.6) are respectively split into three integrals over the intervals $(-1,p(x))\,,\ (p(x),c)\,,\ (c,1)$ and $(-1,c),\ (c,q(x))\,,\ (q(x),1)$, and we rewrite it as

$$-\mu_0 a^{1-2a_0}e^{b_0\pi}\int\limits_{-1}^{p(x)}\frac{\tau'(t)dt}{(p(x)-t)^{1-2a_0}}+\mu_0 a^{1-2a_0}e^{-b_0\pi}\int\limits_{p(x)}^{c}\frac{\tau'(t)dt}{(t-p(x))^{1-2a_0}}+$$

$$+\mu_0 a^{1-2a_0}e^{-b_0\pi}\int\limits_{c}^{1}\frac{\tau'(t)dt}{(t-p(x))^{1-2a_0}}+\mu_1 b^{1-2a_0}e^{b_0\pi}\int\limits_{-1}^{c}\frac{\tau'(t)dt}{(q(x)-t)^{1-2a_0}}+$$

$$+\mu_1 b^{1-2a_0}e^{b_0\pi}\int\limits_{c}^{q(x)}\frac{\tau'(t)dt}{(q(x)-t)^{1-2a_0}}-\mu_1 b^{1-2a_0}e^{-b_0\pi}\int\limits_{q(x)}^{1}\frac{\tau'(t)dt}{(t-q(x))^{1-2a_0}}=$$

$$=\gamma_0\left[\mu_0 D_{-1,x}^{1-2a_0}a_1\tau\left(p(x)\right)-\mu_1 D_{x,1}^{1-2a_0}a_1\tau\left(q(x)\right)\right]+\Psi_3(x),\ x\in(-1,1), \tag{3.7}$$

where $\gamma_0=\frac{2\gamma}{a_1b_1k_2(1-\beta_0)(m+2)},\ \ \Psi_3(x)=\frac{2\Psi_2(x)}{b_1k_2(1-\beta_0)(m+2)}$.

In the integrals of (3.7) over the intervals $(-1,p(x))$, $(p(x),c)$, $(-1,c)$, by making the substitution of the integration variable $t=p(s)$, and in the integrals over the intervals $(c,q(x))$, $(q(x),1)$, $(c,1)$, by making the substitution $t=q(s)$, while taking into account the relation $a\tau^{-'}(p(x))=-b\tau^{-'}(q(x))+f'(x)$ equation (3.7) is rewritten in the form

$$\left(\mu_0 e^{b_0\pi}-\mu_1 e^{-b_0\pi}\right)\left(\int\limits_{-1}^{x}\frac{dt}{(x-t)^{2a_0}}\int\limits_{-1}^{t}\frac{b\tau'\left(q(s)\right)ds}{(t-s)^{1-2a_0}}\right)-\left(\mu_0 e^{-b_0\pi}-\mu_1 e^{b_0\pi}\right)\cdot$$

$$\cdot\left(\int\limits_{-1}^{x}\frac{dt}{(x-t)^{2a_0}}\int\limits_{t}^{1}\frac{b\tau'\left(q(s)\right)ds}{(s-t)^{1-2a_0}}\right)+\mu_0 a^{1-2a_0}e^{-b_0\pi}\left(\int\limits_{-1}^{x}\frac{dt}{(x-t)^{2a_0}}\int\limits_{-1}^{1}\frac{b\tau'\left(q(s)\right)ds}{(1-at-bs)^{1-2a_0}}\right)-$$

$$-\mu_1 b^{1-2a_0} e^{b_0\pi}\left(\int\limits_{-1}^{x}\frac{dt}{(x-t)^{2a_0}}\int\limits_{-1}^{1}\frac{b\tau'\left(q(s)\right)ds}{(1-bt-as)^{1-2a_0}}\right)=$$

$$=\gamma_0\Gamma(1-2a_0)\left[\mu_0 D_{-1,x}^{2a_0-1}D_{-1,x}^{1-2a_0}\tau\left(q(x)\right)-\mu_1 D_{-1,x}^{2a_0-1}D_{x,1}^{1-2a_0}\tau\left(q(x)\right)\right]+$$

$$+\Psi_4(x),\ x\in(-1,1). \tag{3.8}$$

By virtue of the following easily provable identities

$$1.\int\limits_{-1}^{x}\frac{dt}{(x-t)^{2a_0}}\int\limits_{-1}^{t}\frac{b\tau'\left(q(s)\right)ds}{(t-s)^{1-2a_0}}=-\frac{\pi}{sin(2a_0\pi)}\tau\left(q(x)\right).$$

$$2.\int\limits_{-1}^{x}\frac{dt}{(x-t)^{2a_0}}\int\limits_{t}^{1}\frac{b\tau'\left(q(s)\right)ds}{(s-x)^{1-2a_0}}=-\pi\cot(2a_0\pi)\tau\left(q(x)\right)-\int\limits_{-1}^{1}\left(\frac{1+x}{1+t}\right)^{1-2a_0}\frac{\tau\left(q(t)\right)dt}{t-x}$$

$$3.\int\limits_{-1}^{x}\frac{dt}{(x-t)^{2a_0}}\int\limits_{-1}^{1}\frac{b\tau'\left(q(s)\right)ds}{(1-at-bs)^{1-2a_0}}=\int\limits_{-1}^{1}\left(\frac{1+x}{1+a-bs}\right)^{1-2a_0}\frac{b\tau\left(q(s)\right)ds}{1-ax-bs}$$

$$4.\int\limits_{-1}^{x}\frac{dt}{(x-t)^{2a_0}}\int\limits_{-1}^{1}\frac{b\tau'\left(q(s)\right)ds}{(1-bt-as)^{1-2a_0}}=\int\limits_{-1}^{1}\left(\frac{1+x}{1+b-as}\right)^{1-2a_0}\frac{a\tau\left(q(s)\right)ds}{1-bx-as}$$

we rewrite equation (3.8) in the form

$$\tau_1(x)-\lambda\int\limits_{-1}^{1}\left(\frac{1+x}{1+t}\right)^{1-2a_0}\frac{\tau_1(t)dt}{t-x}=\lambda_1\int\limits_{-1}^{1}\frac{\tau_1(s)ds}{1-ax-bs}+\lambda_2\int\limits_{-1}^{1}\frac{\tau_1(s)ds}{1-bx-as}+$$

$$+R_1[\tau_1]+\Psi_5(x),x\in(-1,1), \tag{3.9}$$

where $\tau_1(x)=\tau(q(x))$,

$$\lambda_0=-\frac{\pi}{sin(2a_0\pi)}(\mu_0 e^{b_0\pi}-\mu_1 e^{-b_0\pi})+\pi ctg(2a_0\pi)(\mu_0 e^{-b_0\pi}-\mu_1 e^{b_0\pi})+$$

$$+\gamma_0\mu_1\Gamma(1-2a_0)cos((1-2a_0)\pi),$$

$$\lambda=\frac{\mu_0 e^{-b_0\pi}-\mu_1 e^{b_0\pi}-\gamma_0\mu_1\Gamma(1-2a_0)}{\gamma_0},\quad \lambda_1=-\mu_0 e^{-b_0\pi}/\lambda_0,\quad \lambda_2=\mu_1 e^{b_0\pi}/\lambda_0,$$

$$R_1[\tau_1]=\lambda_1 a^{1-2b_0}\int\limits_{-1}^{1}\left[\left(\frac{1+x}{1+a-bs}\right)^{1-2a_0}-\left(\frac{1}{a}\right)^{1-2a_0}\right]\frac{\tau_1(s)ds}{1-ax-bs}+$$

$$+\lambda_2 b^{1-2a_0}\int\limits_{-1}^{1}\left[\left(\frac{1+x}{1+b-as}\right)^{1-2a_0}-\left(\frac{1}{b}\right)^{1-2a_0}1\right]\frac{\tau_1(s)ds}{1-bx-as},$$

-a regular operator, $\Psi_5(x)=\Psi_4(x)/\lambda_0$ -is a known function.

The equation (3.9) is a singular integral equation with a non-Fredholm operator on the right-hand side of the equation, because, due to the equality $a+b=1$, these kernels have isolated singularities of the first order at the point $(x,s) = (1,1)$ and therefore they are singled out separately.

3.2. Derivation and Analysis of the Wiener-Hopf Integral Equation.

Temporarily considering the right-hand side of equation (3.9 as a known function, we write it in the form of the singular integral equation

$$\tau_1(x) - \lambda \int_{-1}^{1} \left(\frac{1+x}{1+t}\right)^{1-2a_0} \frac{\tau_1(t)dt}{t-x} = g(x),\ x \in (-1,1), \tag{3.10}$$

where

$$g(x) = \lambda_1 \int_{-1}^{1} \frac{\tau_1(s)ds}{1-ax-bs} + \lambda_2 \int_{-1}^{1} \frac{\tau_1(s)ds}{1-bx-as} + R_1[\tau_1] + \Psi_5(x),\ x \in (-1,1). \tag{3.11}$$

He following holds:

Theorem 3.2. *If the function $g(x) \in L_p(-1,1)$, $p > 1$, satisfies the Holder condition for $x \in (-1,1)$, then for the solution $\tau_1(x)$ of equation* (3.10) *in the class of functions $H(-1,1)$, the formula*

$$\tau_1(x) = \frac{g(x)}{1+\lambda^2\pi^2} + \frac{\lambda}{1+\lambda^2\pi^2} \int_{-1}^{1} \left(\frac{1+x}{1+t}\right)^{1-2a_0-\delta} \left(\frac{1-x}{1-t}\right)^{\delta} \frac{g(t)dt}{t-x}, \tag{3.12}$$

is valid, where $\delta = arctg(\lambda\pi)/\pi$.

The method of proving Theorem 3.2 is identical to the method described [21]. Substituting the expression for $g(x)$ from (3.11) into (3.12) we obtain

$$\tau_1(x) = \frac{\lambda_1}{1+\lambda^2\pi^2} \int_{-1}^{1} \frac{\tau_1(s)ds}{1-ax-bs} + \frac{\lambda_2}{1+\lambda^2\pi^2} \int_{-1}^{1} \frac{\tau_1(s)ds}{1-bx-as} + \frac{\lambda(1+x)^{1-2a_0-\delta}(1-x)^{\delta}}{1+\lambda^2\pi^2} \times$$

$$\times \int_{-1}^{1} \tau_1(s)ds \int_{-1}^{1} \frac{(1+t)^{\delta+2a_0-1}}{(1-t)^{\delta}} \cdot \left(\frac{\lambda_1}{1-at-bs} + \frac{\lambda_2}{1-bt-as}\right) \frac{dt}{t-x} +$$

$$+ R_2[\tau_1] + \Psi_6(x), x \in (-1,1), \tag{3.13}$$

where

$$R_2[\tau_1] = \frac{1}{1+\lambda^2\pi^2} R_1[\tau_1] + \frac{\lambda}{1+\lambda^2\pi^2} \int_{-1}^{1} \left(\frac{1+x}{1+t}\right)^{1-2a_0-\delta} \left(\frac{1-x}{1-t}\right)^{\delta} \frac{R_1[\tau_1]}{t-x} dt-$$

regular operator,

$$\Psi_6(x) = \frac{1}{1+\lambda^2\pi^2} \Psi_5(x) + \frac{\lambda}{1+\lambda^2\pi^2} \int_{-1}^{1} \left(\frac{1+x}{1+t}\right)^{1-2a_0-\delta} \left(\frac{1-x}{1-t}\right)^{\delta} \frac{\Psi_5(t)dt}{t-x}$$

known function.

By evaluating the inner integral in equation (3.13), we rewrite this equation in the form (3.14).

$$\tau_1(x) = \frac{\lambda\lambda_1\pi}{sin(\delta\pi)}\frac{a^{1-2a_0}}{b^\delta}\int\limits_{-1}^{1}\left(\frac{1-x}{1-s}\right)^\delta\frac{\tau_1(s)ds}{1-ax-bs}+$$

$$+\frac{\lambda\lambda_2\pi}{sin(\delta\pi)}\frac{b^{1-2a_0}}{a^\delta}\int\limits_{-1}^{1}\left(\frac{1-x}{1-s}\right)^\delta\frac{\tau_1(s)ds}{1-bx-as}+R_3[\tau_1]+\Psi_6(x),\ x\in(-1,1), \tag{3.14}$$

where

$$R_3[\tau_1] = R_2[\tau_1]+\frac{\lambda\lambda_1}{sin(\delta\pi)}\frac{a^{1-2a_0}}{b^\delta}\int\limits_{-1}^{1}\left[\left(\frac{1+x}{1+a+bs}\right)^{1-2a_0-\delta}-1\right]\left(\frac{1-x}{1-s}\right)^\delta\frac{\tau_1(s)ds}{1-ax-bs}+$$

$$+\frac{\lambda\lambda_2}{sin(\delta\pi)}\frac{b^{1-2a_0}}{a^\delta}\int\limits_{-1}^{1}\left[\left(\frac{1+x}{1+a+bs}\right)^{1-2a_0-\delta}-1\right]\frac{\tau_1(s)ds}{1-bx-as}$$

-regular operator.

The equation (3.14) with respect to the unknown function $\tau_1(x)$ is not a Fredholm equation because the kernels of this equation have isolated singularities of the first order at the point $(x,s)=(1,1)$. Considering the equalities

$$1-bx-as = b(1-x)+a(1-s), \quad 1-ax-bs = a(1-x)+b(1-s)$$

in (3.14), we make the change of variables $x = 1-2e^{-y}, \quad s = 1-2e^{-t}$ and introduce the notations $\rho(y) = e^{-(0,5-\delta)y}\tau_1(1-2e^{-y})$,

$$K_0(x) = \sqrt{2\pi}\left(\frac{\lambda_1^*}{ke^{-x/2}+e^{x/2}}+\frac{\lambda_2^*}{e^{-x/2}+ke^{x/2}}\right),\ k=\frac{a}{b},$$

$\lambda_1^* = \frac{\lambda\lambda_1\pi}{sin(\delta\pi)}\frac{a^{1-2a_0}}{b^{1+\delta}}$, $\lambda_2^* = \frac{\lambda\lambda_2\pi}{sin(\delta\pi)}\frac{b^{-2a_0}}{a^\delta}$, we write equation (3.15) in the form [5], [12], [13], [16], [18], [14].

$$\rho(y) = \frac{1}{\sqrt{2\pi}}\int\limits_{0}^{+\infty}K_0(y-t)\rho(t)dt = R_5[\rho]+\Psi_7(y),\ y\in[0,+\infty), \tag{3.15}$$

where $R_5[\rho] = e^{-(0,5-\delta)y}R_4[\rho]-$ regular operator, $\Psi_7(y) = e^{-(0,5-\delta)y}\Psi_6(y)-$ known function.

Equation 3.15 is a Wiener-Hopf integral equation [5].

The function $K_0(x)$ has an exponential order of decay at infinity, and $K_0'(x)\in C[0,+\infty)$. Consequently, $K_0(x)\in L_2\cap H_\alpha$ [16,c.12]. [16,c.56].

The Fredholm theorems for convolution-type integral equations are applicable only when the index of these equations is zero. The index of equation (3.15) is the index of the expression $1-K^\wedge(x)$ taken with the opposite sign: $\chi = -Ind(1-K^\wedge(x))$ [5], where

$$K^\wedge(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{-ixt}K_0(t)dt = \int\limits_{-\infty}^{+\infty}\left\{\frac{\lambda_1^*}{e^{t/2}+ke^{-t/2}}+\frac{\lambda_2^*}{ke^{t/2}+e^{-t/2}}\right\}e^{-ixt}dt. \tag{3.16}$$

It is well known [20] that

$$\int_{-\infty}^{+\infty} \frac{e^{-ixt}dt}{ke^{t/2}+e^{-t/2}} = \frac{\pi e^{ixlnk}}{\sqrt{k}ch(\pi x)}. \tag{3.17}$$

By virtue of equality (3.17) from the representation (3.16) it is easy to obtain that

$$K^\wedge(x) = \frac{\pi(\lambda_1^*+\lambda_2^*)cos(xlnk)}{\sqrt{k}ch(\pi x)} - i\frac{\pi(\lambda_1^*-\lambda_2^*)sin(xlnk)}{\sqrt{k}ch(\pi x)}. \tag{3.18}$$

By virtue of the condition (3.1) of Theorem 3.1 we obtain

$$\frac{\pi(\lambda_1^*+\lambda_2^*)}{\sqrt{k}} = \frac{\lambda\pi^2}{\sqrt{k}sin(\delta\pi)}\left(\lambda_1\frac{a^{1-2a_0}}{b^{1+\delta}}+\lambda_2\frac{b^{-2a_0}}{a^\delta}\right) =$$

$$= \frac{-\lambda\pi^2\sqrt{b}}{\lambda_0\sqrt{a}\sin(\delta\pi)}\left[\mu_0\frac{a^{2-4a_0}}{b^{1+\delta}e^{b_0\pi}} - \mu_1\frac{b^{1-4a_0}}{a^\delta e^{-b_0\pi}}\right] < 1, \tag{3.19}$$

Due to the inequality (3.19), the inequality $Re(1-\hat{K}(x)) > 0$ follows from (3.18), and moreover, $Re\hat{K}(x) = O\left(\frac{1}{\mathrm{ch}(\pi x)}\right)$ for sufficiently large $|x|$.

Note that if $z = x+iy$ is a complex variable, then its argument $\arg z = \arctan\left(\frac{y}{x}\right)$ when $Re(z) = x > 0$.

Thus, according to [5], taking into account $Re\hat{K}(\pm\infty) = 0,\ Im\hat{K}(\pm\infty) = 0$, we obtain that

$$\chi = -\operatorname{Ind}(1-\hat{K}(x)) = -\frac{1}{2\pi}\left[\arg(1-\hat{K}(x))\right]\Big|_{-\infty}^{+\infty} = -\frac{1}{2\pi}\left[\arctan\frac{Im(1-\hat{K}(x))}{Re(1-\hat{K}(x))}\right]\Bigg|_{-\infty}^{+\infty} =$$

$$= -\frac{1}{2\pi}\left[\arctan\frac{0}{1} - \arctan\frac{0}{1}\right] = 0,$$

i.e., the change in the argument of the expression $1-\hat{K}(x)$ on the real axis, expressed in full rotations, is zero, meaning the index $\chi = 0$. Therefore, equation (3.15) is uniquely reduced to the Fredholm integral equation of the second kind [19], whose unique solvability follows from the uniqueness of the solution to the BSF problem.

Thus, Theorem 3.1 is proven.

References

[1] Bitsadze, A. V.; On some simplest generalizations of linear elliptic boundary value problems. Doklady AN SSSR. 1969. Vol. 185, No. 4. pp. 739–740.

[2] Bitsadze, A. V.; Some classes of partial differential equations. Moscow, 1981. 448 p.

[3] Devingtal, Yu. V.; On the existence and uniqueness of a solution to one of F. I. Frankl's problems. Izvestiya VUZ. Mathematics. 1958. Vol. 2, No. 3. pp. 39–51.

[4] Frankl, F. I.; The flow around profiles with a local supersonic zone ending with a straight shock wave. Applied Mathematics and Mechanics. 1956. Vol. 20, No. 2. pp. 196-202.

[5] Gakhov, F. D., Chersky, Yu. I.; Convolution-type equations. Moscow, 1978.

[6] Karatoprakliev, G. D.; On a generalization of the Tricomi problem. // Doklady AN SSSR. 1964. Vol. 158. pp. 271–274.

[7] Lin Jian-bin.; On some Frankl problems. Vestnik of Leningrad State University. Mathematics, Mechanics, Astronomy. 1961. Vol. 3, No. 13. pp. 28–39.

[8] Mirsaburov, M., Turaev, R. N.; A problem in an unbounded domain with the Bitsadze-Samarsky condition on a part of the boundary characteristic and its parallel internal characteristic for the Gellerstedt equation with singular coefficients. Differential Equations. 2024. Vol. 60, No. 8. pp. 1074-1086.

[9] Mirsaburov, M., Bobomurodov U.; A problem with the Frankl and Bitsadze-Samarsky conditions on the degeneration line and on parallel characteristics for the Gellerstedt equation with a singular coefficient. Differential Equations. 2012. Vol. 48, No. 11. pp. 730–737.

[10] Mirsaburov M.; Problem with analogs of the Franklcondition on a characteristic and the degeneration segment for an equation of mixed type with a singular coefficient. Differential Equations. 2017. No. 53. pp. 773–783.

[11] Mirsaburov, M.; A problem with a missing shift condition for the Gellerstedt equation with a singular coefficient. Izvestiya VUZ. Mathematics. 2018. No. 5. pp. 52–63.

[12] Mirsaburov, M.; Boundary problem for a class of mixed-type equations with the Bitsadze-Samarsky condition on parallel characteristics. Differential Equations. 2001. Vol. 37, No. 9. pp. 1281–1284.

[13] Mirsaburov, M., Begaliev, O., Khurramov, N. Kh.; On a generalization of the Tricomi problem. Differential Equations. 2019. Vol. 55, No. 8. pp. 1117–1126.

[14] Mirsaburova, G. M. A problem with an analogue of the Bitsadze-Samarsky condition for a class of degenerate hyperbolic equations // Izvestiya VUZ. Mathematics. — 2022. — No. 6. — pp. 54-59.

[15] Mirsaburova, U. M.; A problem with a shift on internal characteristics in an unbounded domain for the Gellerstedt equation with a singular coefficient. Izvestiya VUZ. Mathematics. 2022. No. 9. pp. 70–82.

[16] Polosin, A. A.; On the unique solvability of the Tricomi problem for a special domain. Differential Equations. 1996. Vol. 32, No. 3. pp. 394-401.

[17] Prudnikov, A. P., Brychkov, Yu. A., Marichev, O. I.; Integrals and series. Moscow: Nauka, 1981. 800 p.

[18] Ruziyev, M. Kh.; Boundary problem for a mixed-type equation with singular coefficients. Izvestiya VUZ. Mathematics. 2022. No. 7. pp. 18–29.

[19] Sabitov, K. B.; Functional, differential, and integral equations. Moscow: Vysshaya Shkola, 2005.

[20] Salakhitdinov, M. S., Mirsaburov, M.; Nonlocal problems for mixed-type equations with singular coefficients. Tashkent: University, 2005. 224 p.

[21] Smirnov, M. M.; Mixed-type equations. Moscow, 1985. 304 p.

[22] Zhegalov, V. I.; Boundary problem for a mixed-type equation with boundary conditions on the transition line. Kazan University Scientific Notes. 1962. Vol. 122, No. 3. pp. 3–16.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.185-192
DOI: 10.29229/uzmj.2025-2-17

Forward Problems for Fractional Wave equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. We study a forward problem for a fractional wave equation involving the RiemannLiouville fractional derivative, where the elliptic part is given by the Laplace operator. The equation is considered in an arbitrary multidimensional domain with a sufficiently smooth boundary. By employing the Fourier method, we establish theorems on the existence and uniqueness of the classical solution to the forward problem.

Keywords: Forward problems; Direct problems; Fractional Wave equation; Fourier method; Classical solution; Sobolev spaces.

MSC (2020): 35R11, 74S25.

1. Introduction

The study of fractional partial differential equations has garnered significant attention in recent years due to their applicability in modeling anomalous diffusion, viscoelasticity, and other complex physical phenomena. Among various types of fractional derivatives, the Riemann-Liouville derivative is particularly important, though more intricate to handle due to its non-local and singular kernel. In this section, we review several key contributions in the field that are closely related to our study.

The time fractional string vibration equation that we have presented, involving the RiemannLiouville fractional derivative of order $1 < \rho < 2$, has been the subject of various research studies. Researchers have explored both the direct problem, solving the equation for a given fractional order.

Trifce Sandev and Zivorad Tomovski [12] studied a general time-fractional wave equation for a vibrating string. They obtained solutions in terms of MittagLeffler-type functions and a complete set of eigenfunctions of the SturmLiouville problem. Their approach utilized the Caputo fractional derivative and employed the separation of variables methods and the Laplace transform. Their work provides insights into modeling processes in complex or viscoelastic media. Allen et al. [2] consider a boundary value problem involving both the Riemann-Liouville and the AtanganaBaleanu fractional derivatives. They obtain analytic solutions in the form of infinite series by employing the Laplace transform. While the problem formulation differs from ours, their analytical approach provides valuable insight into the structure of fractional models with mixed derivatives.

Ashurov and Fayziev [3] address an inverse problem related to determining the order of the Riemann-Liouville derivative in a time-fractional wave equation. Their study is formulated in an abstract Hilbert space setting and relies on the operator-theoretic framework. Although our research focuses on direct problems, the theoretical implications of inverse problems contribute significantly to understanding solution behavior with respect to fractional order sensitivity. In [8] , Kian and Yamamoto study a class of semilinear fractional wave equations with Caputo derivatives. They establish existence and uniqueness results for weak solutions, and explore the regularity properties under different assumptions on the nonlinearity and source term. While our work focuses on the RiemannLiouville derivative and classical solutions, this study provides an important comparative reference for different types of fractional operators.

Our study differs in several crucial aspects: we consider the classical solution of a linear time-fractional wave equation with Riemann-Liouville derivative of order $1 < \rho < 2$, in a general N-dimensional domain with sufficiently smooth boundary. We rigorously establish existence and uniqueness theorems using the Fourier method, which allows for precise spectral analysis of the Laplace operator. Such a formulation not only extends the classical theory but also emphasizes the role of the Riemann-Liouville derivative in higher-dimensional settings, which has been comparatively less explored in the literature.

2. Formulation of the problem

Consider the time-fractional wave equation with the Riemann-Liouville fractional derivative of order $1 < \rho < 2$:

$$\partial_t^\rho u(x,t) - \Delta u(x,t) = f(x,t), \quad 0 < t \le T, \ x \in \Omega \subset R^N \tag{2.1}$$

with the following initial conditions

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \ \lim_{t\to 0} \partial_t^{\rho-2} u(x,t) = \psi(x), \ \ x \in \Omega \tag{2.2}$$

and boundary condition

$$u(x,t)|_{\partial\Omega} = 0 \tag{2.3}$$

where $\varphi(x)$, $\psi(x)$ and $f(x,t)$ are given smooth functions, $\triangle = \sum\limits_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator.

The fractional integration of order ρ of a function f defined in $[0,\infty)$ in the Riemann - Liouville sense is defined by the formula

$$I^\rho f(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-r)^{\rho-1} f(r) dr, \quad t > 0,$$

provided the right-hand side exists. Here, $\Gamma(\rho)$ is Euler's gamma function. Using this definition, one can define the Riemann - Liouville fractional derivative of order ρ, $k-1 < \rho \le k$, $k \in \mathbb{N}$, as (see, for example, [11], p. 14)

$$\partial_t^\rho f(t) = \frac{d^k}{dt^k} I^{k-\rho} f(t).$$

Definition 2.1. A function $t^{2-\rho} u(x,t) \in C(\bar{\Omega} \times [0,T])$ with the properties

(1) $\partial_t^{\rho-1} u(x,t), \quad \partial_t^{\rho-2} u(x,t) \in C(\Omega \times [0,T])$,

(2) $\partial_t^{\rho} u(x,t), \quad \triangle u(x,t) \in C(\bar{\Omega} \times (0,T])$,

and satisfying conditions (2.1)-(2.3) is called ***the classical solution*** of the forward problem (2.1)-(2.3).

We will apply the Fourier method, which will lead us to consider the following spectral problem

$$\begin{cases} -\triangle v(x) = \lambda v(x), \quad x \in \Omega; \\ v(x)\big|_{\partial\Omega} = 0. \end{cases} \tag{2.4}$$

Since the boundary $\partial\Omega$ is sufficiently smooth, then this problem has a complete in $L_2(\Omega)$ set of orthonormal eigenfunctions $\{v_k(x)\}$, $k \ge 1$, and a countable set of positive eigenvalues $\{\lambda_k\}$, (see, e.g. [6] - [10]). It is convenient to assume that $0 < \lambda_1 \le \lambda_2 \cdots \to +\infty$.

We note right away that the method proposed here, based on the Fourier method, is applicable to equation (2.1) with an arbitrary elliptic differential operator $A(x,D)$, if only the corresponding spectral problem has a complete system of orthonormal eigenfunctions in $L_2(\Omega)$.

3. Preliminaries

In this section, we formulate the lemma from the book by Krasnoselskii et al. (see, e.g., [9]), the fundamental result of V.A. Ilyin (see, e.g.,[7]) on the convergence of the Fourier coefficients and recall some properties of the Mittag-Leffler function.

Let Ω be an arbitrary N-dimensional domain with a sufficiently smooth boundary $\partial\Omega$.

Let A stand for the operator acting in $L_2(\Omega)$ as $Ag(x) = -\triangle g(x)$ in the domain of definition $D(A) = \{g \in C^2(\overline{\Omega}) : g(x) = 0, x \in \partial\Omega\}$. We denote the self-adjoint extension of A in $L_2(\Omega)$ by $\hat{A}$.

In order to formulate the indicated lemma, it is necessary to introduce the power of the operator $\hat{A}$.

Let σ be an arbitrary real number. The power of operator A, acting in $L_2(\Omega)$ is defined as:

$$\hat{A}^\sigma g(x) = \sum_{k=1}^{\infty} \lambda_k^\sigma \, g_k \, v_k(x), \quad g_k = (g, v_k),$$

and the domain of definition has the form

$$D(\hat{A}^\sigma) = \{g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 < \infty\}.$$

For elements of $D(\hat{A}^\sigma)$ we introduce the norm

$$||g||_\sigma^2 = \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 = ||\hat{A}^\sigma g||^2,$$

where $|| \cdot ||$ is the norm of $L_2(\Omega)$.

The following lemma plays an essential role in our reasoning (see, e.g., [9], p. 453).

Lemma 3.1. *Let $\sigma > \frac{N}{4}$. Then operator $\hat{A}^{-\sigma}$ continuously maps the space $L_2(\Omega)$ into $C(\overline{\Omega})$, and moreover, the following estimate holds*

$$\|\hat{A}^{-\sigma} g\|_{C(\Omega)} \le C \|g\|_{L_2(\Omega)}.$$

When proving the existence of solutions to forward problem, it is necessary to study the convergence of series of the form:

$$\sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2, \quad \tau > \frac{N}{2}, \tag{3.1}$$

where h_k is the Fourier coefficient of function $h(x)$. In the case of integers τ, the conditions for the convergence of such series in terms of the membership of the function $h(x)$ in classical Sobolev spaces $W_2^1(\Omega)$ were obtained in the work of V.A. Ilin (see, e.g.,[7]). To formulate these conditions, we introduce the class $\dot{W}_2^1(\Omega)$ as the closure in the $W_2^1(\Omega)$ norm of the set of all functions that are continuously differentiable in Ω and vanish near the boundary of Ω.

So, if function $h(x)$ satisfies the conditions

$$h(x) \in W_2^{[\frac{N}{2}]+1}(\Omega) \quad \textit{and} \quad h(x), \Delta h(x), ..., \Delta^{[\frac{N}{4}]} \in \dot{W}_2^1(\Omega), \tag{3.2}$$

then the number series (3.1) (we can take $\tau = \frac{N}{2} + 1$ if N is even, and $\tau = \frac{N+1}{2}$ if N is odd) converges.

4. Main results

We search for the solution of problem (2.1)-(2.3) using the Fourier method on the form of the following series:

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t) v_j(x) \tag{4.1}$$

We substitute (4.1) into equation (2.1):

$$\partial_t^\rho \left(\sum_{j=1}^{\infty} u_j(t)v_j(x)\right) - \triangle \left(\sum_{j=1}^{\infty} u_j(t)v_j(x)\right) = \sum_{j=1}^{\infty} f_j(t)v_j(x).$$

From this, we obtain the equality

$$\sum_{j=1}^{\infty} \left[\partial_t^\rho u_j(t) + \lambda_j u_j(t)\right] v_j(x) = \sum_{j=1}^{\infty} f_j(t)v_j(x).$$

Therefore

$$\partial_t^\rho u_j(t) + \lambda_j u_j(t) = f_j(t). \tag{4.2}$$

According to problem (2.1), we obtain the following problem

$$\begin{cases} \partial_t^\rho u_j(t) + \lambda_j u_j(t) = f_j(t); \\ \lim_{t\to 0} \partial_t^{\rho-1} u_j(t) = \varphi_j; \\ \lim_{t\to 0} \partial_t^{\rho-2} u_j(t) = \psi_j, \end{cases} \tag{4.3}$$

where the Fourier coefficients of the functions $f(x,t)$, $\varphi(x)$ end $\psi(x)$ are denoted by $f_j(t)$, φ_j, end ψ_j, respectively, in accordance with the system of eigenfunctions $\{v_j(x)\}$. Thus, the solution of problem (4.3) has the following form (see, for example, [4], p. 173):

$$u_j(t) = \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) + \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) + \int_0^t f_j(t-\xi)\xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi. \tag{4.4}$$

Then, substituting expression (4.4) into (4.1), we obtain the following formal solution:

$$u(x,t) = \sum_{j=1}^{\infty} \big[\varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) + \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) + \tag{4.5}$$

$$+ \int_0^t f_j(t-\xi)\xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi\big] v_j(x),$$

where $E_{\rho,\mu}(z)$ is the Mittag-Leffler function, which has the following form:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \tag{4.6}$$

and the estimate:

$$0 < |E_{\rho,\mu}(-t)| \le \frac{C}{1+t}, \quad t > 0. \tag{4.7}$$

We will also use the following formula

$$\int_0^t \xi^{\rho-1} E_{\rho,\rho}(-\lambda \xi^\rho) d\xi = t^\rho\, E_{\rho,\rho+1}(-\lambda t^\rho). \tag{4.8}$$

Theorem 4.1. *Assume that (3.2) holds under the condition $\tau > \frac{N}{2}$, let $\varphi(x)$ and $\psi(x)$ functions satisfy conditions (3.2) Moreover, let $f(x,t)$ as a function of x satisfy conditions (3.2) for all $t \in [0,T]$. Then there exists a unique solution of the forward problem (2.1)-(2.3) and this classical solution has the representation*

$$u(x,t) = \sum_{j=1}^{\infty} \left[\varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^{\rho}) + \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^{\rho}) + \int_0^t f_j(t-\xi)\xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^{\rho}) d\xi\right] \upsilon_j(x), \tag{4.9}$$

where φ_j, ψ_j and $f_j(t)$ are the Fourier coefficients of functions $\varphi(x)$, $\psi(x)$ $f(x,t)$ respectly.

5. Proof of the Uniqueness of the solution.

Let $u_1(x,t)$ and $u_2(x,t)$ be solutions that satisfy the given conditions of the problem. We need to prove that $u_1(x,t) = u_2(x,t)$, i.e., $u(x,t) = u_1(x,t) - u_2(x,t) \equiv 0$. Since the considered problem is linear, we arrive at the following problem for the function $u(x,t)$:

$$\begin{gathered} \partial_t^{\rho} u(x,t) - \Delta u(x,t) = 0, \quad 0 < t \le T, \quad x \in \Omega \subset R^N \\ u|_{\partial\Omega} = 0, \\ \lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = 0, \quad \lim_{t\to 0} \partial_t^{\rho-2} u(x,t) = 0, \quad x \in \Omega \end{gathered}$$

Let $u(x,t)$ be a classical solution of the problem (2.2) – (2.3). We consider the following function:

$$\omega_j(t) = \int_{\Omega} u(x,t)\, \upsilon_j(x)\, dx \tag{5.1}$$

here, $\upsilon_j(x)$ is an arbitrary eigenfunction corresponding to the eigenvalue λ_j of the problem (4.4). According to the definition of a classical solution, the equation (4.5) can be written as follows:

$$\partial_t^{\rho} \omega_j(t) = \int_{\Omega} \partial_t^{\rho} u(x,t)\, \upsilon_j(x)\, dx = \int_{\Omega} \Delta u(x,t)\, \upsilon_j(x)\, dx$$

or

$$\partial_t^{\rho} \omega_j(t) = \int_{\Omega} \Delta u(x,t)\, \upsilon_j(x)\, dx = -\lambda_j \int_{\Omega} (x,t)\, \upsilon_j(x)\, dx = -\lambda_j \omega_j(t), \quad t > 0.$$

The Cauchy problem for the function $\omega_j(t)$ takes the following form:

$$\begin{aligned} &\partial_t^{\rho} \omega_j(t) + \lambda_k \omega_j(t) = 0, \quad t > 0, \\ &\lim_{t\to 0} \partial_t^{\rho-1} \omega_k(t) = 0, \qquad \lim_{t\to 0} \partial_t^{\rho-2} \omega_k(t) = 0. \end{aligned}$$

This problem has a unique solution. Thus, the function defined by equation (5.1) is identically equal to zero: $\omega_j(t) \equiv 0$ (see, e.g.,[5]). Since the set of eigenfunctions $\{\upsilon_j\}$ forms a complete system in $L_2(\Omega)$, and for all $x \in \Omega$, $t > 0$, we have $u(x,t) = 0$.

6. Proof of the Existence of a Solution.

Expressing solution (3.14) in the form $u(x,t) = S_1 + S_2 + S_3$,

$$S_1^n = \sum_{j=1}^{n} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^{\rho}) \upsilon_j(x),$$

$$S_2^n = \sum_{j=1}^{n} \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^{\rho}) \upsilon_j(x),$$

$$S_3^n = \sum_{j=1}^{n} \left(\int_0^t f_j(t-\xi) \cdot \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right) v_j(x).$$

We define the partial sums of S_j as S_k^n and examine the convergence of each of these sums. First, we examine the convergence of the sum:

$$S_1^n = \sum_{j=1}^{n} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) v_j(x). \tag{6.1}$$

According to the Krosnoselskii lemma, we transition from the norm in the space $C(\Omega)$ to the norm in the space $L_2(\Omega)$, and using Parseval's inquality, we obtain the following:

$$\| -\Delta S_1^n \|^2_{C(\Omega)} = \| -\Delta \sum_{j=1}^{n} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) v_j(x) \|^2_{C(\Omega)} =$$

$$= \| \hat{A}^{-\sigma} \sum_{j=1}^{n} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) v_j(x) \lambda_j \lambda_j^\sigma \|^2_{C(\Omega)} \leq$$

$$\leq C_1 \left\| \sum_{j=1}^{n} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) v_j(x) \lambda_j^{\sigma+1} \right\|^2_{L_2(\Omega)} \leq C_1 \sum_{j=1}^{n} \left| \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) v_j(x) \lambda_j^{\sigma+1} \right|^2 \leq$$

$$\leq C_1 \sum_{j=1}^{n} \left| \varphi_j t^{\rho-1} \cdot \frac{C_2}{1+\lambda_j t^\rho} \cdot \lambda_j^{\sigma+1} \right|^2 \leq C_1 \sum_{j=1}^{n} \left| \varphi_j t^{\rho-1} \cdot \frac{C_2}{1+\lambda_j t^\rho} \cdot \lambda_j^{1+\sigma} \right|^2 \leq$$

$$\leq \frac{C}{t^2} \sum_{j=1}^{n} |\varphi_j \lambda_j^\sigma|^2, \quad t > 0.$$

According to the Il'ins result, if $\varphi \in W_2^{[\frac{N}{2}]+1}(\Omega)$ and $\varphi, \Delta\varphi, ..., \Delta^{\frac{N}{2}} \varphi \in \dot{W}_2^1(\Omega)$, then the series $\sum_{j=1}^{n} \left| \varphi_j \lambda_j^\sigma \right|^2$ will converge, where $\tau = \frac{N}{2} + \varepsilon$, and $\varepsilon > 0$ is a sufficiently small numbur. Thus, series (6.1) is convergent. Now, let us consider the sum S_2^n:

$$S_2^n = \sum_{j=1}^{n} \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) v_j(x) \tag{6.2}$$

As above, according to the Krasnoselskii lemma, we transition from the norm in the space $C(\Omega)$ to the norm in the space $L_2(\Omega)$, and by Parseval's inequality, we obtain

$$\| -\Delta S_2^n \|^2_{C(\Omega)} = \| -\Delta \sum_{j=1}^{n} \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) v_j(x) \|^2_{C(\Omega)} =$$

$$= \| \hat{A}^{-\sigma} \sum_{j=1}^{n} \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) v_j(x) \lambda_j \lambda_j^\sigma \|^2_{C(\Omega)} \leq$$

$$= C_1 \| \sum_{j=1}^{n} \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) v_j(x) \ \lambda_j^{\sigma+1} \|^2_{L_2(\Omega)} \leq C_1 \sum_{j=1}^{n} \left| \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) v_j(x) \lambda_j^{\sigma+1} \right|^2 \leq$$

$$\leq C_1 \sum_{j=1}^{n} \left| \psi_j t^{\rho-2} \cdot \frac{C_2}{1+\lambda_j t^\rho} \cdot \lambda_j^{\sigma+1} \right|^2 \leq C_1 \sum_{j=1}^{n} \left| \psi_j t^{\rho-2} \cdot \frac{C_2}{1+\lambda_j t^\rho} \cdot \lambda_j^{1+\sigma} \right|^2 \leq$$

$$\leq C_1 \sum_{j=1}^{n} \left| \psi_j t^{\rho-2} \frac{C_2}{\lambda_j t^\rho} \cdot \lambda_j^{1+\sigma} \right|^2 = \sum_{j=1}^{n} \left| \psi_j \cdot \frac{C_2}{t^2} \cdot \lambda_j^{\sigma} \right|^2 = \frac{C}{t^4} \sum_{j=1}^{n} |\psi_j|^2 \lambda_j^{2\sigma}, \quad t > 0.$$

Thus, according to Il'in's result, if $\psi \in W_2^{[\frac{N}{2}]+1}(\Omega)$ and $\psi, \Delta\psi, ..., \Delta^{\frac{N}{2}}\psi \in \dot{W}_2^1(\Omega)$, then the series $\sum\limits_{j=1}^{n} |\psi_j|^2 \lambda_j^{2\sigma}$ will be convergent. Thus, as we can see, the series (5.2) is convergent. Let us check the convergence of the sum S_3^n:

$$S_3^n = \sum_{j=1}^{n} \left(\int_0^t f_j(t-\xi) \cdot \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right) v_j(x) \tag{6.3}$$

$$\| -\Delta S_3^n \|_{C(\Omega)} = \left\| -\Delta \sum_{j=1}^{n} \left(\int_0^t f_j(t-\xi) \cdot \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right) v_j(x) \right\|_{C(\Omega)} =$$

$$= \left\| \sum_{j=1}^{n} \lambda_j \left(\int_0^t f_j(t-\xi) \cdot \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right) v_j(x) \right\|_{C(\Omega)}.$$

According to the Krasnoselskii lemma and Parseval's inequality, we have

$$\| -\Delta S_3^n \|_{C(\Omega)}^2 = \left\| -\Delta \sum_{j=1}^{n} \left(\int_0^t f_j(t-\xi) \cdot \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right) v_j(x) \right\|_{C(\Omega)}^2 =$$

$$= \left\| \hat{A}^{-\sigma} \sum_{j=1}^{n} \lambda_j^{\sigma+1} \left(\int_0^t f_j(t-\xi) \cdot \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right) v_j(x) \right\|_{C(\Omega)}^2 \leq$$

$$\leq C \left\| \sum_{j=1}^{n} \lambda_j^{\sigma+1} v_j(x) \int_0^t f_j(t-\xi) \cdot \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right\|_{L_2(\Omega)}^2 \leq$$

$$\leq C \sum_{j=1}^{n} \left| \lambda_j^{\sigma+1} \int_0^t f_j(t-\xi) \cdot \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right|^2 \leq$$

Here, we estimate the Mittag-Leffler function from above by 1 and by applying the Cauchy-Schwarz inequality, we obtain the following:

$$\leq C_1 \sum_{j=1}^{n} \lambda_j^{2(\sigma+1)} \int_0^t |f_j(t-\xi)|^2 d\xi \cdot \int_0^t \left|\xi^{\rho-1}\right|^2 d\xi \leq$$

$$\leq C_1 \int_0^t \left[\sum_{j=1}^{n} \lambda_j^{2(\sigma+1)} |f_j(t-\xi)|^2 \right] d\xi \cdot \int_0^t \left|\xi^{\rho-1}\right|^2 d\xi \leq C_1 T^{2\rho} \int_0^t \left[\sum_{j=1}^{n} \lambda_j^{2(\sigma+1)} |f_j(t-\xi)|^2 \right] d\xi.$$

Thus, according to Il'in's result, if $f \in W_2^{[\frac{N}{2}]+1}(\Omega)$ and $f, \Delta f, ..., \Delta^{\frac{N}{2}} f \in \dot{W}_2^1(\Omega)$, then the series $\sum\limits_{j=1}^{n} \lambda_j^{2(\sigma+1)} |f_j(t-\xi)|^2$ is convergent, here $\tau = \sigma + 1 > \frac{N}{4} + 1$.

Hence, summing up the estimates of all three terms in (3.14), we obtain $\Delta u(t) \in C(\bar{\Omega}\times(0,T])$.

In addition, equation (2.1) implies $\partial_t^\rho S_k^n(t) = -\Delta S_k^n(t)$. Therefore, from the above reasoning, we finally have $\partial_t^\rho u(t) \in C(\bar{\Omega} \times (0,T])$.

A simple calculation shows the fulfillment of the initial conditions $\partial_t^{\rho-1}u(x,t), \quad \partial_t^{\rho-2}u(x,t) \in C(\Omega \times [0,T])$.

Thus, Theorem (3.2) is completely proved.

7. Acknowledgement.

The authors are grateful to professor R.R. Ashurov and Y.E. Fayziev for discussions of these results.

References

[1] Agmon S.; On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, *Comm. Pure and Appl. Math.* 1962, 15, 119-141.

[2] Allen I.K., Duggal D., Nasir S., Karimov E.T.; On a boundary value problem for a time-fractional wave equation with the Riemann-Liouville and Atangana-Baleanu derivatives. Bulletin of the Institute of Mathematics 2020, No.1, 1-9.

[3] Ashurov R., Fayziev Y.; Inverse problem of determining an order of the Riemann-Liouville derivative in time. Eurasian Mathematical Journal, Vol.13, No. 1, 2022,19-31.

[4] Gorenflo R., Kilbas A.A., Mainardi F., Rogozin S.V.; *Mittag-Leffler functions, related topics and applications*. Springer. 2014.

[5] Hatano Y., Nakagawa J., Wang, S., Yamamoto M.; Determination of order in fractional diffusion equation, *J. Math-for-Ind.*, 2013, 5A, 51-57.

[6] Il'in V.A.; Fundamental systems of functions of the Laplace operator in an arbitrary domain and asymptotic estimates for fundamental numbers / V.A. Il'in // Doklady Akademii Nauk Russian Academy of Sciences. - 1967, Vol. 177, No. 1, 25-28 [in Russian].

[7] Il'in V.A.; On the solvability of mixed problems for hyperbolic and parabolic equations / V.A. Il'in // Russian Math. Surveys, 1960, Vol.15, No 2, 97-154 [in Russian].

[8] Kian Y., Yamamoto M.; On existence and uniqueness of solutions for semilinear fractional wave equations. Fractional Calculus and Applied Analysis, Vol. 20, No. 1, 2017, 117138.

[9] Krasnoselski M.A., Zabreyko P.P., Pustilnik E.I., Sobolevski P.S.; Integral operators in the spaces of integrable functions / M.A. Krasnoselski, P.P. Zabreyko, E.I. Pustilnik, P.S. Sobolevski — M. Nauka. 1966 [in Russian].

[10] Ladyzhenskaya O.A.; Mixed problem for a hyperbolic equation. / O.A. Ladyzhenskaya — Gostexizdat. 1953.

[11] Pskhu A.V.; Fractional partial differential equations / A.V. Pskhu — M.: NAUKA, 2005 [in Russian].

[12] Sandev T., Tomovski Z.; Wave Equation for a Vibrating String in Presence of a Fractional friction and an External Force, Symposium on Fractional Signals and Systems Lisbon09. M. Ortigueira et.al. Lisbon, Portugal, November 4-6, 2009, 11p.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.193-199
DOI: 10.29229/uzmj.2025-2-18

Solution of a nonlocal boundary value problem of diffusion-convective transfer of radioactive substances in the lithosphere-atmosphere system

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. The article studies a boundary value problem for describing non-local and non-stationary diffusion-convective transfer of radioactive material in the lithosphere-atmosphere system. The non-locality of the transfer is due to heredity effects in the upper layer of the lithosphere and is described from a mathematical point of view using fractional Gerasimov-Caputo derivatives with respect to time and fractional Riemann-Liouville derivatives with respect to the spatial coordinate. Boundary conditions are specified both at the outer boundaries and at the inner lithosphere-atmosphere boundary. The solution to the problem is found using the integral Laplace transform and is given in terms of the generalized Wright function.

Keywords: nonlocal transport, Gerasimov-Caputo fractional derivatives, Riemann-Liouville fractional derivatives, generalized Wright function, diffusion convection, lithosphere, atmosphere

MSC (2020): 26A33, 44A10

1. Introduction

The study of radioactive substance transfer processes in the lithosphere-atmosphere system is of great importance. On the one hand, such problems arise in ecology and are related to forecasting the distribution of radioactive substances in the atmosphere [16, 4], on the other hand, radioactive substances can be indicators of the stress-strain state of the geoenvironment and affect other geophysical fields [14].

The processes of radioactive substance or emanations transfer are studied within the framework of the theory of the emanation method. According to this theory, the main mechanisms of radioactive substance transfer are diffusion and convection. Then, using mathematical physics, diffusion-convection equations with the corresponding initial and boundary conditions are constructed. As a rule, these are linear equations that allow finding a solution using integral transformations. Within the framework of the classical theory of the emanation method, model equations contain integer derivatives, which in turn limits the application of diffusion-convective transfer equations [17, 13]. This is due to the fact that radioactive substances have a limited diffusion length. Therefore, using classical transport equations, it is impossible to describe anomalous effects in the values of radioactive substance concentration on the earth's surface in the absence of a strong radioactive source at some depth in the ground.

Therefore, a non-classical theory of the emanation method began to develop [9, 10], in the equations of which fractional derivatives began to be used [7, 5]. Fractional derivatives made it possible to describe the effect of non-locality (heredity) of the medium, which is associated with its permeability.

It is known that fractional derivatives describe anomalous transport processes [15]. For example, the introduction of a fractional derivative with respect to time (non-locality with respect to time) leads to subdiffusion - a slower process than ordinary diffusion [1, 2], and a fractional derivative with respect to spatial coordinates (non-locality with respect to space) leads to superdiffusion - a faster process than ordinary diffusion [6].

In this paper, we study the non-stationary process of diffusion-convection of a radioactive substance in the lithosphere-atmosphere system taking into account non-locality. Non-locality is taken into account in the upper layer of the lithosphere (soil); in the atmosphere, heredity effects are absent.

2. Statement of the problem

It is necessary to find the solution $A(z,t)$ in the domain $(t>0, -\infty<z<\infty)$:

$$\lim_{z\to\infty} A(z,t)=0,$$

$$\frac{\partial A(z,t)}{\partial t}=D_a\frac{\partial^2 A(z,t)}{\partial z^2}+v_a\frac{\partial A(z,t)}{\partial z}-\lambda A(z,t), \quad z>0 \tag{2.1}$$

$$\lim_{z\to 0-0} D_{z0}^{\alpha-2}A(z,t)=\left.A(z,t)\right|_{z=0+0},$$

$$\lim_{z\to 0-0}\left[D_g D_{z0}^{\alpha-1}A(z,t)+v_g D_{z0}^{\beta-1}A(z,t)\right]=D_a\left.\frac{\partial A(z,t)}{\partial z}\right|_{z=0+0}+v_a\left.A(z,t)\right|_{z=0+0},$$

$$\frac{\partial^\gamma A(z,t)}{\partial t^\gamma}=D_g D_{z0}^{\alpha}A(z,t)+v_g D_{z0}^{\beta}A(z,t)-(\lambda A(z,t)-A_\infty), z<0,$$

$$\lim_{z\to-\infty} A(z,t)=A_\infty.$$

where $D_{z0}^{\alpha}, D_{z0}^{\beta}$ are Riemann-Liouville fractional differentiation operators $(1<\alpha<2, 0<\beta<1)$:

$$D_{z0}^{\alpha}A(z,t)=\frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dz^2}\int\limits_z^0\frac{A(y,t)\,dy}{(z-y)^{\alpha-1}}, D_{z0}^{\beta}A(z,t)=\frac{1}{\Gamma(1-\beta)}\frac{d}{dz}\int\limits_z^0\frac{A(y,t)\,dy}{(z-y)^{\beta}},$$

where $\Gamma(\cdot)$ is the gamma function and ∂_{0t}^{γ} is the operator in the sense of Gerasimov-Caputo [8, 3]:

$$\frac{\partial^\gamma A(z,t)}{\partial t^\gamma}=\frac{1}{\Gamma(1-\gamma)}\int\limits_z^0\frac{A'_\theta(z,\theta)\,d\theta}{(t-\theta)^\gamma}, 0<\gamma<1,$$

D_a, D_g are the diffusion coefficients of matter in the atmosphere and loose sediments, m^2/s, m^α/s; λ is the decay constant of matter, 1/s; A_∞ is the volumetric activity of matter in radioactive equilibrium with the emanation source at a given depth, Bq/m^3; $A(z,t)$ – volumetric activity of the substance in loose sediments, Bq/m^3, $A_\infty = KA_M\rho(1\text{-}\eta)/\eta$, where K – substance emanation coefficient, rel. units; A_M – specific activity of the emanation source, Bq/kg; ρ – density of solid particles of loose sediments, kg/m^3; η – porosity of loose sediments.

Remark 2.1. The boundary value problem (2.1) describes non-local and non-stationary diffusion-convective transfer of radioactive substance in the lithosphere-atmosphere system. We will not discuss the derivation of model equations (2.1) here. We will say that the introduction of fractional derivatives with respect to time and space to describe the diffusion-convective transfer in the soil indicates the non-locality of this medium. The non-locality of the medium in our case is associated with its permeability, which is responsible for the intensity of the transfer process.

Remark 2.2. External boundary conditions indicate that at some depth in the soil there is a source of radioactive substance, and in the atmosphere with height due to turbulent diffusion there is a sharp decrease in its concentration. Internal boundary conditions — equality of concentrations and flows at the boundary of the media provide the necessary condition for gluing solutions.

Remark 2.3. It should be noted that the paper [11] considered the boundary value problem (2.1) for radioactive radon gas under the condition $\alpha = 2, \beta = \gamma = 1$. The analytical solution to the problem was obtained using the Laplace integral equation in terms of integral error functions. The paper [12] considered the stationary case of the problem (2.1), when the partial derivatives with respect to time on the right-hand side are equal to zero.

3. Solution method

Let us introduce at the characteristic time t_0 both the scale z_0 and the corresponding dimensionless coordinates $\tau = t/t_0, \xi = z/z_0$. Then the equation (2.1) will be written in dimensionless coordinates as:

$$\frac{\partial A(\xi,\tau)}{\partial \tau} = \bar{D}_a \frac{\partial^2 A(\xi,\tau)}{\partial \xi^2} + \bar{v}_a \frac{\partial A(\xi,\tau)}{\partial \xi} - \bar{\lambda} A(\xi,\tau), \quad \xi > 0 \tag{3.1}$$

$$\frac{\partial^\gamma A(\xi,\tau)}{\partial \tau^\gamma} = \bar{D}_g D^\alpha_{\xi 0} A(\xi,\tau) + \bar{v}_g D^\beta_{\xi 0} A(\xi,\tau) - (\bar{\lambda} A(\xi,\tau) - A_\infty), \quad \xi < 0$$

$$\lim_{\xi \to 0-0} D^{\alpha-2}_{0\xi} A(\xi,\tau) = A(\xi,\tau)|_{\xi=0+0},$$

$$\lim_{\xi \to 0-0} \left[\bar{D}_g D^{\alpha-1}_{\xi 0} A(\xi,\tau) + \bar{v}_g D^{\beta-1}_{\xi 0} A(\xi,\tau)\right] = \bar{D}_a \left.\frac{\partial A(\xi,\tau)}{\partial \xi}\right|_{\xi=0+0} + \bar{v}_a A(\xi,\tau)|_{\xi=0+0},$$

Transfer parameters in the soil and in the atmosphere will also be dimensionless quantities.

We will seek a solution to equation (3.1) in the form of a traveling wave with a velocity of V. We will make a replacement in (3.1): $A(\xi,\tau) = f(x), x = \xi - V\tau$, $W_a = V - \bar{v}_a$, $s = -V\tau$. Then equation (3.1) will become the equation:

$$\bar{D}_a \frac{d^2 f(x)}{dx^2} + W_a \frac{df(x)}{dx} - \bar{\lambda} f(x) = 0, \quad x > s \tag{3.2}$$

$$\bar{D}_g D^\alpha_{xs} f(x) + \bar{v}_g D^\beta_{xs} f(x) + V^\gamma \frac{d^\gamma f(x)}{dx^\gamma} - (\bar{\lambda} f(x) - A_\infty) = 0, \quad x < s$$

$$\lim_{x \to s-0} D^{\alpha-2}_{xs} f(x) = f(x)|_{x=s+0},$$

$$\lim_{x \to s-0} \left[\bar{D}_g D^{\alpha-1}_{xs} f(x) + \bar{v}_g D^{\beta-1}_{xs} f(x)\right] = \bar{D}_a \left.\frac{df(x)}{dx}\right|_{x=s+0} + \bar{v}_a f(x)|_{x=s+0}.$$

The solution of the first equation (3.2) for the atmosphere is known, it can be written as follows:

$$f(x) = C_1 e^{x\left(\frac{-r_1 + \sqrt{r_1^2 + 4r_2}}{2}\right)} + C_2 e^{-x\left(\frac{r_1 + \sqrt{r_1^2 + 4r_2}}{2}\right)}, \quad r_1 = W_a/\bar{D}_a, r_2 = \bar{\lambda}/\bar{D}_a. \tag{3.3}$$

From physical considerations, at $x \to \infty$ the solution of equation (3.3) tends to zero. Indeed, the radon concentration decreases toward the earth's surface, and in the atmosphere it is close to zero. Therefore, the constant C_1=0 and then the boundary conditions at the soil–atmosphere boundary are rewritten as follows:

$$\lim_{x\to s-0} D_{xs}^{\alpha-2} f(x) = R_1 C_2,$$

$$\lim_{x\to s-0} \left[\bar{D}_g D_{xs}^{\alpha-1} f\left(x\right) + \bar{v}_g D_{xs}^{\beta-1} f\left(x\right)\right] = R_2 C_2,$$

$$R_1 = \exp\left(-s\left(r_1 + \sqrt{r_1^2 + 4r_2}\right)\Big/ 2\right), R_2 = -R_1(\bar{D}_a(r_1 + \sqrt{r_1^2 + 4r_2})\Big/ 2 + \bar{v}_a)$$

Let us consider the second equation (3.2) in more detail. Applying the integral Laplace transform to it, we obtain the following equation:

$$f(p) = \frac{C_2 R_2 p + p^2 C_2 R_1 - \bar{\lambda} A_\infty}{p\left(\bar{D}_g p^\alpha + V^\beta p^\gamma - \bar{v}_g p^\beta - \bar{\lambda}\right)}. \tag{3.4}$$

Let us separately consider the expression: $1/(\bar{D}_g p^\alpha + V^\gamma p^\gamma - \bar{v}_g p^\beta - \bar{\lambda})$. Here the following cases are possible: $\gamma < \beta, \gamma = \beta, \gamma > \beta$ The case when $\gamma = \beta$ leads us to the solution obtained in [12], but with slightly different coefficients. Let $\gamma > \beta$, then we can write

$$1/(p^\alpha - \sigma p^\gamma - \mu p^\beta - b) = p^{-\gamma}/\left[(p^{\alpha-\gamma} - \sigma)(1 - (\mu p^{\beta-\gamma} + b p^{-\gamma})/(p^{\alpha-\gamma} - \sigma))\right],$$

$$\sigma = -V^\gamma/\bar{D}_g, \quad \mu = \bar{v}_g/\bar{D}_g, \quad b = \bar{\lambda}/\bar{D}_g.$$

Taking into account the condition $\left|(\mu p^{\beta-\gamma} + b p^{-\gamma})/(p^{\alpha-\gamma} + \sigma)\right| < 1$ it can be written as follows [?]:

$$\sum_{n=0}^{\infty} p^{-\gamma}\left(\mu p^{\beta-\gamma} + b p^{-\gamma}\right)^n \Big/ (p^{\alpha-\gamma} + \sigma)^{n+1} = \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{n!\mu^m b^l}{m!l!} p^{-1-l-(\gamma-\beta)m} \Big/ (p^{\alpha-\gamma} + \sigma)^{n+1}.$$

According to the work [5], the inverse Laplace transform of this expression for the first term of the equation (3.4) gives:

$$f_1\left(x\right) = \frac{C_2 R_2}{\bar{D}_g} \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{n!\mu^m b^l x^{(\alpha-\gamma)n+\alpha-2+l+(\gamma-\beta)m}}{m!l!} F_1,$$

$$F_1 = {}_1\Psi_1\left[\begin{array}{l}(n+1,1)\\((\alpha-\gamma)n+\alpha-1+l+(\gamma-\beta)m, \alpha-\gamma)\end{array}\middle| (\sigma x^{\alpha-\gamma})\right],$$

where ${}_1\Psi_1\left(x\right)$ is the generalized Wright function [5]. Similarly for the second and third terms of the equation (3.4):

$$f_2\left(x\right) = \frac{C_2 R_1}{\bar{D}_g} \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{n!\mu^m b^l x^{(\alpha-\gamma)n+\alpha-1+l+(\gamma-\beta)m}}{m!l!} F_2,$$

$$F_2 = {}_1\Psi_1\left[\begin{array}{l}(n+1,1)\\((\alpha-\gamma)n+\alpha+l+(\gamma-\beta)m, \alpha-\gamma)\end{array}\middle| (\sigma x^{\alpha-\gamma})\right]$$

$$f_3\left(x\right) = -A_\infty b \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{n!\mu^m b^l x^{(\alpha-\gamma)n+\alpha+l+(\gamma-\beta)m}}{m!l!} F_3,$$

$$F_3 = {}_1\Psi_1 \left[\begin{array}{l} (n+1,1) \\ ((\alpha-\gamma)n+\alpha+1+l+(\gamma-\beta)m, \alpha-\gamma) \end{array} \middle| \left(\sigma x^{\alpha-\gamma}\right) \right]$$

The solution in this case is a superposition:

$$f(x) = f_1(x) + f_2(x) + f_3(x)\,. \tag{3.5}$$

Let's find the constant C_2. To do this, we use the method proposed in [12]. We write the equation (3.4) as follows:

$$f(p) = \frac{C_2 R_1}{\bar{D}_g p} \frac{(p^2 + R_2/R_1 p - A_\infty b/(R_1 C_2))}{p^\alpha - \mu p^\beta - \sigma p^\gamma - b} = \tag{3.6}$$

$$= \frac{C_2 R_1}{\bar{D}_g p} \frac{\left[p - \left(\dfrac{-R + \sqrt{R^2 + 4A_\infty b/(R_1 C_2)}}{2}\right)\right]\left[p - \left(\dfrac{-R - \sqrt{R^2 + 4A_\infty b/(R_1 C_2)}}{2}\right)\right]}{p^\alpha - \mu p^\beta - \sigma p^\gamma - b},$$

$$R = R_2/R_1.$$

Since the denominator of the equation (3.6) $p^\alpha - \mu p^\beta - \sigma p^\gamma - b$ is an infinitely increasing function, and its values are in the right half-plane, then the equation has a positive root. Let us denote this root as K, then the denominator of the equation (3.5) can be expanded in powers of $p - K$:

$$p^\alpha - \mu p^\beta - \sigma p^\gamma - b = \sum_{i=0}^{\infty} c_i (p-K)^i \tag{3.7}$$

Substituting (3.6) into (3.5) we obtain:

$$f(p) = \frac{C_2 R_1}{\bar{D}_g p} \frac{\left[p + \left(\dfrac{R - \sqrt{R^2 + 4A_\infty b/(R_1 C_2)}}{2}\right)\right]\left[p + \left(\dfrac{R + \sqrt{R^2 + 4A_\infty b/(R_1 C_2)}}{2}\right)\right]}{(p-K)\sum\limits_{i=0}^{\infty} c_i (p-K)^{i-1}} \tag{3.8}$$

In order for $f(p)$ to be analytic in the right half-plane, it is necessary to choose the constant C_2 in such a way that the first bracket in the numerator is canceled with the bracket in the denominator. This is achieved when

$$C_2 = \frac{A_\infty b}{(K^2 + RK) R_1} \tag{3.9}$$

Then the solution (3.6) according to (3.9) will be written as follows:

$$A(\xi,\tau) = \frac{A_\infty b e^{-(\xi - V\tau)\left(\frac{r_1 + \sqrt{r_1^2 + 4r_2}}{2}\right)}}{(K^2 + RK) R_1}, \quad \xi > 0$$

$$A(\xi,\tau) = A_\infty b \left[\frac{R_2\theta_2 + R_1\theta_1}{\bar{D}_g (K^2 R_1 + K R_2)} - \theta_0\right], \xi < 0$$

$$\theta_i = \sum_{n=0}^{\infty} \sum_{m+l=n} \frac{\mu^m b^l (\xi - V\tau)^{(\alpha-\gamma)n+\alpha-2+i+l+(\gamma-\beta)m}}{m!l!} F_i, \tag{3.10}$$

$$F_i = {}_1\Psi_1\left[\begin{array}{l}(n+1,1)\\((\alpha-\gamma)n+\alpha-2+i+l+(\gamma-\beta)m,\alpha-\gamma)\end{array}\middle|\sigma\left(\xi-V\tau\right)^{\alpha-\gamma}\right],$$

$$R_1=\exp\left(V\tau\left(r_1+\sqrt{r_1^2+4r_2}\right)\Big/2\right), R_2=-R_1(\bar{D}_a(r_1+\sqrt{r_1^2+4r_2})\Big/2+\bar{v}_a),$$

K is the solution of the equation $y^\alpha-\sigma y^\gamma-\mu y^\beta-b=0$, $i=1,2,3$.

Proposition 3.1. *Similar considerations can be used for the case $\gamma<\beta$. The parameters in (3.10) will be redefined.*

Proposition 3.2. *It can be shown using the methodology of the works* [12]*, that in the case when $\alpha=2,\beta=\gamma=1$ solution (3.10) goes over to the classical solution obtained in the article* [11].

4. Conclusion

In this paper, an analytical solution was obtained for the problem of non-stationary radon transfer from loose sediments to the atmospheric surface layer in the anomalous diffusion and advection mode, which has an explicit form. This solution is expressed through the generalized Wright function depending on the parameters of the environment α,β,γ.

The next stage of the development of the developed model is the study of its solution for various values of α,β,γ, as well as a comparison of the modeling results with experimental data.

Another direction of research may be related to solving inverse problems with the aim of identifying the values of orders of fractional derivatives based on experimental data [2].

Funding. The work was carried out within the framework of the Agreement between the V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan and the Federal State Budgetary Scientific Institution "Institute of Cosmophysical Research and Radio Wave Propagation of the Far Eastern Branch of the Russian Academy of Sciences" on cooperation in the field of mathematical research (No. 117 of April 28, 2022), as well as within the framework of the State Assignment of the Institute of Cosmophysical Research and Radio Wave Propagation of the Far Eastern Branch of the Russian Academy of Sciences (topic registration number 124012300245-2).

References

[1] Alimov Sh.A., Ashurov R.R.; On the backward problems in time for time-fractional subdiffusion equations. Fract. Differ. Calc., –2021. –11. –P. 203 217.

[2] Ashurov R.R., Shakarova M.D.; Inverse problem for the subdiffusion equation with fractional Caputo derivative, Ufa Mathematical Journal, –2024. –16. –P. 111-126.

[3] Caputo M., Fabrizio M.; On the notion of fractional derivative and applications to the hysteresis phenomena. Meccanica, –2017. –52. –P. 30433052.

[4] Geraskin I.N., Krayushkin, A.V., Smirnova, A.K.; RC Code for Calculating Doses from Radioactive Releases to the Atmosphere. Phys. Atom. Nuclei,–2023.–86.–P. 18181826.

[5] Kilbas A.A., Srivastava H.M., Trujillo J.J.; Theory and Applications of Fractional Differential Equations. Elsevier. –2006.

[6] Monroy D.A., Raposo E.P.; Solution of the space-fractional diffusion equation on bounded domains of superdiffusive phenomena. Phys. Rev. E, –2024. –110. –054119.

[7] Nakhushev A.M.; Fractional calculus and its application. Fizmatlit. – 2003.

[8] Novozhenova O.G. Life And Science of Alexey Gerasimov, One of the Pioneers of Fractional Calculus in Soviet Union. Fract. Calc. Appl. Anal., —2017. –20. –P. 790809.

[9] Parovik R.I., Shevtsov B.M.; Radon transfer processes in fractional structure medium. Mathematical Models and Computer Simulations, –2010. –2. – P. 180-185.

[10] Parovik R.I.; Mathematical models of the non-classical theory of the emanation method. Vitus Bering Kamchatka State University. – 2014.

[11] Parovik R.I.; Mathematical model of non-stationary diffusionadvection of radon in the soilatmosphere system. E3s web of conferences, –2023. –462. -02023.

[12] Parovik R.I.; Nonlocal model of diffusion-advection radon in ground-atmosphere. Mat. Model., –2010. –22. –P. 95106.

[13] Stajic J.M., Nikic B., Gulan L. et al.; Estimation of radon emanation power: a comparison of different methods. Air Qual Atmos Health, –2023. –16. –P. 20532061.

[14] Tverdyi D., Makarov E., Parovik R.; Estimation of Radon Flux Density Changes in Temporal Vicinity of the Shipunskoe Earthquake with Mw = 7.0, 17 August 2024 with the Use of the Hereditary Mathematical Model. Geosciences, –2025. –15(1). –30.

[15] Uchaikin V.V.; Fractional phenomenology of cosmic ray anomalous diffusion. Phys.-Usp., –2013. –56. –1074.

[16] Yakovleva V., Zelinskiy A., Parovik R., et al.; Model for reconstruction of γ-background during liquid atmospheric precipitation. Mathematics. –2021. –9, –P. 1636.

[17] Yakovleva V.S., Parovik R.I.; Solution of diffusion-advection equation of radon transport in many-layered geological media. Nukleonika, –2010. –55, – P. 601-606.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.200-209
DOI: 10.29229/uzmj.2025-2-19

On the generalized continuity of a function at the points of convergence of its spectral decomposition

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this paper, we study the necessary conditions for the summability of spectral expansions in eigenfunctions of the Schrödinger operator with a singular potential on an arbitrary $N-$ dimensional domain. It is proved that if the spectral expansion of an arbitrary function at some point is summed by Riesz means, then its mean value over the sphere at the specified point has generalized continuity.

Keywords: eigenfunctions, eigenvalues, spectrum, Fourier series, convergence.
MSC (2020): 34L10,34L15, 47A10,42A16,42A20

1. Introduction

Sufficient conditions for the convergence of spectral expansions associated with elliptic operators have been well studied by many mathematicians [1, 2, 10, 3, 13, 4, 9, 12, 16, 15, 17, 5]. In particular, the convergence of spectral expansions in terms of the eigenfunctions of the Schrödinger operator can be found in works [1, 2, 10, 3, 13, 4, 9, 12].

In [3] the Schrödinger operator with potential $q(x)$, singular at the point x_0, is considered. In this paper conditions are proved that it is ensure the summability at each point $x \in \Omega$ by Riesz means of the Fourier series in to eigenfunctions of the Schrödinger operator. For the Schrödinger operator with smooth potential it is considered in [13], and in [4] for the case of potentials having a first-order singularity on manifolds of dimension no greater than $N-3$. In [9] the uniform convergence and the convergence in mean of Riesz means of spectral expansions of the Schrödinger operator with singular potential satisfying the Stummel type condition are established.

Let Ω be an arbitrary bounded domain in $R^N (N \geq 3)$ with a C^∞-smooth boundary and let $q(x)$ be an arbitrary non-negative function from the class $L_2(\Omega)$. Consider the Schrödinger operator

$$L = -\Delta + q(x) \tag{1.1}$$

with domain $C_0^\infty(\Omega)$. In the domain, the operator is symmetric and positive. Let $\hat{L}$ denote an arbitrary positive self-adjoint extension of the operator L with discrete spectrum. According to classical theorem of K.O. Friedrichs [16, 15], such a self-adjoint extension exists.

Let denote the sequence $0 < \lambda_1 \leq \lambda_2 \leq ...$ of eigenvalues of the operator $\hat{L}$, and $\{u_n\}_{n=1}^\infty$ the complete orthonormal system of the corresponding eigenfunctions in $L_2(\Omega)$. The spectral decomposition of an arbitrary function $f \in L_2(\Omega)$ has the form

$$E_\lambda f(x) = \sum_{\lambda_n < \lambda} f_n u_n(x) \tag{1.2}$$

and their Riesz means of order s are defined by the equality

$$E_\lambda^s f(x) = \sum_{\lambda_n < \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x) \;=\; \int_0^\lambda \left(1 - \frac{t}{\lambda}\right)^s dE_t f(x). \tag{1.3}$$

Note that for $s = 0$ the Riesz means coincides with the partial sums of the series (1.2). In the present paper, we study the behavior of a function f in a neighborhood of the point at which its spectral expansion or Riesz means are convergent. The problem of convergence of spectral expansions and their Riesz means at an arbitrary inner point of the domain has been studied in sufficient detail for arbitrary elliptic operators (see reviews [[1, 2, 10, 3]]).

In this paper, we assume that the potential q is spherically symmetric. Namely, let $a \in C^\infty(0,\infty)$ be a non-negative function satisfying

$$t^k|a^{(k)}(t)| \leq C_\tau t^{\tau-1} \quad (t>0; \quad k=0,1,...,[N/2]) \tag{1.4}$$

for some $\tau > 0$. If $N = 3$, then it is assumed that $\tau > 1/2$. In particular, we have

$$a(t) \leq C_\tau t^{\tau-1} \quad (t>0). \tag{1.5}$$

The constant C_τ, depends only on τ. Now assume that the potential q has the form

$$q(x) = \frac{a(|x-x_0|)}{|x-x_0|}, \quad x \in \Omega,$$

where $x_0 \in \Omega$ is an arbitrary but fixed point.

The average value of order $\alpha > 0$ of a function $f(x)$ over a ball of radius r centered at point x is defined as follows:

$$S_r^\alpha f(x) \;=\; \frac{1}{\omega_N(\alpha) r^N} \int\limits_{|y|\leq R} \left(1 - \frac{|y|^2}{r^2}\right)^{\alpha-1} f(x+y)dy, \tag{1.6}$$

where

$$\omega_N(\alpha) \;=\; \int\limits_{|y|<1} \left(1-|y|^2\right)^{\alpha-1} dy.$$

In this section, the main result of the paper is formulated in the following theorem.

Theorem 1.1. *Suppose that numbers $\alpha > 0$ and $s \geq 0$ satisfy the condition*

$$s - \alpha < (N-3)/2. \tag{1.7}$$

Let the expansion of a function $f \in L_2(\Omega)$ at some point $x_0 \in \Omega$ be summed by Riesz means of order s:

$$\lim_{\lambda\to\infty} E_\lambda^s f(x_0) = f(x_0). \tag{1.8}$$

Then the following equality holds

$$\lim_{r\to 0} S_r^\alpha f(x_0) = f(x_0).$$

Similar theorems are proved for the Laplace and $B-$ elliptic operators in the works [6], [7]. Also in the work [8] for a $B-$ elliptic operator with a Bessel operator in one of the variables it is proved that if the spectral decomposition of an arbitrary function at some point of a given hypersurface is summed by Riesz means, then its average value over a hemisphere with the center at the specified point has generalized smoothness.

2. Relationship between ball means and Riesz means.

By integrating the spectral expansion

$$f(x+y)=\sum_{k=1}^{\infty} f_k u_k(x+y),$$

over the ball $|y|<r$ after multiplication by an appropriate factor, we obtain the representation

$$S_r^{\alpha} f(x)=\sum_{k=1}^{\infty} f_k S_r^{\alpha} u_k(x+y). \tag{2.1}$$

Next, by definition (1.5), we have

$$S_r^{\alpha} u_k(x)=\frac{1}{\omega(N,\alpha) r^N} \int\limits_{|y|\leq r} \left(1-\frac{|y|^2}{r^2}\right)^{\alpha-1} u_k(x+y)dy. \tag{2.2}$$

Next, we will use the following formula for the average value (see [4]):

$$\int\limits_{\theta} u_n(x+\rho\theta)d\theta = u_n(x)\left\{2^{N/2-1}\Gamma(N/2)(\rho\mu_n)^{1-N/2}J_{N/2-1}(\rho\mu_n)+\psi(\rho,\mu_n)\right\}, \tag{2.3}$$

where $|\psi(\rho,\mu_n)|\leq C\mu_n^{-\tau}\omega(\rho\mu_n)$, $\omega(t)=\min(1,t^{(1-N)/2})$. Substituting formula (2.3) into integral (2.2), we obtain the equality

$$S_r^{\alpha} u_k(x)=u_k(x)\frac{2^{N/2-1}\Gamma(N/2)}{\omega(N,\alpha)r^N}\int\limits_0^r \left(1-\frac{\rho^2}{r^2}\right)^{\alpha-1} J_{N/2-1}(\rho\sqrt{\mu_k})(\rho\sqrt{\mu_k})^{1-N/2}\rho^{N-1}d\rho+$$

$$+u_k(x)\frac{1}{\omega(N,\alpha)r^N}\int\limits_0^r \left(1-\frac{\rho^2}{r^2}\right)^{\alpha-1}\rho^{N-1}\psi(\rho,\mu_k)d\rho.$$

According to the well-known formula (see [11], p.717) for the Bessel function, the first integral on the right-hand side is equal to the expression $2^{\alpha-1}\Gamma(\alpha)r^N(r\mu_n)^{1-\alpha-N/2}J_{N/2+\alpha-1}(r\mu_n)$, and the second integral is estimated by $r^N\xi_k(x,r)$, where $\xi_k(x,r)=O(1)\mu_k^{-\tau}$.

Therefore, the equality is satisfied

$$S_r^{\alpha} u_k(x)=B(N,\alpha)u_k(x)\frac{J_{N/2+\alpha-1}(r\mu_n)}{(r\mu_n)^{N/2+\alpha-1}}+\frac{1}{\omega(N,\alpha)}u_k(x)\xi_k(x,r), \tag{2.4}$$

where $\xi_k(x,r)=O(1)\mu_k^{-\tau}$, $B(N,\alpha)=2^{\alpha-1}\Gamma(\alpha)\frac{2^{N/2-1}\Gamma(N/2)}{\omega(N,\alpha)}$.

Then from equality (2.1) and (2.4) we obtain the representation

$$S_r^{\alpha} f(x)=B(N,\alpha)\sum_{k=1}^{\infty} f_k u_k(x)\frac{J_{N/2+\alpha-1}(r\mu_n)}{(r\mu_n)^{N/2+\alpha-1}}+\sum_{k=1}^{\infty} f_k u_k(x)\xi_k(x,r). \tag{2.5}$$

Let us consider separately each of the two terms on the right-hand side of (2.5).

$$\Lambda_r^{\alpha} f(x)=B(N,\alpha)\sum_{k=1}^{\infty} f_k u_k(x)\frac{J_{N/2+\alpha-1}(r\mu_n)}{(r\mu_n)^{N/21+\alpha-1}} \tag{2.6}$$

and we denote

$$I_r^\alpha f(x) = \sum_{k=1}^{\infty} f_k u_k(x)\xi_k(x,r). \tag{2.7}$$

Then

$$S_R^\alpha f(x) = \Lambda_R^\alpha f(x) + I_R^\alpha f(x). \tag{2.8}$$

Lemma 2.1. *Let conditions (1.7) be satisfied, and let the Fourier coefficients of a function $f \in L_2(\Omega)$ satisfy the condition*

$$\sum_{n=1}^{\infty} |f_n|^2 \lambda_n^m < \infty \tag{2.9}$$

for any positive integer m. Then the relation

$$\Lambda_r^\alpha f(x) = \frac{C(N,\alpha,s)}{r^{N/2+\alpha-s-2}} \int_0^\infty (\sqrt{\lambda})^{s-\alpha-N/2} J_{N/2+\alpha+s}(r\sqrt{\lambda}) E_\lambda^s f(x) d\lambda, \tag{2.10}$$

holds for an arbitrary point $x \in \Omega$ and for each r in the interval $0 < r < dis\{x, \partial\Omega\}$, where $J_\nu(z)$ is a Bessel function of the first kind and $C(N,\alpha,s)$ is some positive constant.

Proof. Condition (2.9) and the uniform convergence of the series

$$\sum_{n=0}^{\infty} |u_n(x)|^2 \lambda_n^{-\beta}, \quad \beta > N/2 \tag{2.11}$$

on any compact set $K \subset \Omega$ imply the uniform convergence of the spectral expansion of the function f on compact sets and, in particular, the boundedness of the Riesz means $E_\lambda^s f(x)$ of arbitrary order at any point $x \in \Omega$. Let us consider the integral on the right side of equality (2.10):

$$I(r) = \int_0^\infty (\sqrt{\lambda})^{s-\alpha-N/2} J_{\alpha+s+N/2}(r\sqrt{\lambda}) E_\lambda^s f(x) d\lambda. \tag{2.12}$$

According to the well known estimates for Bessel functions (see [18], chapter 7)

$$J_\nu(z) = O(z^\nu), \quad J_\nu(z) = O(z^{-1/2}), \tag{2.13}$$

valid for $\nu \geq -1/2$ and $z > 0$, the integral

$$\int_0^\infty |J_{\alpha+s+N/2}(t)| t^{1+s-\alpha-N/2} dt$$

is converges for all $\alpha \geq 0$ and $s \geq 0$ satisfying the condition $1 + s - \alpha - N/2 < -1/2$. This condition obviously coincides with (1.7). It follows directly from this that the integral (2.10) for any function $f \in L_2(\Omega)$ converges absolutely and uniformly on any compact set $K \subset \Omega$.

For any $A > 0$, we introduce the partial integral

$$I_A(r) = \int_0^A (\sqrt{\lambda})^{s-\alpha-N/2} J_{\alpha+s+N/2}(r\sqrt{\lambda}) E_\lambda^s f(x) d\lambda. \tag{2.14}$$

It is clear that

$$\lim_{A\to\infty} I_A(r) = I(r). \tag{2.15}$$

We set

$$a_+^s = \begin{cases} a^s, & \text{if} \quad a > 0, \\ 0, & \text{if} \quad a \le 0 \end{cases}$$

and transform the partial integral (2.14) with regard of definition (1.3) as follows:

$$I_A(r) = \int\limits_0^A (\sqrt{\lambda})^{N/2+\alpha+s} J_{N/2+\alpha+s}(r\sqrt{\lambda}) \sum_{\lambda_n<\lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x) d\lambda =$$

$$= \int\limits_0^A (\sqrt{\lambda})^{N/2+\alpha+s} J_{N/2+\alpha+s}(r\sqrt{\lambda}) [\sum_{\lambda_n<\lambda} (\lambda - \lambda_n)^s f_n u_n(x)] \lambda^{-s} d\lambda =$$

$$= \int\limits_0^A (\sqrt{\lambda})^{-s-\alpha-N/2} J_{N/2+\alpha+s}(r\sqrt{\lambda}) [\sum_{n=1}^{\infty} (\lambda - \lambda_n)_+^s f_n u_n(x)] d\lambda.$$

By changing the order of integration and summation, we obtain the relation

$$I_A(r) = \sum_{n=1}^{\infty} f_n u_n(x) \int\limits_0^A (\lambda - \lambda_n)^s (\sqrt{\lambda})^{-s-\alpha-N/2} J_{N/2+\alpha+s}(r\sqrt{\lambda}) d\lambda.$$

Note that in the case when $\lambda_n < A$, the integration under the sign of the sum is essentially performed over the interval $\lambda_n \le \lambda \le A$. Further, in the case when $\lambda_n \ge A$, the integrals are equal to 0. Therefore,

$$I_A(r) = \sum_{\lambda_n<A} f_n u_n(x) \int\limits_{\lambda_n}^A (\lambda - \lambda_n)^s (\sqrt{\lambda})^{-s-\alpha-N/2} J_{N/2+\alpha+s}(r\sqrt{\lambda}) d\lambda. \tag{2.16}$$

Making the change of variables $z = \lambda/\lambda_n$ and using the inequality $\lambda_n < A$, we obtain

$$\int\limits_{\lambda_n}^A (\lambda - \lambda_n)^s (\sqrt{\lambda})^{-s-\alpha-N/2} J_{N/2+\alpha+s}(r\sqrt{\lambda}) d\lambda =$$

$$= \lambda_n^{s+1} (\sqrt{\lambda_n})^{-s-\alpha-N/2} \int\limits_1^{A/\lambda_n} (z-1)^s (\sqrt{z})^{-s-\alpha-N/2} J_{N/2+\alpha+s}(r\sqrt{\lambda_n z}) dz.$$

Next, using the formula (see [11], p. 717)

$$\int\limits_1^{\infty} (z-1)^s z^{-\nu/2} J_\nu(a\sqrt{z}) dz = 2^{s+1}\Gamma(s+1) \frac{J_{\nu-s-1}(a)}{a^{s+1}},$$

where $s - \nu/2 - 1/4 < 0$, we obtain the relation

$$\int\limits_1^{A/\lambda_n} (z-1)^s (\sqrt{z})^{-s-\alpha-N/2} J_{N/2+\alpha+s}(r\sqrt{\lambda_n z}) dz =$$

$$= 2^{s+1}\Gamma(s+1)\frac{J_{N/2+\alpha-1}(r\sqrt{\lambda_n})}{(r\sqrt{\lambda_n})^{s+1}} \; - \; L_A(r,\lambda_n),$$

where

$$L_A(r,\lambda_n) = \int\limits_{A/\lambda_n}^{\infty} (z-1)^s(\sqrt{z})^{-s-\alpha-N/2} J_{N/2+\alpha+s}(r\sqrt{\lambda_n z})dz.$$

As a result, from equality (2.16) we obtain

$$I_A(r) = 2^{s+1}\Gamma(s+1)r^{N/2+\alpha-s-2}\sum_{\lambda_n<A} f_n u_n(x)\frac{J_{N/2+\alpha-1}(r\sqrt{\lambda_n})}{(r\sqrt{\lambda_n})^{N/2+\alpha-1}} \; -$$

$$- \sum_{\lambda_n<A} f_n u_n(x)(\sqrt{\lambda_n})^{2+s-\alpha-N/2}L_A(r,\lambda_n). \tag{2.17}$$

Let's enter a number,

$$\varepsilon = \frac{1}{2}[(N-3)/2+\alpha-s] \; > \; 0,$$

which is positive due to condition (1.7). Then

$$|L_A(r,\lambda_n)| \; \le \; C{\lambda_n}^{-1/4}\int\limits_{A/\lambda_n}^{\infty}\frac{dz}{z^{1+\varepsilon}} \; = \; C_1\frac{{\lambda_n}^{\varepsilon-1/4}}{A^{\varepsilon}}.$$

Let us take an arbitrary sufficiently large natural number m and apply the Cauchy-Bunyakovsky inequality. Then

$$\sum_{\lambda_n<A} f_n u_n(x)(\sqrt{\lambda_n})^{2+s-\alpha+N/2}L_A(r,\lambda_n) \; =$$

$$= \; \frac{O(1)}{A^{\varepsilon}}\left(\sum_{\lambda_n<A}|f_n|^2\lambda_n^{2m}\right)^{1/2}\left(\sum_{\lambda_n<A}|u_n(x)|^2\lambda_n^{s-\alpha+1/2+\varepsilon-1/2-2m}\right)^{1/2}.$$

From condition (2.9) it follows that for sufficiently large values of m, both factors in brackets are bounded by $A>0$. Therefore, as A tends to $+\infty$, from (2.15) and (2.17) we obtain:

$$I(r) = 2^{s+1}\Gamma(s+1)r^{N/2+\alpha-s-2}\sum_{n=1}^{\infty} f_n u_n(x)\frac{J_{N/2+\alpha-1}(r\sqrt{\lambda_n})}{(r\sqrt{\lambda_n})^{N/2+\alpha-1}}.$$

Comparing this equality with (2.6), we obtain:

$$\Lambda_r^{\alpha} f(x) = \frac{B(N,\alpha)}{2^{s+1}\Gamma(s+1)}r^{s+2-\alpha-N/2}I(r).$$

From here, taking into account the definition (2.12) of the integral $I(r)$, the required equality (2.10) follows with the constant $C = 2^{-s-1}B(N,\alpha)/\Gamma(s+1)$. The proof of Lemma 2.1 is complete

$\square$

The following lemmas are related to the necessity to remove the constraint (2.9), which substantially restricts the class of considered functions. To this end, we introduce the function

$$F(x,h)=\sum_{n=1}^{\infty} f_n e^{-\lambda_n h} u_n(y), \quad y\in\Omega, \quad h>0. \tag{2.18}$$

The Riesz means of the spectral expansion of this function have the form

$$E_\lambda^s F(y,h)=\sum_{\lambda_n<\lambda}\left(1-\frac{\lambda_n}{\lambda}\right)^s f_n e^{-\lambda_n h} u_n(y). \tag{2.19}$$

Lemma 2.2. *Let $f\in L_2(\Omega)$ and let conditions* (1.7) *be satisfied. If at some point $x\in\Omega$ the value $E_\lambda^s f(x)$ is bounded by $\lambda>0$, then equality* (2.10) *is satisfied uniformly in r on any compact subset of the interval $0<r<dis\{x,\partial\Omega\}$.*

Proof. The proof of Lemma 2.2 is given in [6]. □

The following statement is used in the proof.

Proposition 2.3. *For each $s\geq 0$, there exists a function $\psi_s(t,h,\lambda)$ satisfying the estimate*

$$\int_0^\lambda t^s|\psi_s(t,h,\lambda)|dt \;\leq\; C\lambda^s, \tag{2.20}$$

uniformly with respect to h, $0<h<1$, for all $\lambda>0$ and such that

$$E_\lambda^s F(x,h)=E_\lambda^s f(x)e^{-\lambda h}+\frac{1}{\lambda^s}\int_0^\lambda t^s\psi_s(t,h,\lambda)E_\lambda^s f(x)dt. \tag{2.21}$$

The validity of this statement follows from the results of work [5].

Lemma 2.4. *Let $f\in L_2(\Omega)$ and let conditions* (1.7) *be satisfied. If equality* (1.8) *is satisfied at some point $x\in\Omega$, then*

$$\lim_{r\to 0}\Lambda_r^\alpha f(x)=f(x). \tag{2.22}$$

Proof. According to Lemmas 2.1 and 2.2, equalities (2.10) hold for the function $f\in L_2(\Omega)$. In equalities (2.10), after replacing the variable $t=r\sqrt{\lambda}$, we obtain the equality

$$\Lambda_r^\alpha f(x)=C(N,\alpha,s)\int\limits_0^\infty t^{s-\alpha-N/2}J_{N/2+\alpha+s}(t)E_{t^2/r^2}^s f(x)dt.$$

At each fixed point the equality (1.8) is satisfied. Due to the boundedness of $|E_{t^2/r^2}^s f(x)|\leq M(x)$ and the convergence of the integral under conditions (1.7)

$$\int\limits_0^\infty t^{s-\alpha-N/2}|J_{N/2+\alpha+s}(t)|dt<\infty$$

using Lebesgue's theorem, we obtain equality (2.22). □

Since $\hat{L}$ is an arbitrary positive self-adjoint extension of the operator L, then for any $\sigma > 0$ the operator $\hat{L}$ is an integral operator

$$\hat{L}^{-\sigma}u(x) = \int\limits_{\Omega} G_\sigma(x,y)u(y)dy,$$

whose kernel $G_\sigma(x,y)$ has the following expansion in a Fourier series in eigenfunctions:

$$G_\sigma(x,y) = \sum_{n=1}^{\infty} \frac{u_n(x)u_n(y)}{\lambda^\sigma}.$$

For a negative degree self-adjoint extension of the operator under consideration, the following estimate is obtained:

Lemma 2.5. *Let $\sigma > N/4$. Then for any $f \in L_2(\Omega)$ and on each compact set $K \subset \Omega$ we have the estimate*

$$\|\hat{L}^{-\sigma}f\|_{L_\infty(K)} \le C(K)\|f\|_{L_2(\Omega)}.$$

Proof. See [3]. □

Corollary 2.6. *For any $\sigma > N/4$ the estimate*

$$\sum_{n=1}^{\infty} |u_n(x)|^2\lambda_n^{-2\sigma} \le C(K), \quad x \in K, \tag{2.23}$$

is satisfied uniformly on each compact subset $K \subset \Omega$.

Proof. The proof of the corollary follows from the following estimate

$$\sum_{|\sqrt{\lambda_n}-\mu|\le 1} |u_n(x)|^2 \le C(K)\mu^{N-1}, \quad \mu > 1.$$

Then

$$\sum_{n=1}^{\infty} \frac{|u_n(x)|^2}{\lambda_n^{2\sigma}} = \sum_{k=1}^{\infty} \sum_{k\le\sqrt{\lambda_n}\le k+1} \frac{|u_n(x)|^2}{\lambda_n^{2\sigma}} = \sum_{k=1}^{\infty} k^{-4\sigma} \sum_{k\le\sqrt{\lambda_n}\le k+1} |u_n(x)|^2 \le$$

$$\le C(K)\sum_{k=1}^{\infty} k^{-4\sigma}(k+1)^{N-1} \sim \int\limits_1^{\infty} \frac{dt}{t^{4\sigma-N+1}}.$$

The integral converges for $\sigma > N/4$. □

Corollary 2.7. *For the quantity $I_r^\alpha f(x)$ introduced by equality (**??**), for any $\tau > N/4$ the estimate*

$$|I_r^\alpha f(x)| \le C(K)\|f\|_{L_2(\Omega)} \tag{2.24}$$

is satisfied uniformly on each compact subset $K \subset \Omega$.

Lemma 2.8. *Let $f \in L_2(\Omega)$ then*

$$\lim_{R\to 0} |S_R^\alpha f(x) - \Lambda_R^\alpha f(x)| = 0. \tag{2.25}$$

Proof. According to Corollary 2.7 of lemma 2.5, for any function $f \in L_2(\Omega)$ the estimate (2.24) is satisfied. We show that this estimate implies the equality

$$\lim_{R\to 0} I_R^\alpha f(x) = 0. \tag{2.26}$$

This equality obviously holds if $f \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$, then for any $\varepsilon > 0$ there exist functions $f_1 \in C_0^\infty(\Omega)$ and $f_2 \in L_2(\Omega)$ such that $f = f_1 + f_2$ and

$$\|f_2\|_{L_2(\Omega)} \le \varepsilon.$$

In this case, from (2.24) we obtain

$$|I_R^\alpha f(x)| \le |I_R^\alpha f_1(x)| + |I_R^\alpha f_2(x)| \le |I_R^\alpha f_1(x)| + C\varepsilon.$$

Hence,

$$\overline{\lim_{R\to 0}} |I_R^\alpha f(x)| \le C\varepsilon,$$

and due to the arbitrariness of $\varepsilon > 0$, the required equality (2.26) follows from here, which, according to relation (2.8), means the fulfillment of equality (2.25). □

The proof of the theorem follows directly from lemmas 2.4 and 2.8.

References

[1] Alimov Sh. A., Ilin V. A., Nikishin E. M.; Convergence problems of multiple trigonometric series andspectral decompositions. I, Russ. Math. Surv. –1976. -31. -6. –P. 29–86.

[2] Alimov Sh. A., Ashurov R. R, Pulatov A. K.; Multiple Fourier series and Fourier integrals. Commutative harmonic analysis. IV. Encyclopedia Math. Sci.,Berlin-Heidelberg, Springer Verlag. –1991. - 42. –P. 1–95.

[3] Alimov Sh. A. Joo I.; On the Riesz summability of eigenfunction expansions. Acta Sci.Math. –1983. -45. –P. 5–18.

[4] Alimov Sh. A., Kholmukhamedov A. R.; Mixed type equations and free boundary problems (Equations of Mixed Type and Problems with Free Boundary). Tashkent. –1987. –P. 147–156.

[5] Alimov Sh. A.; On the summation of spectral expansions by the Riesz and Abel methods. Uzbek. Math. J. –2011. -3. –P. 20–35.

[6] Alimov Sh. A.; On the smoothness of mean values of functions with summable spectral expansion. Differential Equations. –2012. -48. -4. –P. 506–516.

[7] Alimov Sh. A., Pirmatov Sh. T.; On the Smoothness of a Function at the Convergence Point of Its Spectral Expansion Associated with B-Elliptic Operators. Lobachevskii Journal of Mathematics. –2023. -Vol. 44. -No. 8. –P. 3207–3217.

[8] Alimov Sh. A., Pirmatov Sh. T.; On the Pinsky Phenomenon for B-Elliptic Operators. Differential Equations. – 2023. -Vol. 59. -No. 5. –P. 606–617.

[9] Ashurov R. R., Faiziev Yu. E.; On Eigenfunction Expansions Associated with the Schrodinger Operator with a Singular Potential. Differential Equations, –2005. -Vol. 41. -No. 2. –P. 254–263.

[10] Brandolini L., Colzani L.; Localization and convergence of eigenfunction expansions. Journal of Fourier Analysis and Applications, –1999. -vol.5(5). –P. 431–447.

[11] Gradshtein I. S., Ryzhik I. M.; Tablitsy integralov, summ, ryadov i proizvedenii (Tables of Integrals, Sums, Series, and Products). Moscow, –1963.

[12] Khalmukhamedov A. R.; Expansions in eigenfunctions of the Schrodinger operator with singular potential.Differentsialnye Uravneniya. –1984. -No.9, –P.1642–1645.

[13] Levitan B. M.; On expansions according to eigenfunctions of the equation $\Delta u + \lambda - q(x_1, x_2, .., x_n) = 0$. Izv. Akad. Nauk SSSR Ser. Mat.–1956.-20. –P.437–468.

[14] Pirmatov Sh. T.; Necessary conditions of decomposition by Bessel operator's eigenfunctions. Uzbek Mathematical Journal. –2020. -No. 1. –P.97–101.

[15] Riesz F., Sz.-Nagy.; Lecons d'analyse finctionnelle. Budapesht: Akademiai Kiado. –1952.

[16] Sz.-Nagy.; Spektraldarstellung linear Transformationen des IIiberstchen Raumes. Ergebnisse der MAthematik, V/5. 1942. Berlin: Springer-Verlag.

[17] Titchmarsh E.C. Eigenfunction Expansions Associated with Second Order Differential Equations. Part II.Oxford: Clarendon Press.–1958.

[18] Watson G. N.; A Treatise on the Theory of Bessel Functions. 2nd ed. London: Cambridge University Press. –1966. –P. 804.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.210-221
DOI: 10.29229/uzmj.2025-2-20

Nonlocal Regularization of Protter-Morawetz BVP for the 3-D Tricomi equation

Popivanov N.

Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. This is a work in progress with my coauthor Barbara Keyfitz, Professor of Mathematics in Ohio State University. In this paper some boundary-value problems (BVPs) for partial differential equations are discussed. The situation is rather surprising and there is no general understanding even more than sixty years after Murray Protter (Berkeley) formulated them. These are multidimensional analogues of classical BVPs on the plane and intuitively the initial expectations was that their properties would be similar. Unexpectedly, it turned out that unlike the two-dimensional variants, the Protter problems are not well-posed. The generalized solution is uniquely determined, but it may have a strong singularity at an isolated boundary point even for the wave equation with infinitely smooth right-hand side. In this paper will be introduced a new BVP which will be nonlocal and in some sence is a "regularizer" for the probably strongly overdetermined original Protter-Morawetz problem for the 3-D Tricomi equation. Let note also that we put a long list of bibliography inside.

Keywords: ill-posed problems, regularization methods, boundary-value problems, nonlocal problems, mixed type equations

MSC (2020): 35M12, 35M30, 35L02, 35Q35

1. Guderley-Morawetz problem in $\mathbb{R}^2$

Consider the two dimensional problem for the Gellerstedt equation

$$t|t|^{m-1}u_{xx} - u_{tt} = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \sigma \cup A_1C_1 \cup A_2C_2,$$

where $m > 0$, Ω is a mixed type domain in the plane.

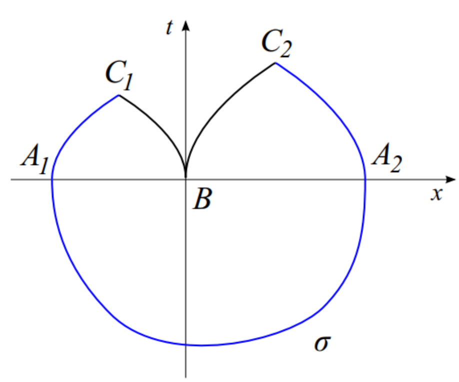

Figure 2. The domain for Guderley-Morawetz problem

In the Guderley-Morawetz problem one takes a simply bounded open and connected set (containing the origin, say) and removes the *solid characteristic cone with vertex at the origin* (look to Fig. 2). The curves A_1C_1 and A_2C_2 are characteristic.

When $t < 0$ the equation is elliptic and describes subsonic flows, while for $t > 0$ the equation is hyperbolic that corresponds to supersonic flows. The topic was extensively studied in the 1950s and 1960s with the development of supersonic aircrafts. The need of practical applications, as for efficient airplane design (a connection first established by Frankl), was the driving force and many prominent scientists from USSR and USA (especially from the Courant Institute) were involved in the research. In particular the Guderley-Morawetz problem models flows around airfoils. More explanations about this statement of the problem one could find in L. Bers [5], Some results for existence of week solution and uniqueness for a strong solution in weighted Sobolev spaces you can find in C. Morawetz [24]. Finally, a common result about "week solutions are strong ones" was proved by P. Lax and R. Phillips [22].

An historic survey from Canthleen Morawetz one could find in [25]. Let also mention here that according to Cathleen Morawetz, another very interesting domain from the physical point of view is with the closed line from the hyperbolic part of the domain... The answer of this question is still open.

2. Protter-Morawetz multidimensional problem

M.K. Protter [35] proposed a multidimensional analogue to the two-dimensional Guderley-Morawetz problem for the Gellerstedt equation of hyperbolic-elliptic type.

$$Lu \equiv t|t|^{m-1}\left(u_{x_1x_1} + u_{x_2x_2}\right) - u_{tt} = 0 \quad \text{in } \Omega,$$

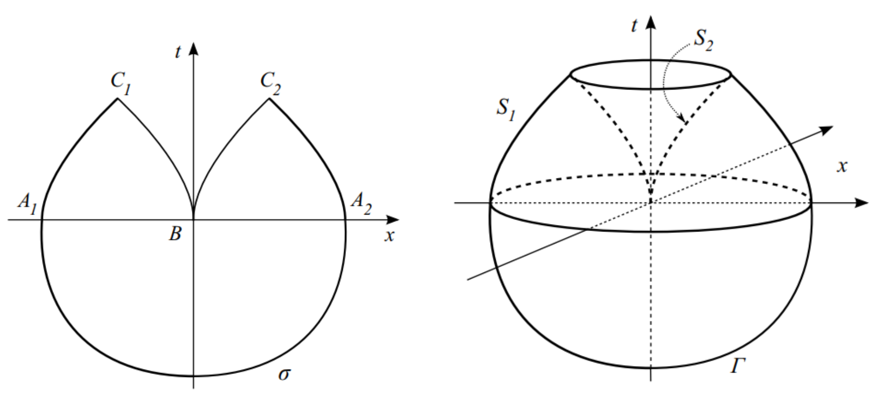

Figure 3. The domain for Protter-Morawetz problem

Remark 2.1. Actually, if on the left picture in Fig 3 the curve σ is symmetric according to the t-axes, the right picture will come after rotation the left one around t-axes.

Later, A.K.Azis and M. Schneider [3] proved the uniqueness of quasi-regular solutions. A.V. Bitsadze in [4] studied the existence of the solution in the axis-symmetric case. A lot of publications in 1960s and and 1970s – USA, USSR, Germany. The next step came in 1957, when appeared a result shows that the multidimensional case is quite different from 2-D case.

In that example for the Protter's problem in the hyperbolic part of the domain only have been shown the infinite dimensions of the cokernel of Protter problem.

To explain more precise the situation in the Mixed type equations, we will show it on the case of the wave equation in next Sections, which statement have been given also by M. Protter [35].

3. Protter problems for the (3+1)-D wave equation

Consider the wave equation in $\mathbb{R}^4$

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x,t)$$

with points $(x,t) = (x_1, x_2, x_3, t)$, in the domain

$$\Omega = \{(x,t)\ :\ 0 < t < 1/2,\ t < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - t\}.$$

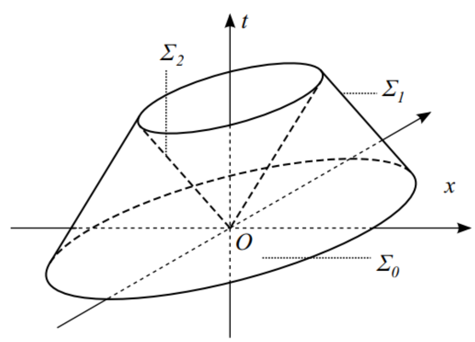

Figure 4. The domain for Protter problem

The boundary of Ω consists of two characteristic parts – the cones

$$\Sigma_1 = \{0 < t < 1/2,\ |x| = 1 - t\},$$

$$\Sigma_2 = \{0 < t < 1/2,\ |x| = t\}$$

and a non-characteristic part – the ball

$$\Sigma_0 = \{t = 0,\ |x| < 1\},$$

centered at the origin $O : x = 0, t = 0$.

Consider the following Protter problems for the inhomogeneous wave equation:

Problem P1. Find a solution of the wave equation in Ω which satisfies the boundary conditions

$$u|_{\Sigma_0} = 0, \quad u|_{\Sigma_1} = 0.$$

Problem P1*. Find a solution of the wave equation in Ω which satisfies the adjoint boundary conditions

$$u|_{\Sigma_0} = 0, \quad u|_{\Sigma_2} = 0.$$

3.1. **Solutions of the homogeneous adjoint problem.**
P. Garabedian [12] proved uniqueness of a classical solution of Protter problem. What about existence? Protter's advice: use of the Asgeirsson principle. Very short time later appears the following very surprising result:

Theorem 3.1 (Tong Kwang Chang [47])**.** *There are infinite number of linearly independent classical solutions to the homogeneous problem $P1^*$.*

Let us mention, that such result is true for the case of Gellerstedt and Keldish equations also (see [13], [15], [14], [29], [30]).

If $\mathbf{v}(x,t)$ is a classical solution to the homogeneous adjoint problem $P1^*$, then **a necessary condition** for the existence of classical solution for the Problem $P1$ is the orthogonality of the right-hand side function f to the function $\mathbf{v}$:

$$(f, \mathbf{v}) = (\Box u, \mathbf{v}) = (u, \Box \mathbf{v}) = 0.$$

To avoid such infinite number of necessary conditions, Popivanov and Schneider introduced **generalized solutions** for the problem $P1$, eventually with a singularity at the origin O.

Definition 3.2 ([36])**.** A function $u = u(x,t)$ is called a **generalized solution** of the problem $P1$ in Ω, if:

1) $u \in C^1(\overline{\Omega} \backslash O)$, $u|_{\Sigma_0 \backslash O} = 0, u|_{\Sigma_1} = 0$, and

2) the identity

$$\int_\Omega (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - u_{x_3} w_{x_3} - f w) dx dt = 0$$

for all $w \in C^1(\overline{\Omega})$ such that $w = 0$ on Σ_0 and **in a neighborhood of Σ_2**.

Let mention here a series of paper by S. Aldashev since 1990: singular solutions for hyperbolic and mixed-type equations.

The obvious question: how to solve Problem $P1$ with right-hand side function $f = v(x,t)$ – a solution of the homogeneous Problem $P1^*$?

Popivanov and Schneider [37, 38]:

- Uniqueness
- For each $n \in \mathbb{N}$ there exists a right-hand side function $f \in C^n(\bar{\Omega})$, for which the generalized solution has a strong power-type singularity like $|x|^{-n}$.
- This singularity **is isolated at the vertex O and does not propagate along the bi-characteristics at the light characteristic cone**.

- Khe Kan Cher [18, 19, 20]: new solutions of the homogenous adjoint problem $P1^*$.
- S. Aldashev: some singular solutions
- Korean mathematicians Jong Duek Jeon et al. [16], Jong Bae Choi, Jong Yeoul Park [17]: singular solutions with power-type singularity **without mentioning any orthogonality conditions**.
- N.P, T. Popov [32]: the influence of the orthogonality conditions for the right-hand side function, on the singularity of the generalized solution of Problem $P1$.
- The exact asymptotic behaviour [34], [6], [33].

3.2. **Solutions with exponential growth.**

If only finite number of the orthogonality conditions are not fulfilled, the generalized solution has power-type singularity Natural question: are there solutions with stronger singularity in the case when the right-hand side is not orthogonal to infinite number of the solutions of the homogeneous adjoint problem? We are able to construct an infinitely smooth in $\overline{\Omega}$ right-hand side function $f(x,t)$, such that the corresponding generalized solution has exponential singularity.

Theorem 3.3 (Popivanov, Popov, Witt, [33])**.** *There exists a function $f \in C^\infty(\overline{\Omega})$ and a positive number $\delta < 1/2$, such that the unique generalized solution $u(x,t) \equiv u(x_1,x_2,x_3,t) \in C^1(\overline{\Omega}\backslash O)$ of the problem P1 for the wave equation with right-hand function f, satisfies the estimates*

$$u(0,0,t,t) \geq \exp(t^{-1}) \quad \textit{for} \quad t \in (0,\delta),$$

and for some constant C

$$|u(x,t)| \leq C\exp(2|x|^{-1}) \quad \textit{for} \quad (x,t) \in \Omega.$$

4. Study of singularity

According to the previous results, the generalized solution of the Problem P1 is smooth in $\overline{\Omega} \setminus O$ and has possible singularity only at O. To explain better such behavior we are looking for an extension problem:

Problem E. Extend the generalized solution of Problem P1, as a solution of the wave equation with smooth right-hand side function inside of the light cone Σ_2.

We hope this extension will give more information about possible singularity at O. This sometimes is possible, sometimes not!

4.1. **Polynomial growth of the generalized solution at the vertex O.** Actually, this happens if only finitely many of these orthogonality conditions are violated. In this case the generalized solutions are smooth up to the light cone, while the data, being conormal with respect to the light cone Σ_2, help us to extend the general solution of Protter problem across the light cone to a solution of the wave equation. Using the microlocal technique it is possible to build such kind of solution $\widetilde{\mathbf{u}}$ in the frame of distribution theory. In this case the properties of the new solution $\widetilde{\mathbf{u}}$ inside of the cone Σ_2 looks almost the same: they are smooth inside Σ_2, but with some singularity on Σ_2, the order of which might be calculated. This is a work in progress.

4.2. **The case of exponential growth solution at the vertex O.** Actually, in this case the solution in the Section 4 above is found as a superposition of polynomially growing generalized solutions. The latter are smooth up to the light cone, while the data being conormal with respect to the light cone Σ_2, could be extended across the light cone to a solution of the wave equation. The order of conormality increases as more terms are added to the right side, that violate the orthogonality conditions. Eventually, it yields the strong exponential singularity. Actually, this exponential singularity is only on some directions on the light cone Σ_2 (which is obvious on the visualization), but to build such extension across the light cone to a solution of the wave equation will be not possible in the frame of the distribution theory in the common case.

The obvious question is: **How to proceed in such cases?**

5. Nonlocal statement of the Problem

Now, we will introduce a new **nonlocal Problem B**, following our joint with David Edmunds (Sussex University, UK) paper [9]. Let also mention here that before our join result for the Tricomi equation, the author had published a series of papers (see [39], [42]), with this approach for the case of wave equation and for the Tricomi equation.

A key point in the present work is that we make progress by relating the Gellerstedt equation to a first-order system – but to a particular system derived artificially and equipped with boundary conditions that make it well-posed, and inequivalent to the original problem.

Let $K : \mathbb{R} \to \mathbb{R}$ be of class C^1 and such that $tK(t) > 0$ for $t \neq 0$ and $K'(t) > 0$, $t \in \mathbb{R}$. An obvious example is K(t)=t, which choise leading directly to the Tricomi equation.

Denote by $G = G_+ \cup S_0 \cup G_-$ a bounded simply connected region, where

$$G_+ = \left\{ (x_1, x_2, t) \in \mathbb{R}^3 : 0 < t < d,\ \int_0^t \sqrt{K(\tau)} d\tau < \rho < 1 - \int_0^t \sqrt{K(\tau)} d\tau \right\},$$

with $2\int_0^d \sqrt{K(\tau)} d\tau = 1$; S_0 is the disc $S_0 = \{x : t = 0,\ 0 \le \rho \le 1\}$ and

$$G_- = \{x \in \mathbb{R}^3 : 0 < \rho < 1,\quad g(\rho) < t < 0\}.$$

We assume that the rotating surface $\Gamma = \{\ 0 \le \rho \le 1, t = g(\rho)\ \}$ is C^2, lies in the half space $\{t < 0\}$ and meets the plane $\{t = 0\}$ at the curve $\{\rho = 1,\ t = 0\}$, i.e. $g \in C^2[0,1]$, $g(\rho) < 0$, $0 \le \rho < 1$, $g(1) = 0$, $g'(0) = g''(0) = 0$.

Let consider in G the so called Gellerstedt equation

$$Lu := K(t)\left(u_{x_1x_1} + u_{x_2x_2}\right) - u_{tt} \equiv K(t)\left\{\rho^{-1}\left(\rho u_\rho\right)_\rho + \rho^{-2}u_{\varphi\varphi}\right\} - u_{tt} = f(x,t), \tag{5.1}$$

where f is a given function. Note that the surfaces

$$S_1 = \left\{0 \le t \le d, \rho = 1 - \int_0^t \sqrt{K(\tau)} d\tau\right\}, S_2 = \left\{0 \le t \le d, \rho = \int_0^t \sqrt{K(\tau)} d\tau\right\}$$

are characteristics of (5.1). The equation (5.1) is of changing type in G: it is elliptic in G_-, hyperbolic in G_+ and parabolic on S_0.

Problem P. Find a solution of (5.1) in G, which satisfies the condition

$$u = 0 \quad \text{on} \quad S_1 \cup \Gamma. \tag{5.2}$$

Problem P*. Find a solution of (5.1) in G, with

$$u = 0 \quad \text{on} \quad S_2 \cup \Gamma. \tag{5.3}$$

There are two well known models for changing type equations: the Tricomi equation (with $K(t) \equiv t$), and Lavrent'ev - Bitsadze equation ($K(t) \equiv \text{sgn}t$). Bitsadze and Didenko investigated for both the axially symmetric cases of Problem P. Didenko proved the existence of solutions, but in a function space with a degenerating weight function on $\{t = 0\}$. Some uniqueness results for quasi - regular solutions for both cases were obtained by Aziz, Schneider [3] and Karatoprakliev [21] . The Protter problems have also been investigated by Salzman, Schneider, Papadakis, Aldashev, Sorokina, Popivanov, Schneider and others (see [1], [2], [31], [44], [45]). Sorokina [45] got some interesting results, which we will discuss in Section 5. Let fix that for now there are no general existence results for the Problem P in G.

Note. What is the situation for Protter Problem P now:

(**i**) The Problem P has at most one quasi-regular solution (see [3])

(**ii**) We do not know any kind of general solvability results for some wide class of right - hand side functions f.

(**iii**) We do not know whether or not there exists any nontrivial solution to the adjoint Problem P^* (like infinite number of solutions for the Problem $P1^*$).

(**iv**) We have a feeling that the Problem P is also strongly overdetermined. That is the reason to look for another Problem, which can give us more information about the Problems P and P*.

(**v**) Instead of considering the ill-posed Problems $P1$ and $P1^*$ in G_+, Edmunds, Popivanov [9] studied for the Gellersteadt equation an appropriate new problem in G_+. This nonlocal regularization of the Problem $P1$ connects points from G_+ with points on the characteristic cone S_2. The new problem is such that its solutions are free of singularities on S_2 and in some sense it is a regularizer of the ill-posed Problem $P1$. In our case of changing type equation we look for some new equation, instead of (5.1).

Key: We actually do not have any information about well-posedness or ill–posedness of the Protter Problem P. There are many different ways to proceed in such situation (let say for "regularization of such improperly posed problems") see site [9]. Now, we suggest some approach, formulating a new BVP, named B (with a nonlocal operator) for which we show that it is well-posed. Finally, the solution of the original Protter Problem P will coincide with the solution of the well posed Problem B for a wide class of right-hand side functions $f(x,t)$. The results for this nonlocal Problem B will give some more information even for the adjoint Problem P*.

Now, having this situation in mind (with regard to the work on ill-posed problems), with the same reason as in [9] we investigate some non-local problems in the domain G. To explain them, let α be a small positive parameter. Given a point $(t_0,\ \rho_0,\varphi_0) \in G_+$ we consider the curve $t = C_0\rho^{-\alpha} - \alpha(1+\alpha)^{-1}\rho, \varphi = \varphi_0$ through this point, i.e. with $C_0 = t_0\rho_0^{\alpha} + \alpha(1+\alpha)^{-1}\rho_0^{1+\alpha}$ (look to the Fig. 5 below). Now let $(p_\alpha(t_0,\rho_0), q_\alpha(t_0,\rho_0))$ be the point of intersection of two curves

$$\rho = \int_0^t \sqrt{K(\tau)}d\tau, \quad t = C_0\rho^{-\alpha} - \alpha(1+\alpha)^{-1}\rho.$$

For enough small, but fixed, positive parameter α this construction connect one arbitrary fixed point $(t_0,\ \rho_0,\varphi_0) \in G_+$ with a unique determined point on the cone S_2.

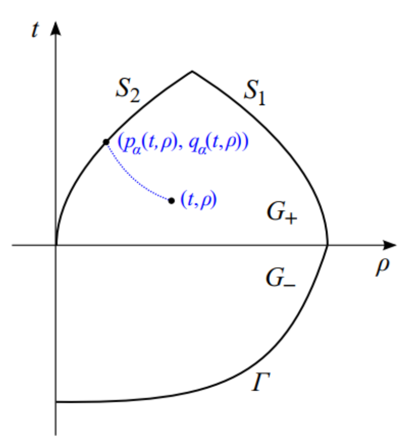

FIGURE 5. Nonlocal equation

Let note that in the case of Tricomi equation (i.e. $K(t) = t$) is enough $\alpha \leq 1/3$. In the elliptic part G_- the condition on Γ is:

E1. There exists a positive number α_1, for which

$$\rho g'(\rho) + \alpha_1(\rho + |g(\rho)|) > 0, \tag{5.4}$$

$$\alpha_1(1 + |K(d_1)|) < 2, \text{ where } d_1 := \min\{g(\rho) : 0 \leq \rho \leq 1\}. \tag{5.5}$$

Instead of studying Problem P we now consider

Problem B. Find a solution $\omega(t, \rho, \varphi)$ of the equation

$$\begin{cases} (L\omega)(t,\rho,\varphi) - \rho^{-2}K(t)\omega_{\varphi\varphi}\left(p_\alpha\left(t,\rho\right), q_\alpha\left(t,\rho\right), \varphi\right) = f(t,\rho,\varphi) & \text{in } G_+, \\ (L\omega)(t,\rho,\varphi) = f(t,\rho,\varphi) \quad \text{in } G_-, \end{cases} \tag{5.6}$$

which satisfies the boundary conditions (5.2), i.e.

$$\omega = 0 \quad \text{on} \quad S_1 \cup \Gamma.$$

Remark 5.1. Since this point we will working with Tricomy equation, i.e. with $K(t) = t$. Note that in this case it is enough $0 < \alpha < 1/3$. Obviously, the condition (5.4) is satisfied if $g'(\rho) \geq 0$.

Remark 5.2. Note, that the equation (5.6) is not a standard one. Actually, we save the equation (5.1) in its elliptic part G_- of the domain G, but modify it in the hyperbolic part G_+ to the non-local equation (5.6), not of ΨDO type, for which the classification does not apply. Finally, the equation (5.6) has continuous coefficients in G. More precisely, the equation (5.6) is non-local in the hyperbolic part G_+ of the domain G, because it involves points with coordinates (t, ρ, φ) and $(p_\alpha(t,\rho), q_\alpha(t,\rho), \varphi)$. We remark here merely that in our non-local Problem B, in the additional term

$$\rho^{-2}K(t)\omega_{\varphi\varphi}\left(p_\alpha\left(t,\rho\right), q_\alpha\left(t,\rho\right), \varphi\right)$$

of (5.6), the point $(p_\alpha, q_\alpha, \varphi)$ lies just on the characteristic cone S_2, where the singularity at the point $(0, 0, 0)$ appears in the '*generalized solution*' of the original Problem $P1$. The derivative $\omega_{\varphi\varphi}$ is tangential to S_2 at that point. Also, the equation (5.6) is local and elliptic in G_-, local and parabolic on $\{t = 0\}$ and nonlocal in $\bar{G}_+ \backslash S_2$. Finally, on the characteristic cone S_2 it is

$$L_1 u\Big|_{S_2} \equiv K(t)\rho^{-1}\left(\rho u_\rho\right)_\rho - u_{tt} = f\ ,$$

i.e. obviously it is local there, but not hyperbolic. Let mention also here that such kind equations in Russian literature are known as "loaded equations" (see for example [28], [26], [27]).

6. Results on the nonlocal problem

Following the work of Morawetz [24] in the two-dimensional case and Sorokina [45] in the multi-dimensional case, we introduce the weighted Sobolev space

$$\widetilde{W}_2^1(G) := \left\{\omega : \|\omega\|_{\widetilde{W}_2^1(G)} = \left(\int_G \left(\omega^2 + \omega_t^2 + r\left(\omega_{x_1}^2 + \omega_{x_2}^2\right)\right) dx\right)^{1/2} < \infty\right\}, \tag{6.1}$$

where $r = \sqrt{x_1^2 + x_2^2 + t^2}$. (The weight in Sorokina is different.)

Definition 6.1. A function $\omega(t,\rho,\varphi)$ is called a generalized solution of Problem B if: (a) ω, $\dfrac{\partial\omega}{\partial t}\in L_2(G)$, (b) $\omega\in\widetilde{W}_2^1(G_-)$; $\sqrt{r}\dfrac{\partial\omega}{\partial\rho}$, $\sqrt{r}\rho^{-1}\dfrac{\partial\omega^1}{\partial\varphi}\in L_2(G_+)$, where

$$\omega^1(t,\rho,\varphi):=\omega(t,\rho,\varphi)-\omega\left(p_\alpha\left(t,\rho\right),\ q_\alpha\left(t,\rho\right),\varphi\right); \tag{6.2}$$

(c) $\omega=0$ *on* $S_1\cup\Gamma$; (d) for any function $v\in C^*:=\{v\in C^1(\overline{G}): v=0$ on $S_2\cup\Gamma$ and in some neighborhood of $(0,0,0)\}$, the equality

$$\begin{aligned}&\int_G\left(\frac{\partial\omega}{\partial t}\frac{\partial v}{\partial t}-K(t)\frac{\partial\omega}{\partial\rho}\frac{\partial v}{\partial\rho}-fv\right)\rho d\rho d\varphi dt\\ -&\int_{G_-}\frac{K\left(t\right)}{\rho^2}\frac{\partial\omega}{\partial\varphi}\frac{\partial v}{\partial\varphi}\rho d\rho d\varphi dt-\int_{G+}\frac{K\left(t\right)}{\rho^2}\frac{\partial\omega^1}{\partial\varphi}\frac{\partial v}{\partial\varphi}\rho d\rho d\varphi dt=0\end{aligned} \tag{6.3}$$

holds.

Theorem 6.2. *Suppose that the positive parameter α is enough small and the surface Γ is such that the conditions (5.4) and (5.5) are satisfied. Then for any $f\in L_2\left(G\right)$, there exists a generalized solution of problem B. This generalized solution is at most one and satisfies the a priori estimate*

$$\|\omega\|_{\widetilde{W}_2^1(G_-)}+\|\omega\|_{L_2(G)}+\|\omega_t\|_{L_2(G)}+\left\|\sqrt{r}\omega_\rho\right\|_{L_2(G_+)}+\left\|\frac{\sqrt{r}}{\rho}\frac{\partial\omega^1}{\partial\varphi}\right\|_{L_2(G_+)}\le C_\alpha\left\|f\right\|_{L_2(G)}, \tag{6.4}$$

where the constant C_α depends on α, but not on f and ω.

Let formulate a result for infinite smoothness of the generalized solution of Problem B with respect to φ.

Theorem 6.3. *Suppose that the conditions from Theorem 6.2 are satisfied. Let $f\in L_2(G)$ and $\ell\in\mathbb{N}$ be such that $\dfrac{\partial^k f}{\partial\varphi^k}\in L_2(G)$ for $k=1,\dots,\ell$. Then there exists a unique generalized solution ω of Problem B with*

$$\frac{\partial^k\omega}{\partial\varphi^k}\in\widetilde{W}_2^1(G)\ (k=0,\dots\ell-1),\ \frac{\partial^{\ell+1}\omega}{\partial t\partial\varphi^\ell}\in L_2(G),\ \sqrt{r}\frac{\partial^{\ell+1}\omega}{\partial\rho\partial\varphi^\ell}\in L_2(G_+) \tag{6.5}$$

and

$$\sum_{k=0}^{\ell-1}\left\|\frac{\partial^k\omega}{\partial\varphi^k}\right\|_{\widetilde{W}_2^1(G)}+\left\|\frac{\partial^{\ell+1}\omega}{\partial t\partial\varphi^\ell}\right\|_{L_2(G)}+\left\|\sqrt{r}\frac{\partial^{\ell+1}\omega}{\partial\rho\partial\varphi^\ell}\right\|_{L_2(G_+)}\le C_\alpha\sum_{k=0}^{\ell}\left\|\frac{\partial^k f}{\partial\varphi^k}\right\|_{L_2(G)}. \tag{6.6}$$

Corollary 6.4. *Suppose that the conditions from Theorem 6.2 are satisfied. Let $f\in L_2(G)$ and $\dfrac{\partial f}{\partial\varphi}\in L_2(G)$. Then there exists a unique generalized solution $\omega\in\widetilde{W}_2^1(G)$ of Problem B and the a priori estimate*

$$\|\omega\|_{\widetilde{W}_2^1(G)}\le C\left(\|f\|_{L_2(G)}+\|f_\varphi\|_{L_2(G)}\right)$$

holds.

The natural question is: what are the connections between original Protter Problem P and the new nonlocal Problem B? Assume that a solution u_f of the Problem P is such, that it is a solution of the special Problem P (introduced by Didenko), now called

Problem $\mathbf{P}_\varphi$. Find a solution u of Problem P, which satisfy the extra condition $\partial u/\partial\varphi = 0$ on S_2.

We prove that in this case the generalized solution u_f (if it exists for this function f) coincides with the solution ω_f of the Problem B (which always exists if $f \in L_2$), i.e. $u_f \equiv \omega_f$. Accordingly we can say that Problem B is a "non-local regularizer" of the Problems P_φ and P. Using the results about Problem B we prove that the solution u_f of Problem P_φ depends continuously on f. Thus the non-local Problem B (for which Theorem 6.3 holds, ensuring uniqueness, existence and differentiability with respect to φ) is, so to speak, a 'regular continuation' of the strongly over-determined Problem P_φ. In this sense, we may regard Problem B to be a 'non-local regularization' of Problems P and P_φ. All this suggests the following procedure for tackling the probably ill-posed Problem P. For a given function $f \in L_2(G)$ we first try to solve the nonlocal Problem B. To do that, it is possible first to find the solution $\hat{u} = (u_1, u_2, u_3)$ (which always exists) of the local problem for the corresponding system of PDEs (connected to the Problem B) and then to determine a solution ω_f of Problem B by integration of $u_3(t, \rho, \varphi)$. Then we check the value of $(\omega_f)_\varphi$ on the characteristic cone S_2 and if that value is very close to zero, we might conclude that the solution u_f of the Problem P exists and is very close to the function ω_f already found.

Remark 6.5. For a treatment of the famous Hadamard's example about the strongly overdetermined Cauchy problem, Eskin and Vishik changed (see Eskin [10]) the Poisson equation $\Delta_n u = f$ by $\widehat{\Delta}_n u + G(v) = f$, where $G(v)$ is a potential with unknown density v and $\widehat{\Delta}_n$ is a slightly modified Laplas operator Δ_n. In an appropriate Sobolev space they established existence and uniqueness results about the pair of functions (u, v); the addition of the potential $G(v)$ removed the overdeterminacy. In our approach we modify the equation (5.1) to (5.6), but our additional term depends only on the function ω, not on some new function.

Acknowledgments. The part of this paper was prepared when N. Popivanov has visited the Fields Institute, Toronto, Canada. N. Popivanov would like to thank the Fields Institute for the hospitality. The author has been partially supported by the Bulgarian National Science Fund under Grant KP-06-N52/4, 2021.

References

[1] Aldashev S.A.; Multidimensional Darboux problems for the degenerate hyperbolic equation. Differential Equations, –1993. –29. –P. 1829–1834.

[2] Aldashev S.A.; On the well-posedness of multidimensional Darboux problems for the wave equation. Ukrainian Math. J., –1993. –45. –P. 1456–1464.

[3] Aziz A.K., Schneider M.; Frankl-Morawetz problems in R^3. SIAM J. Math. Anal., –1979. –10. –P. 913–921.

[4] Bitsadze A.V.; On some problems about mixed type equation in multidimensional regions. Doklady ANSSR, –1956. –110. –P. 1021–1024.

[5] Bers L.; Mathematical Aspects of Subsonic and Transonic Gas Dynamics. John Wiley & Sons, New York, –1958. (Republished: Dover Publications, New York, –2016.)

[6] Dechevsky L., Popivanov N., Popov T.; Exact asymptotic expansion of singular solutions for (2+1)-D Protter problem. Abstract and Applied Analysis, –2012. –Article ID 278542.

[7] Didenko V.P.; On boundary-value problems for multidimensional hyperbolic equation with degeneration. Soviet Math. Dokl., –1972. –13. –P. 998–1002.

[8] Didenko V.P.; Boundary-value problems for three-dimensional hyperbolic equation of mixed type. Differential Equations, –1975. –11. –P. 25–28.

[9] Edmunds D., Popivanov N.; A nonlocal regularization of some over-determined boundary-value problems I. SIAM J. Math. Anal., –1998. –29, No.1. –P. 85–105.

[10] Eskin G.I.; Boundary-value problems for elliptic pseudodifferential equations. Transl. Math. Monographs, 52, Amer. Math. Soc., Providence, Rhode Island, –1981.

[11] Friedrichs K.O.; Symmetric positive linear differential equations. Comm. Pure Appl. Math., –1958. –11. –P. 333–418.

[12] Garabedian P.R.; Partial differential equations with more than two variables in the complex domain. J. Math. Mech., –1960. –9. –P. 241–271.

[13] Hristov T.; Singular solutions to Protter problem for Keldysh type equations. AIP Conf. Proc., –2014. –1631(1). –P. 255–262. https://doi.org/10.1063/1.4902484

[14] Hristov T.; On the Generalized Solvability of 3-D Protter-Morawetz Problem for Equations of Keldysh Type. AIP Conf. Proc., Proceedings of the 50th International Conference on Application of Mathematics in Engineering and Economics, Sozopol, Bulgaria, –2024. (In print)

[15] Hristov T., Nikolov A.; Behaviour of Singular solutions to Protter problem for (3+1)-D Keldysh type equations. Compt. Rend. Acad. Bulg. Sci., –2017. –70(2). –P. 167–174.

[16] Jeon J.D., Cher K.K., Park J.H., Jeon Y.H., Choi J.B.; Protter's conjugate boundary value problems for the two dimensional wave equation. J. Korean. Math. Soc., –1996. –33. –P. 857–863.

[17] Choi J.B., Park J.Y.; On the conjugate Darboux-Protter problems for the two dimensional wave equations in the special case. J. Korean Math. Soc., –2002. –39(5). –P. 681–692.

[18] Cher K.K.; Nonuniqueness of solutions of the Darboux problem. Siberian Math. J., –1985. –26. –P. 286–288.

[19] Cher K.K.; An estimate of the solution of Darboux-Protter problems for the two-dimensional wave equation. Soviet Math. Dokl., –1991. –43. –P. 887–891.

[20] Cher K.K.; Nontrivial solutions of some homogeneous boundary value problems for a many-dimensional hyperbolic Euler-Poisson-Darboux equation in an unbounded domain. Differential Equations, –1998. –34(1). –P. 139–142.

[21] Karatoprakliev G.D.; Uniqueness of solutions of certain boundary-value problems for equations of mixed type and hyperbolic equations in space. Differential Equations, –1982. –18. –P. 49–53.

[22] Lax P.D., Phillips R.S.; Local boundary conditions for dissipative symmetric linear differential operators. Comm. Pure Appl. Math., –1960. –13. –P. 427–455.

[23] Lupo D., Payne K., Popivanov N.; Nonexistence of nontrivial solutions for supercritical equations of mixed elliptic-hyperbolic type. Progress in Non-Linear Differential Equations and Their Applications, Birkhuser, –2005. –66. –P. 371–390.

[24] Morawetz C.S.; A weak solution for a system of equations of elliptic-hyperbolic type. Comm. Pure Appl. Math., –1958. –11. –P. 315–331.

[25] Morawetz C.S.; Mixed equations and transonic flow. J. Hyperbolic Differ. Equ., –2004. –1(1). –P. 1–26.

[26] Nakhushev A.M.; On the theory of linear boundary value problem for hyperbolic and mixed type equations. D.Sc. Thesis, Novosibirsk, –1971.

[27] Nakhushev A.M.; Nonlocal boundary problems with displacement and their relation to loaded equations. Differential Equations, –1985. –21. –P. 74–81.

[28] Nakhushev A.M.; Criteria for continuity of the gradient of the solution to the Darboux problem for the Gellerstedt equation. Differential Equations, –1992. –28. –P. 1445–1457.

[29] Nikolov A.; Improved Asymptotic Representation of the Singular Solutions of a 4-D Problem for Keldysh-Type Equations. AIP Conf. Proc., –2018. –2048. –P. 040019-1–040019-6.

[30] Nikolov A.; On the Generalized Solutions of a Boundary Value Problem for Multidimensional Hyperbolic and Weakly Hyperbolic Equations. Research monograph, Publishing house of Technical University of Sofia, –2018. ISBN: 978-619-167-349-0.

[31] Papadakis J.S.; A boundary-value problem for equations of mixed type. Bull. Soc. Math. Grce, –1973. –14. –P. 157–171.

[32] Popivanov N., Popov T.; Exact Behavior of Singularities of Protters Problem for the 3-D Wave Equation. In: Herzberger J. (ed.) Inclusion Methods for Nonlinear Problems. Computing Supplementa, –2002. –16. –P. 213–236.

[33] Popivanov N., Popov T., Witt I.; Solutions with exponential singularity for (3+1)-D Protter problems. J. Hyperbolic Differential Equations, –2023. –20(2). –P. 475–498.

[34] Popivanov N., Popov T., Scherer R.; Asymptotic expansion of singular solutions for (3+1)-D Protter problems. J. Math. Anal. Appl., –2007. –331. –P. 1093–1112.

[35] Protter M.H.; New boundary value problems for the wave equation and equations of mixed type. J. Rat. Mech. Anal., –1954. –3. –P. 435–446.

[36] Popivanov N.I., Schneider M.; The Darboux problems in $\mathbb{R}^3$ for a class of degenerated hyperbolic equations. C. R. Acad. Bulgare Sci., –1988. –41(11). –P. 7–9.

[37] Popivanov N.I., Schneider M.; The Darboux problem in $\mathbb{R}^3$ for a class of degenerating hyperbolic equations. J. Math. Anal. Appl., –1993. –175. –P. 537–579.

[38] Popivanov N.I., Schneider M.; On M.H. Protter problems for the wave equation in $\mathbb{R}^3$. J. Math. Anal. Appl., –1995. –194. –P. 50–77.

[39] Popivanov N.; Well-posedness of some nonlocal boundary value problems in three-dimensional domains. Questions of well-posedness and methods for the investigation of inverse problems, Siberian Branch of the Academy of Sciences of USSR, Novosibirsk, –1986. –P. 20–33.

[40] Popivanov N.; On nonlocal regularization of boundary value problems for the wave equation. Dokl. Akad. Nauk SSSR, –1986. –289(5). –P. 1060–1063.

[41] Popivanov N.; A nonlocal analogue of a three-dimensional FranklMorawetz problem. Dokl. Math., –1988. –36(2). –P. 268–272.

[42] Popivanov N.; Overdetermined problems for the wave equation and their nonlocal regularization. Differential Equations, –1988. –24. –P. 1301–1312.

[43] Popivanov N.; Nonlocal regularization of some overdetermined boundary value problems. In: Proc. 1987 EQUADIFF Conference, Lecture Notes in Pure and Appl. Math., –1989. –118. –P. 581–586.

[44] Salzman H., Schneider M.; Schwache Lsungen des Frankl-Morawetz Problems in $\mathbb{R}^3$. Mh. Math., –1977. –84. –P. 237–246.

[45] Sorokina N.G.; Multidimensional generation of a theorem of C. Morawetz for weak solvability of differential equations of mixed type. Differential Equations, –1977. –13. –P. 28–134.

[46] Sorokina N.G.; Multidimensional analogues of the Tricomi problem. In: Proc. Conference on PDE, Moscow State University, –1978. –P. 452–454. (In Russian)

[47] Tong K.C.; On a boundary-value problem for the wave equation. Science Record, New Series, –1957. –1(1). –P. 1–3.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.222-231
DOI: 10.29229/uzmj.2025-2-21

On overdetermination conditions in order determination inverse problems for fractional diffusion equations

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. The fractional diffusion equation in an infinite layer is considered, and an integral identity is proved that relates the solution of the Cauchy problem for the equation under consideration with entire harmonic functions. Based on this identity, overdetermination conditions are proposed in inverse problems of determining the order for the fractional diffusion equation. In these inverse problems, in addition to the desired solution, the order of fractional differentiation is unknown and must be determined. The proposed overdetermination conditions are spatially oriented: these conditions use information about the desired solution at fixed points in time. Theorems on the uniqueness and solvability of inverse problems with the proposed overdetermination conditions are proved.

Keywords: fractional diffusion equation, Cauchy problem, inverse problem, order determination problem, overdetermination condition.

MSC (2020): 35R11, 35K57, 35R30

1. Introduction

Consider the equation

$$\left(\frac{\partial^\alpha}{\partial y^\alpha} - \Delta_x\right) u(x,y) = f(x,y) \tag{1.1}$$

where $\frac{\partial^\alpha}{\partial y^\alpha} = D_{0y}^\alpha$ is the Riemann–Liouville fractional derivative of order α with the origin at the point $y = 0$ (see the definition below); $\alpha \in (0,1)$; and Δ_x is the Laplace operator with respect to $x = (x_1, ..., x_n) \in \mathbb{R}^n$, i.e.

$$\Delta_x = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

The equation (1.1), which coincides with the diffusion equation for $\alpha = 1$ and is called the fractional diffusion equation in the case $0 < \alpha < 1$, has been actively studied in recent decades. These studies began in [23, 22, 15, 11]. Since then, a large number of works have appeared devoted to the study of various problems for fractional diffusion equations, and various approaches to their solution have been developed. Brief overviews of works on this topic are given in [18, 19]. More detailed and complete surveys can be found in the article [13] and monographs [20, 14, 16].

In the last few years, interest in the study of inverse problems for fractional-order equations in which the order of fractional differentiation is unknown has increased greatly. The study of such problems for the fractional diffusion equation was apparently begun in the works of [10, 12]. Today this scientific direction has an extensive bibliography. Detailed surveys of the results obtained in this direction can be found in [17, 3, 2].

For such problems, overdetermination conditions for finding the unknown order α play an important role. In the first works on inverse problems with unknown order, the overdetermination conditions were time-oriented, i.e. conditions that use use behavior of the sought solution along the time variable at a fixed point in space.

In the works [4, 1], spatially oriented overdetermination conditions were first proposed, i.e. conditions information about the sought solution at a fixed point in time. This approach is currently being actively developed [5, 6, 7, 8, 9, 3].

The aim of this paper is to discuss new spatially oriented overdetermination conditions for the equation (1.1). First, we prove an integral identity relating solutions of the Cauchy problem for the equation (1.1) to entire harmonic functions. As a consequence of this identity, we obtain relations between the sought solution and the unknown order of the equation, from which the latter can be uniquely determined. Based on these relations, we present three overdetermination conditions generated by the obtained integral identity.

The article is structured as follows. In Section 2 we give the definitions of the fractional integral and derivative, recall some facts concerning the solvability of the Cauchy problem for the equation (1.1), and prove some auxiliary statements about the properties of its solutions. In Section 3, an integral identity is proved that connects solutions of the Cauchy problem and entire harmonic functions, and its corollaries are given. The formulation and proof of the main results of the article are given in Section 4. Here we formulate theorems on the solvability of inverse problems with overdetermination conditions generated by the proven integral identity. Section 5 contains conclusions and comments on the results obtained.

2. Preliminaries

2.1. **Integrals and derivatives of fractional order.** The Riemann–Liouville fractional integral of order β $(\beta < 0)$ with respect to y, and with origin at $y = 0$ is defined as follows

$$D_{0y}^{\beta} g(y) = \frac{1}{\Gamma(-\beta)} \int_0^y g(t)(y-t)^{-\beta-1} dti. \tag{2.1}$$

The Riemann–Liouville fractional derivative has the form

$$D_{0y}^{\beta} g(y) = \frac{d^m}{dy^m} D_{0y}^{\beta-m} g(y).$$

Here $\beta > 0$, $m \in \mathbb{N}$ and is chosen to satisfy the inclusion $\beta \in (m-1, m]$. It is also assumed that

$$D_{0y}^{0} g(y) = g(y).$$

2.2. **Cauchy problem and fundamental solution.** We consider the equation (1.1) in an infinite layer and use the following notation:

$$\Omega = \mathbb{R}^n \times (0, T), \qquad \Omega_0 = \mathbb{R}^n \times [0, T).$$

Definition 2.1. A function $u(x,y)$ is called a regular solution of the equation (1.1) in the domain Ω if $y^{1-\alpha}u(x,y) \in C(\Omega_0)$, $D_{0y}^{\alpha-1}u(x,y) \in C_y^1(\Omega)$, $u(x,y) \in C_x^2(\Omega)$, and $u(x,y)$ satisfies the equation (1.1) in Ω.

The Cauchy problem for the equation (1.1) is formulated as follows: *find a regular solution of* (1.1) *in* Ω *that satisfies the initial condition*

$$\lim_{y \to 0} D_{0y}^{\alpha-1} u(x,y) = \tau(x) \qquad (x \in \mathbb{R}^n). \tag{2.2}$$

The fundamental solution of (1.1) can be written as [21]

$$\Gamma_{\alpha,n}(x,y) = 2^{-n} \pi^{\frac{1-n}{2}} y^{\frac{\alpha}{2}(2-n)-1} f_{\frac{\alpha}{2}} \left(|x| y^{-\frac{\alpha}{2}}; n-1, \frac{\alpha}{2}(2-n) \right)$$

where

$$f_{\beta}(z; \varepsilon, \delta) = \begin{cases} \dfrac{2}{\Gamma\left(\frac{\varepsilon}{2}\right)} \displaystyle\int_1^{\infty} \phi(-\beta, \delta; -zt) \left(t^2-1\right)^{\frac{\varepsilon}{2}-1} dt, & \text{if} \quad \varepsilon > 0, \\ \phi(-\beta, \delta; -z), & \text{if} \quad \varepsilon = 0, \end{cases} \qquad \beta \in (0,1),$$

and

$$\phi(\xi, \eta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\xi k + \eta)}$$

is the Wright function [24].

The two lemmas below reveal the question of solvability and uniqueness of problems (1.1) and (2.2) (proofs can be found in [21]).

In what follows, the set of functions growing (decreasing) no faster than $\exp(p|x|^q)$ as $|x| \to \infty$ is denoted by $\mathcal{E}(p,q)$ ($p \in \mathbb{R}$, $q \geq 0$), or more precisely

$$\mathcal{E}(p,q) = \left\{ g(x) : \lim_{r\to\infty} \exp\left(-p\, r^q\right) \sup_{|x|=r} |g(x)| = 0 \right\}. \tag{2.3}$$

Lemma 2.2. *Let $\tau(x) \in C(\mathbb{R}^n)$, $y^{1-\mu} f(x,y) \in C(\Omega_0)$, and*

$$\tau(x) \in \mathcal{E}\left(\rho, \frac{2}{2-\alpha}\right), \qquad \sup_{y\in[0,T]} y^{1-\mu} f(x,y) \in \mathcal{E}\left(\rho, \frac{2}{2-\alpha}\right)$$

for some $\mu > 0$ and $\rho < \frac{2-\alpha}{2}\left(\frac{\alpha}{2T}\right)^{\frac{\alpha}{2-\alpha}}$; and let $y^{1-\mu} f(x,y)$ be locally Hölder continuous with respect to x in Ω_0. Then the function

$$u(x,y) = \int_{\mathbb{R}^n} \tau(s)\, \Gamma_{\alpha,n}(x-s,y)\; ds + \int_0^y \int_{\mathbb{R}^n} f(s,t)\, \Gamma_{\alpha,n}(x-s,y-t)\; ds\, dt,$$

is a regular solution of the problem (1.1) *and* (2.2).

Lemma 2.3. *The problem* (1.1) *and* (2.2) *has at most one solution in the class of functions satisfying*

$$\sup_{y\in[0,T]} y^{1-\alpha} u(x,y) \in \mathcal{E}\left(\sigma, \frac{2}{2-\alpha}\right) \qquad \text{for some } \sigma > 0. \tag{2.4}$$

Now let us recall the estimates for the function $\Gamma_{\alpha,n}(x,y)$ and its derivatives, which we will use later. The proof of these estimates can be found in [21].

In what follows, the letter C denotes positive constants that are different in different cases, and if necessary, the parameters on which they may depend are indicated in parentheses; e.g. $C = C(\alpha, \beta, ...)$.

Lemma 2.4. *Let $\zeta \in \mathbb{R}$ and $\beta = \frac{\alpha}{2}$. Then*

$$\left| D^{\zeta}_{0y} \Gamma_{\alpha,n}(x,y) \right| \leq C y^{\beta(2-n)-\zeta-1} \gamma_p\left(|x|y^{-\beta}\right) E(|x|y^{-\beta}, \sigma), \tag{2.5}$$

$$\left| \frac{\partial}{\partial x_k} D^{\zeta}_{0y} \Gamma_{\alpha,n}(x,y) \right| \leq C\, |x_k|\, y^{-\beta n-\zeta-1} \gamma_{p+2}\left(|x|y^{-\beta}\right) E(|x|y^{-\beta}, \sigma),$$

$$\left| \frac{\partial^2}{\partial x_k^2} D^{\zeta}_{0y} \Gamma_{\alpha,n}(x,y) \right| \leq C\, y^{-\beta n-\zeta-1} \gamma_q\left(|x|y^{-\beta}\right) E(|x|y^{-\beta}, \sigma), \quad (k = 1, 2, ..., n),$$

for any $\sigma \in (0, \sigma_0)$, where $\sigma_0 = (1-\beta)\beta^{\beta/(1-\beta)}$, and

$$E(z,\sigma) = \exp\left(-\sigma z^{1/(1-\beta)}\right), \quad \gamma_n(z) = \begin{cases} 1, & n \leq 3, \\ |\ln z| + 1, & n = 4, \\ z^{4-n}, & n \geq 5, \end{cases}$$

$$p = \begin{cases} n, & \text{if} \quad \zeta \in \mathbb{N} \cup \{0\}, \\ n+2, & \text{if} \quad \zeta \notin \mathbb{N} \cup \{0\}, \end{cases} \qquad q = \begin{cases} n+2, & \text{for} \quad \zeta \in \mathbb{N} \cup \{0\} \quad \text{or} \quad n = 1, \\ n+4, & \text{for} \quad \zeta \notin \mathbb{N} \cup \{0\} \quad \text{and} \quad n \geq 2. \end{cases}$$

Here $C = C(n, \alpha, \sigma)$, and σ can be chosen (by choosing C) as close as desired to σ_0.

2.3. **Auxiliary assertions.** Here we prove auxiliary statements necessary for the further exposition.
In what follows, by $\mathcal{H}$ we denote the set of entire harmonic functions, i.e.

$$\mathcal{H} := \left\{ v(x) \in C^2\left(\mathbb{R}^n\right) : \, \Delta_x v(x) = 0, \quad x \in \mathbb{R}^n \right\}.$$

Lemma 2.5. *Let $v(x) \in \mathcal{H} \cap \mathcal{E}(\sigma, p)$ for some $\sigma > 0$ and $p \le \frac{2}{2-\alpha}$. Then*

$$\int_{\mathbb{R}^n} v(s)\, \Gamma_{\alpha,n}\left(x - s, y\right)\, ds = \frac{y^{\alpha-1}}{\Gamma(\alpha)} v(x), \tag{2.6}$$

for every $y \in (0, y_0)$, where $y_0 = \frac{\alpha}{2}\left(\frac{2\sigma}{2-\alpha}\right)^{\frac{\alpha}{2-\alpha}}$ if $p = \frac{2}{2-\alpha}$, and $y_0 = \infty$ else.

Proof. First, we note that the conditions imposed on $v(x)$ and the estimate (2.5) guarantee the convergence of the integral in (2.6).

Let $h(x, y)$ is a solution of the problem

$$\left(D_{0y}^{\alpha} - \Delta_x\right) h(x, y) = 0, \qquad \lim_{y\to 0} D_{0y}^{\alpha-1} h(x, y) = v(x) \qquad \left(x \in \mathbb{R}^n\right). \tag{2.7}$$

Lemma 2.2 gives that

$$h(x, y) = \int_{\mathbb{R}^n} v(s)\, \Gamma_{\alpha,n}\left(x - s, y\right)\, ds \tag{2.8}$$

is a solution of (2.7).

At the same time, given that

$$D_{0y}^{\alpha} y^{\alpha-1} = 0 \qquad \text{and} \qquad D_{0y}^{\alpha-1} y^{\alpha-1} = \Gamma(\alpha),$$

direct computation shows that the function

$$h(x, y) = \frac{y^{\alpha-1}}{\Gamma(\alpha)} v(x) \tag{2.9}$$

satisfies (2.7). Combining Lemma 2.3 and equalities (2.8) and (2.9) proves (2.6). □

In what follows $g * h$ denote the Fourier convolution of the functions g and h, i.e.

$$\left(g * h\right)(x) = \int_{\mathbb{R}^n} g(s) h(x - s)\, ds.$$

Lemma 2.6. *Let $g(x) \in C\left(\mathbb{R}^n\right) \cap \mathcal{E}(-a, p)$, $h(x) \in C\left(\mathbb{R}^n\right) \cap \mathcal{E}(-b, q)$, $a > 0$, $b > 0$, $0 < p \le q$. Then*

$$\left(g * h\right)(x) \in \mathcal{E}(-c, p), \tag{2.10}$$

for every $c < a$ if $p < q$; and $c < \min\{a, b\}$ if $p = q$.

Proof. According to (2.3), we have

$$|g * h| \le C \int_{\mathbb{R}^n} e^{-a|x-s|^p - b|s|^q}\, ds \le C \int_{\mathbb{R}^n} e^{-a\left||x|-|s|\right|^p - b|s|^q}\, ds.$$

By (4.7) we obtain

$$\sup_{|x|=r} |g * h| \le C \int_0^{\infty} z^{n-1} e^{-a|z-r|^p - bz^q}\, dz =$$

$$= C e^{-ar^p} \int_0^r z^{n-1} e^{az^p - bz^q}\, dz + C e^{ar^p} \int_r^{\infty} z^{n-1} e^{-az^p - bz^q}\, dz \le C\left(1 + r^n\right) e^{-ar^p}.$$

This proves (2.10). □

Lemma 2.7. *Let $\tau(x) \in C(\mathbb{R}^n) \cap \mathcal{E}(-a,p)$ for some $a>0$, $p \leq \frac{2}{2-\alpha}$. Then*

$$\int_{\mathbb{R}^n} \tau(s)\,\Gamma_{\alpha,n}(x-s,y)\,ds \in \mathcal{E}(-c,p), \tag{2.11}$$

for every $y \in (0,T)$; and $c<a$ if $p<\frac{2}{2-\alpha}$, and $c < \min\left\{a, \frac{2-\alpha}{2}\left(\frac{\alpha}{2T}\right)^{\frac{\alpha}{2-\alpha}}\right\}$ if $p=\frac{2}{2-\alpha}$.

Proof. The inclusion (2.11) follows from the estimate (2.5) and Lemma 2.6. □

Lemma 2.8. *Let*

$$y^{1-\mu} f(x,y) \in C(\Omega_0), \qquad \sup_{y\in[0,T]} y^{1-\mu} f(x,y) \in \mathcal{E}(-a,p), \tag{2.12}$$

for some $a>0$, $p \leq \frac{2}{2-\alpha}$, and $\mu>0$. Then

$$\int_0^y \int_{\mathbb{R}^n} f(s,t)\,\Gamma_{\alpha,n}(x-s,y-t)\,ds\,dt \in \mathcal{E}(-c,p), \tag{2.13}$$

for every $y \in (0,T)$; and $c<a$ if $p<\frac{2}{2-\alpha}$, and $c < \min\left\{a, \frac{2-\alpha}{2}\left(\frac{\alpha}{2T}\right)^{\frac{\alpha}{2-\alpha}}\right\}$ if $p=\frac{2}{2-\alpha}$.

Proof. Let $g(x) = \sup_{y\in[0,T]} y^{1-\mu} f(x,y)$. By (2.1) and (2.12) we get

$$\left| \int_0^y \int_{\mathbb{R}^n} f(s,t)\,\Gamma_{\alpha,n}(x-s,y-t)\,ds\,dt \right| \leq C \int_{\mathbb{R}^n} g(s) D_{0y}^{-\mu} \Gamma_{\alpha,n}(x-s,y)\,ds.$$

From here, taking into account (2.5) and Lemma 2.6, we obtain (2.13). □

3. Integral identity

Here we prove an integral identity that relates solutions of the Cauchy problem for equation (1.1) to entire harmonic functions, and can be interpreted as a conservation law for solutions of (1.1), (2.2).

Next, for a pair of functions $g(x)$ and $h(x)$ such that $g(x)h(x) \in L(\mathbb{R}^n)$, we set

$$(g,h) := \int_{\mathbb{R}^n} g(x)\,h(x)\,dx.$$

In the case where one of the functions depends on two variables, say $h=h(x,y)$, we write

$$(g,h)(y) := \int_{\mathbb{R}^n} g(x)\,h(x,y)\,dx,$$

assuming that $g(x)h(x,y) \in L(\mathbb{R}^n)$ (as a function of x for each y for which the integral is considered).

Theorem 3.1. *Let*

$$\tau(x) \in C(\mathbb{R}^n) \cap \mathcal{E}(-a,p), \quad y^{1-\mu} f(x,y) \in C(\Omega_0), \quad \sup_{y\in[0,T]} y^{1-\mu} f(x,y) \in \mathcal{E}(-a,p),$$

for some $a>0$, $0<p\leq \frac{2}{2-\alpha}$; and $y^{1-\mu} f(x,y)$ be locally Hölder continuous with respect to x. Let also

$$v(x) \in \mathcal{H} \cap \mathcal{E}(c,p) \qquad \text{for some} \quad c < \begin{cases} a, & \text{if} \quad p < \frac{2}{2-\alpha}, \\ \min\left\{a, \frac{2-\alpha}{2}\left(\frac{\alpha}{2T}\right)^{\frac{\alpha}{2-\alpha}}\right\}, & \text{if} \quad p = \frac{2}{2-\alpha}, \end{cases}$$

and $u(x,y)$ be a solution of the problem (1.1) and (2.2) satisfying (2.4).

Then $v(x)u(x,y) \in L(\mathbb{R}^n)$ for every $y \in (0,T)$ as function of x, and

$$(v,u)(y) = \frac{y^{\alpha-1}}{\Gamma(\alpha)}(v,\tau) + \left(v, D_{0y}^{-\alpha} f\right)(y) \qquad (0 < y < T). \tag{3.1}$$

Proof. By Lemmas 2.2, 2.3, 2.7, and 2.8, we get $u(x,y) \in \mathcal{E}(-c,p)$. On account of the restriction on the growth of $v(x)$, it follows from this inclusion that $v(x)u(x,y) \in L(\mathbb{R}^n)$. The conditions on $\tau(x)$ and $f(x,y)$ allow us to apply Fubini's theorem, so we obtain

$$\int_{\mathbb{R}^n} v(x)u(x,y)\,dx = \int_{\mathbb{R}^n} \tau(s) \int_{\mathbb{R}^n} v(x)\Gamma_{\alpha,n}(x-s,y)\,dx\,ds +$$

$$+ \int_0^y \int_{\mathbb{R}^n} f(s,t) \int_{\mathbb{R}^n} v(x)\Gamma_{\alpha,n}(x-s,y-t)\,dx\,ds\,dt.$$

Taking into account Lemma 2.5, we get (3.1). □

Corollary 3.2. *Let the conditions of Theorem 3.1 be fulfilled and let*

$$w(y) := \left(v, u - D_{0y}^{-\alpha} f\right)(y) \equiv \int_{\mathbb{R}^n} v(x)\left[u(x,y) - D_{0y}^{-\alpha} f(x,y)\right]\,dx \tag{3.2}$$

Then

1) the function $w(y)$ does not change sign on $(0,T)$, namely,

$$\operatorname{sign} w(y) = \operatorname{sign}(v,\tau) \qquad \forall\, y \in (0,T); \tag{3.3}$$

in particular, if $w(y_0) = 0$ for some $y_0 \in (0,T)$ then $w(y) \equiv 0$; and $w(y) = 0$ only if $(v,\tau) = 0$;

2)

$$w(y) \in C^\infty(0,T); \tag{3.4}$$

3) if moreover $(v,\tau) \neq 0$, then

$$\frac{w(y)}{(v,\tau)} = \frac{y^{\alpha-1}}{\Gamma(\alpha)} \qquad \forall\, y \in (0,T); \tag{3.5}$$

$$\frac{w(y_1)}{w(y_2)} = \left(\frac{y_1}{y_2}\right)^{\alpha-1} \qquad \forall\, y_1,\, y_2 \in (0,T); \tag{3.6}$$

and

$$\frac{w'(y)}{w(y)} = \frac{\alpha-1}{y} \qquad \forall\, y \in (0,T). \tag{3.7}$$

Proof. According to the notation (3.2), the relations (3.3), (3.4), (3.5) and (3.6) are simple consequences of the formula (3.1).

Next, from (3.3) it follows that $\frac{w(y)}{(v,\tau)} > 0$ if $(v,\tau) \neq 0$. Taking the logarithmic derivative of both parts of (3.5), we obtain (3.7). □

4. Order determination inverse problems

The equalities (3.5), (3.6) and (3.7) allow us to pose inverse problems for the equation (1.1) in which the order of the fractional derivative, α, is unknown and must be determined.

Let us give a general formulation of **inverse problems** that we consider here: *find a pair* $\{u(x, y),\, \alpha\}$, *where* $u(x, y)$ *is a regular solution of the problem* (1.1), (2.2), *and* $\alpha \in (0, 1)$, *the order of the fractional derivative in* (1.1), *satisfies the overdetermination condition*

$$F(u, \alpha) = d, \tag{4.1}$$

where $F(\cdot, \alpha)$ *is a given functional, and* d *is a given real number.*

Below we present three forms of the functional $F(\cdot, \alpha)$ that can be used as overdetermination conditions in (4.1).

4.1. **One-point condition.** First, let us study some properties of the function (see also [4, 3])

$$H(z, t) := \frac{t^{z-1}}{\Gamma(z)}. \tag{4.2}$$

In what follows, for any positive t we set

$$\eta(t) := \sup_{z>0} H(z, t) \qquad \text{and} \qquad \varphi(t) := \Psi^{-1}(\ln t)\,, \tag{4.3}$$

where $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. From the properties of the functions $\Gamma(z)$ and $\Psi(z)$ it follows that both quantities $\eta(t)$ and $\varphi(z)$ are defined correctly and uniquely for any $t > 0$.

Lemma 4.1. *For each fixed* $t > 0$, *the equation*

$$H(z, t) = d, \tag{4.4}$$

in $z \in \mathbb{R}_+ := (0, \infty)$, *has exactly two distinct solutions if* $0 < d < \eta(t)$; *it has a unique solutions if* $d = \eta(t)$; *and it has no solutions if* $d > \eta(t)$.

Proof. On accounts of the definiition (4.2), it is easy to check that

$$H(z, t) > 0 \qquad (z,\, t > 0); \qquad \lim_{z\to 0} H(z, t) = \lim_{z\to\infty} H(z, t) = 0 \qquad (t > 0), \tag{4.5}$$

$$\frac{\partial}{\partial z} H(z, t) = H(z, t)\left[\ln t - \Psi(z)\right].$$

By (4.3), this equality gives that

$$\operatorname{sign} \frac{\partial}{\partial z} H(z, t) = \operatorname{sign}\left(\varphi(t) - z\right).$$

It follows that the function $H(z, t)$, as a function of z for each positive t, monotonically increases on the interval $0 < z < \varphi(t)$, reaches a maximum at the point $z = \varphi(t)$, i.e.,

$$\eta(t) = H(\varphi(t), t)\,, \tag{4.6}$$

and monotonically decreases for $z > \varphi(t)$. This proves the statement of the lemma. □

Remark 4.2. Note that the equality $H(1, t) = 1$, which is valid for any $t > 0$, shows that $\eta(t) \geq 1$. Moreover, from (4.3), (4.6), and the equality $\Psi(1) = -\gamma$, it follows that $\eta(t) = 1$ if and only if $t = e^{-\gamma}$, where $\gamma = -\Gamma'(1)$ is Euler's constant.

Lemma 4.3. *Let $t > 0$, $d \in (0, \eta(t)]$, $z_1(t)$ and $z_2(t)$ be solutions of (4.4), and $z_1(t) \le z_2(t)$.*

1) If $0 < d < 1$ then

$$0 < z_1(t) < 1 < z_2(t).$$

2) If $1 \le d \le \eta(t)$ then

$$0 < z_1(t) \le z_2(t) \le 1 \qquad \text{for} \quad 0 < t < e^{-\gamma};$$

and

$$1 \le z_1(t) \le z_2(t) \qquad \text{for} \quad t \ge e^{-\gamma}.$$

Moreover, $z_1(t) = z_2(t)$ if and only if $d = \eta(t)$, and $(z_1(t) - 1)(z_2(t) - 1) = 0$ if and only if $d = 1$.

Proof. Taking into account Remark 4.2, Lemma 4.3 follows from Lemma 4.1. □

Theorem 4.4. *Let*

$$\tau(x) \in C(\mathbb{R}^n) \cap \mathcal{E}(-a, p), \quad y^{1-\mu} f(x, y) \in C(\Omega_0), \quad \sup_{y \in [0,T]} y^{1-\mu} f(x, y) \in \mathcal{E}(-a, p),$$

for some $a > 0$, $0 < p \le 1$; and $y^{1-\mu} f(x, y)$ be locally Hölder continuous with respect to x.

Let also $v(x) \in \mathcal{H} \cap \mathcal{E}(c, p)$ for some $c < a$ and $u(x, y)$ be a solution of the problem (1.1) and (2.2) satisfying (2.4); $(v, \tau) \ne 0$, and $y_0 \in (0, T)$.

1) If $0 < d < 1$ or $d = \eta(y_0)$ with $y_0 < e^{-\gamma}$, then the inverse problem (1.1), (2.2), and (4.1) with the overdetermination condition

$$F(u, \alpha) := \frac{\left(v, u - D_{0y}^{-\alpha} f\right)(y_0)}{(v, \tau)} = d, \tag{4.7}$$

has a unique solution.

2) If $1 < d < \eta(y_0)$ and $y_0 < e^{-\gamma}$, then the inverse problem (1.1), (2.2), and (4.1) with the overdetermination condition (4.7), has exactly two solutions.

Proof. The conditions of the theorem guarantee the existence of a unique solution of the problem (1.1), (2.2) for any $\alpha \in (0, 1)$. It remains to determine α satisfying condition (4.1) with $F(u, \alpha)$ given by (4.7). By (3.5), this question is equivalent to the solvability of the equation

$$H(\alpha, y_0) = d.$$

The solvability of this equation under the conditions imposed on y_0 and d follows from Lemma 4.3. □

4.2. **Two-point condition.** As follows from Theorem 4.4, the overdetermination condition given by (4.7) does not always guarantee a unique determination of α for small values of y_0. Here we consider the overdetermination condition that allows us to uniquely determine α, but requires observation data specified at two points in time.

Theorem 4.5. *Let the conditions of Theorem 4.4 imposed on $\tau(x)$, $f(x, t)$, $v(x)$ and $u(x, y)$ be fulfilled, $(v, \tau) \ne 0$, $0 < y_1 < y_2 < T$. Then the problem (1.1), (2.2), and (4.1) with overdetermination condition*

$$F(u, \alpha) := \frac{\left(v, u - D_{0y}^{-\alpha} f\right)(y_1)}{\left(v, u - D_{0y}^{-\alpha} f\right)(y_2)} = d, \tag{4.8}$$

has a unique solution if and only if $1 < d < y_2/y_1$.

Proof. Recall that the condition $(v, \tau) \ne 0$ guarantees that $\left(v, u - D_{0y}^{-\alpha} f\right)(y) \ne 0$ for any $y \in (0, T)$. Further, by virtue of (3.6), (4.1) and (4.8), to determine α it is sufficient to solve the equation

$$\left(\frac{y_1}{y_2}\right)^{\alpha - 1} = d.$$

This equation has a unique solution

$$\alpha = 1 + \frac{\ln d}{\ln y_1 - \ln y_2},$$

which belongs to $(0, 1)$ if and only if $1 < d < y_2/y_1$. □

4.3. **Condition in terms of logarithmic derivative.** Here we will consider another type of overdetermination condition. It is generated by the equality (3.7) and is given using the logarithmic derivative.

Theorem 4.6. *Let the conditions of Theorem 4.4 imposed on $\tau(x)$, $f(x,t)$, $v(x)$ and $u(x,y)$ be fulfilled, $(v,\tau)\neq 0$, $0<y_0<T$. Then the inverse problem* (1.1), (2.2), *and* (4.1) *with overdetermination condition*

$$F(u,\alpha) := \left[\frac{\left(v, u - D_{0y}^{-\alpha} f\right)'(y)}{\left(v, u - D_{0y}^{-\alpha} f\right)(y)}\right]_{y=y_0} = d, \tag{4.9}$$

has a unique solution if and only if $0 < -d < 1/y_0$.

Proof. By (3.7), (4.1), and (4.9), α can be uniquely determined from the equality

$$d = \frac{\alpha - 1}{y_0}.$$

That is $\alpha = 1 + dy_0$, and $\alpha \in (0,1)$ if and only if $0 < -d\,y_0 < 1$. □

5. Conclusion

In the paper, we prove the integral identity (3.1), which connects a solution of the Cauchy problem for the fractional diffusion equation (1.1) with entire harmonic functions. Based on this identity, we propose three overdetermination conditions for the inverse problem, in which, in addition to the desired solution, the order of fractional differentiation is also unknown.

The following features of the presented results should be highlighted.

First, the condition $(v,\tau)\neq 0$ excludes the possibility of considering the discussed problems in the case of zero initial conditions. However, this is compensated by the wide arbitrariness of the choice of harmonic function $v(x)$.

Further, the considered conditions, (4.7), (4.8), and (4.9), work in the case of the Riemann-Liouville derivative. For example, in the case of the Gerasimov-Caputo derivative they are not applicable. It would be interesting to investigate the question of the existence of similar conditions for other forms of fractional differentiation.

Also, the proposed overdetermination conditions require the initial function and the right-hand side to rapidly tend to zero, which also narrows the class of admissible data for the problems under consideration, and it would also be interesting to expand it.

Acknowledgements

This research was partially funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488729, 2024–2026)

References

[1] Alimov Sh., Ashurov R.; Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation. J. Inverse Ill-Posed Probl., 2020, **28**, 651–658.

[2] Alimov Sh., Ashurov R.; On determining the fractional exponent of the subdiffusion equation. Journal of Inverse and Ill-posed Problems, 2025, **33**:2, 171–181.

[3] Ashurov R., Sitnik S.; Identification of the Order of the Fractional Derivative for the Fractional Wave Equation. Fractal Fract., 2023, 7, 67.

[4] Ashurov R., Umarov S.; Determination of the order of fractional derivative for subdiffusion equations. Fract. Calc. Appl. Anal., 2020, **23**, 1647–1662.

[5] Ashurov R., Zunnunov R.; Initial-boundary value and inverse problems for subdiffusion equation in $\mathbb{R}^N$. Fract. Differ. Calc., 2020, **10**, 291–306.

[6] Ashurov R.R., Zunnunov R.; Inverse Problem for Determining the Order of the Fractional Derivative in Mixed-Type Equations. Lobachevskii J. Math., 2021, **42**, 2714–2729.

[7] Ashurov R.R., Fayziev Y.E.; Inverse problem for determining the order of a fractional derivative in a wave equation. Math. Notes, 2021, **110**, 824–836.

[8] Alimov Sh., Ashurov R.; Inverse problem of determining an order of the Riemann-Liouville time-fractional derivative. Progr. Fract. Differ. Appl., 2022, **8**:4, 467–474.

[9] Ashurov R.R., Fayziev Y.E.; Determination of fractional order and source term in a fractional subdiffusion equation. Eur. Math. J., 2022, **13**, 19–31.

[10] Cheng J., Nakagawa J., Yamamoto M., Yamazaki T.; Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. Inverse Problems, 2009, **25**(11), 115002.

[11] Fujita Ya.; Integrodifferential equation which interpolates the heat equation and the wave equation I, II. Osaka J. Math., 1990, **27**, 309–321, 797–804.

[12] Hatano Y., Nakagawa J., Wang S., Yamamoto M.; Determination of order in fractional diffusion equation. J. Math-for-Ind., 2013, 5A, 51–57.

[13] Kilbas A.A.; Partial fractional differential equations and some of their applications. Analysis, 2010, **30**, 35–66.

[14] Kilbas A.A., Srivastava H.M., Trujillo J.J.; Theory and applications of fractional differential equations. Elsevier, Amsterdam, 2006.

[15] Kochubei A.N.; Diffusion of fractional order. Differ. Equ., 1990, **26**, 485–492.

[16] Kochubei A., Luchko Y. (Eds.); Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations. De Gruyter, Berlin, 2019.

[17] Li Z., Liu Y., Yamamoto M.; Inverse problems of determining parameters of the fractional partial differential equations, in A. Kochubei and Y. Luchko, (Eds.) Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations, De Gruyter, Berlin, 2019, 431–442.

[18] Pskhu A.V.; Green Functions of the First Boundary-Value Problem for a Fractional Diffusion–Wave Equation in Multidimensional Domains. Mathematics, 2020, **8**(4), 464.

[19] Pskhu A.V., Rekhviashvili S.Sh.; Fractional Diffusion–Wave Equation with Application in Electrodynamics. Mathematics, 2020,**8**(11), 2086.

[20] Pskhu A.V.; Partial differential equations of fractional order. Nauka, Moscow, 2005.

[21] Pskhu A.V.; The fundamental solution of a diffusion-wave equation of fractional order. Izv. Math., 2009, **73**, 351–392.

[22] Schneider W.R., Wyss W.; Fractional diffusion and wave equations. J. Math. Phys., 1989, **30**, 134–144.

[23] Wyss W.; The fractional diffusion equation. J. Math. Phys., 1986, **27**, 2782–2785.

[24] Wright E.M.; The generalized Bessel function of order greater than one. Quart. J. Math., Oxford Ser., 1940, **11**, 36–48.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.232-245
DOI: 10.29229/uzmj.2025-2-22

On Some Coefficient Inverse Problems of Identification of Thermophysical Parameters in Stratified Media

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. This article addresses the regular solvability in Sobolev spaces of parabolic coefficient inverse problems in stratified media with diffraction-type transmission conditions. The solution posseses all generalized derivatives occurring in the equation summable with some power. The overdetermination conditions are integrals of a solution over the spatial domain with weight. The proof is based on a priori estimates and the fixed point theorem.

Keywords: parabolic equation, inverse problem, initial boundary value problem, existence, uniqueness.

MSC (2020): 35R30, 35K57, 35K20

1. Introduction

We consider the problem of recovering the coefficients and the right-hand in a parabolic equation. Let G be a domain in $\mathbb{R}^n$ with boundary Γ and $Q = (0,T) \times G$. We assume that the domain G is divided into two open sets G^+ and G^-, $\overline{G^-} \subset G$, $\overline{G^+} \cup \overline{G^-} = \overline{G}, G^+ \cap G^- = \emptyset$,. Denote $\Gamma_0 = \partial G^+ \cap \partial G^-$, $S_0 = (0,T) \times \Gamma_0$, $S = (0,T) \times \Gamma$. The parabolic equation has the form

$$u_t + A(t,x,D)u = f(t,x), \quad (t,x) \in Q, \tag{1.1}$$

where the function f and the second order elliptic operator A in $G^\pm$ are representable as

$$-A(t,x,D) = A_0(t,x,D_x) + \sum_{i=1}^{r} q_i(t) A_i(t,x,D_x), \ f = f_0(t,x) + \sum_{i=r+1}^{s} f_i(t,x) q_i(t),$$

$$A_i = \sum_{k,l=1}^{n} a^i_{kl}(t,x)\partial^2_{x_k x_l} + \sum_{k=1}^{n} a^i_k(t,x)\partial_{x_k} + a^i_0.$$

The equation (1.1) is furnished with the initial and boundary conditions

$$u|_{t=0} = u_0, \quad Bu|_S = g(t,x), \tag{1.2}$$

where $Bu = u$ or $Bu = \frac{\partial u}{\partial N} + \sigma u = \sum_{i,j=1}^n a_{ij} u_{x_i} n_j + \sigma u$ and $n = (n_1, n_2, \dots, n_n)$ is the outward unit normal to Γ, and the transmission conditions

$$\frac{\partial u^+}{\partial N}(t,x) = \frac{\partial u^-}{\partial N}(t,x), \ u^+(t,x) = u^-(t,x), \quad (t,x) \in S_0, \tag{1.3}$$

where

$$\frac{\partial u^\pm}{\partial N}(t,x_0) = \lim_{x \in G^\pm,\ x \to x_0 \in \Gamma_0} \sum_{i,j=1}^{n} a_{ij} u_{x_i} \nu_j, \ u^\pm(t,x_0) = \lim_{x \in G^\pm,\ x \to x_0 \in \Gamma_0} u(t,x),$$

$\nu = (\nu_1, \nu_2, \dots, \nu_n)$ is the outward unit normal to G^-. The overdetermination conditions are given in the form

$$\int_G u(t,x)\varphi_j(x)dx = \psi_j(t), \quad j = 1, 2, \dots, s. \tag{1.4}$$

The unknowns in problem (1.1)-(1.4) are a solution u and the functions $q_i(t)$ $(i = 1, 2, \dots, s)$. It is possible that there are many connectedness components of the sets Γ or Γ_0 with their own boundary and transmission conditions. To simplify the presentation, we do not specify this case separately.

Problems of the type (1.1)-(1.4) arise when describing heat and mass transfer, diffusion, filtration processes, in ecology, etc. As an example, we can point out the problems of description of soil temperature regimes of northern territories, where determining thermophysical and mass transfer characteristics becomes crucial. These characteristics are typically specified by solving an associated inverse problem (see [26]). There are few, if any, theoretical results addressing problems (1.1)-(1.4), in contrast to the large number of articles available in the case of a single medium. We describe the results obtained in the latter case. We can refer to the monograph [28] devoted to inverse parabolic problems, and the monographs [5, 14, 20, 22] which contain the main statements of inverse problems, including those in the parabolic case. It is worth mentioning the articles [12, 13, 17, 18] where, in the case $n = 1$, the thermal conductivity is determined as a function of time, and existence and uniqueness theorems are obtained. The additional data used in these articles are the values of a solution at separate points which can be interior or boundary. The thermal conductivity, being independent on one of the spatial variables, and some coefficients with the use of the Cauchy data on the lateral surface of the cylinder and integral data are determined in works [15, 16]. Existence and uniqueness theorems for solutions, as well as stability estimates, have been obtained. In the monograph [5] (see also, for example, the results of work [10] and others) existence and uniqueness theorems for solutions to coefficient inverse problem with coefficients (including the higher-order coefficients) independent of certain spatial variables and the overdetermination data on sections of the spatial domain by planes are established. Due to the specifics of the method, all coefficients also do not depend on some spatial variables. More complete results are obtained in the articles [29]-[31], where the well-posedness of inverse problems for determining coefficients is demonstrated in the case when the solution is given on spatial manifolds or at separate points. Inverse problems with pointwise data are studied in the articles of A. I. Prilepko and his followers and a number of interesting problems are studied in [28]. Similar results but under somewhat different conditions on the data and in other spaces were obtained in articles [32], [33]. Our results are close to those in [34], where inverse problem of determining the higher-order coefficients in the equation in the case of an ordinary initial-boundary value problem (not a transmission problem), are considered.

The numerical solution to problems (1.1)-(1.4) and similar ones has been the subject of a large number of articles. The main numerical methods are based on reducing the problem to an optimal control problem and minimization of the corresponding objective functional (see [35, 2, 25, 36]). Various formulations and results can also be found in the articles [11]-[7]. Additionally, there are several articles devoted to various model problems of this type (see [21]).

Let us describe the content of the work. The second section outlines the conditions for the data of the problem and provides auxiliary results. The third section is devoted to the existence and uniqueness theorems for solutions to the problem (1.1)-(1.4). We consider the cases when the unknown coefficients occur into the higher-order part of the equation and when they appear in the lower-order part.

2. Preliminaries

Let E be a Banach space. Denote by $L_p(G;E)$ (G is the domain in $\mathbb{R}^n$) the space of measurable functions defined on G with values in E and a finite norm $\|\|u(x)\|_E\|_{L_p(G)}$ [37]. The notations for Sobolev spaces $W_p^s(G;E)$, $W_p^s(Q;E)$, etc. are standard (see [8, 9]). If $E=\mathbb{R}$ or $E=\mathbb{R}^n$ then we denote the last space simply by $W_p^s(Q)$. Definitions of Hlder spaces $C^{\alpha,\beta}(\overline{Q}), C^{\alpha,\beta}(\overline{S})$ can be found, for example, in [23]. By the norm of a vector, we mean the sum of the norms of coordinates. Given an interval $J=(0,T)$, denote $W_p^{s,r}(Q)=W_p^s(J;L_p(G))\cap L_p(J;W_p^r(G))$. Similarly, $W_p^{s,r}(S)=W_p^s(J;L_p(\Gamma))\cap L_p(J;W_p^r(\Gamma))$. Let $(u,v)=\int_G u(x)v(x)dx$. All considered spaces and coefficients of equation (1.1) are assumed to be real. Next, we assume that the parameter $p>n+2$ is fixed and

$$\Gamma,\Gamma_0\in C^2. \tag{2.1}$$

The definition of a boundary of class C^s, $s\geq 1$, can be found in [23, Chapter 1]. We introduce the following notations: $Q^\tau=(0,\tau)\times G$, $S_0^\tau=(0,\tau)\times\Gamma_0$, $Q^\pm=(0,T)\times G^\pm$, $Q_\tau^\pm=(0,\tau)\times G^\pm$, $S^\tau=(0,\tau)\times\Gamma$.

The consistency and smoothness conditions for the data can be written as follows:

$$u_0(x)\in W_p^{2-2/p}(G^\pm),\ B(0,x,D)u_0|_\Gamma=g(0,x),\ g\in W_p^{k_0,2k_0}(S),\ p>n+2,$$
$$\frac{\partial u_0^+}{\partial N}(x)=\frac{\partial u_0^-}{\partial N}(x),\ u_0^+(x)=u_0^-(x)\ \ x\in\Gamma_0, \tag{2.2}$$

where $k_0=\beta_0=1-1/2p$ in case $Bu=u$ and $k_0=1/2-1/2p$ otherwise;

$$f_j\in C([0,T];L_p(G)),\quad j=0,r+1,\dots,s; \tag{2.3}$$

$$\psi_j\in C^1([0,T]),\int_G u_0(x)\varphi_j(x)dx=\psi_j(0),\ j\leq s. \tag{2.4}$$

We assume also that

$$a_{ij}^k\in C(\overline{Q^\pm}),\ a_{ij}^k|_{S_0}\in W_p^{s_0,2s_0}(S_0)\ (s_0=1/2-1/p),\ \sigma,a_{ij}^k|_S\in W_p^{s_0,2s_0}(S), \tag{2.5}$$

where the latter condition holds if $Bu\neq u$;

$$\varphi_j\in W_q^{\beta_0}(G)\ (2/p<\beta_0<1-n/p),\ a_{ij}^k\in C([0,\tau];C^{\beta_1}(\overline{G^\pm})),\ a_l^k\in C([0,T];L_p(G)), \tag{2.6}$$

for all $i,j=1,2,\dots,n,\ l=0,1,\dots,n,\ k=0,1,\dots,r$, where $\beta_1>\beta_0$. Consider the matrix B_0 of size $s\times s$, with rows

$$(A_1(0,x)u_0(x),\varphi_j),\dots,(A_r(0,x)u_0(x),\varphi_j),(f_{r+1}(0,x),\varphi_j),\dots,(f_s(0,x),\varphi_j),\ j\leq s.$$

We require that

$$det\,B_0\neq 0. \tag{2.7}$$

Consider the systems of equations

$$B_0\vec{q}_0=\vec{g}_0, \tag{2.8}$$

where

$$\vec{g}_0=(\psi_{1t}(0)-(A_0(0,x)u_0,\varphi_j)-(f_0(0,x),\varphi_j),\dots,\psi_{st}(0)-(A_0(0,x)u_0,\varphi_s)-(f_0(0,x),\varphi_s))^T.$$

In view of (2.7), the system (2.8) has a unique solution $\vec{q}_0=(q_{01},\dots,q_{0s})^T$. Define the coefficients $a_{pl}=\sum_{i=0}^r a_{pl}^i q_{0i}$ and assume that

$$\sum_{p,l=1}^n a_{pl}(t,x)\xi_p\xi_l\geq\delta_0|\xi|^2\ \forall\xi\in\mathbb{R}^n,\forall(t,x)\in Q.$$

where δ_0 is a positive constant. In this case the operator $-A^0 = A_0(t,x,D_x) + \sum_{i=1}^r q_{0i}A_i(t,x,D_x)$ is elliptic and we can consider the problem

$$u_t + A^0(t,x,D_x)u = f \quad ((t,x)\in Q),\ u|_{t=0} = u_0(x), \quad Bu|_S = g,$$
$$\frac{\partial u^+}{\partial N}(x,t) = \frac{\partial u^-}{\partial N}(t,x),\ u^+(t,x) = u^-(t,x),\ (t,x)\in S_0. \quad (2.9)$$

Theorem 2.1. *Let the conditions* (2.1), (2.2), (2.5) *hold and let* $f\in L_p(Q)$. *Then there exists a unique solution* $u\in W_p^{1,2}(Q^+)\cap W_p^{1,2}(Q^-)$ *to the problem* (2.9). *The following estimate holds:*

$$\|u\|_{W_p^{1,2}(Q^+)} + \|u\|_{W_p^{1,2}(Q^-)} \le C\big[\|u_0\|_{W_p^{2-2/p}(Q^+)} + \|u_0\|_{W_p^{2-2/p}(Q^-)} + \|f\|_{L_p(Q)}\big]. \quad (2.10)$$

If $g = 0$, *then the estimate*

$$\|u\|_{W_p^{1,2}(Q_\tau^+)} + \|u\|_{W_p^{1,2m}(Q_\tau^-)} \le C\big[\|u_0\|_{W_p^{2-2/p}(Q_\tau^+)} + \|u_0\|_{W_p^{2-2/p}(Q_\tau^-)} + \|f\|_{L_p(Q^\tau)}\big] \quad (2.11)$$

holds, where the constant C *does not depend on* $u_0, f, \tau\in(0,T]$.

Proof. The first statement of the theorem is proven in the case of coefficients independent of t in the book [27, Chapter 6]. In principle, the assertion of the theorem in the case of arbitrary coefficients follows from the results of this work and standard reasoning, for example, those used in the work [1]. The case of t-dependent coefficients was considered in the article [3, Theorem 7.1]. Unfortunately, these results are mainly devoted to the more complicated case when $\Gamma\cap\Gamma_0\neq\emptyset$ and therefore their presentation and formulations are quite cumbersome. The second assertion and the estimate (2.11) follow from standard arguments, coinciding, for example, with those in the articles [33, theorem 2], [32, theorem 1].

3. Main results

We need the following auxiliary lemma.

Lemma 3.1. *Let* $v\in W_p^{1,2}(Q^+)\cap W_p^{1,2}(Q^-)$, $p>n+2$ *and let the condition* (2.6) *hold. Then the functions* $(a_{ij}^k v_{x_ix_j},\varphi_l)$, $(a_i^k v_{x_i},\varphi_l)$, $(a_0^k v,\varphi_l)$ *are continuous in* t *after possible modification on a set of measure* 0 *and the following estimates hold*

$$\|(a_{ij}^k v_{x_ix_l},\varphi_j)\|_{C([0,\tau])} \le C(\|v\|_{C([0,\tau];W_p^{2-\beta_0}(G^+))} + \|v\|_{C([0,\tau];W_p^{2-\beta_0}(G^-))}), \quad (3.1)$$

$$\|(a_i^k v_{x_j},\varphi_l)\|_{C([0,\tau])} + \|(a_0^k v,\varphi_l)\|_{C([0,\tau])} \le C(\|v\|_{C([0,\tau];W_p^{2-\beta_0}(G^+))} + \|v\|_{C([0,\tau];W_p^{2-\beta_0}(G^-))}). \quad (3.2)$$

where $\tau\in[0,T]$ *is arbitrary and the constant* C *does not depend on* τ.

Proof. Using the Schwarz inequalities we have

$$|(a_{ij}^k v_{x_ix_l},\varphi_j)| \le \|v_{x_ix_j}\|_{W_p^{-\beta_0}(G^+)}\|a_{ij}^k\varphi_j\|_{W_q^{\beta_0}(G^+)} + \|v_{x_ix_j}\|_{W_p^{-\beta_0}(G^-)}\|a_{ij}^k\varphi_j\|_{W_q^{\beta_0}(G^-)} \le$$
$$C(\|v\|_{W_p^{2-\beta_0}(G^+)}\|\varphi_j\|_{W_q^{\beta_0}(G^+)} + \|v\|_{W_p^{2-\beta_0}(G^-)}\|\varphi_j\|_{W_q^{\beta_0}(G^-)}) \le$$
$$C_1(\|v\|_{W_p^{2-\beta_0}(G^+)} + \|v\|_{W_p^{2-\beta_0}(G^-)}) \ \ (1/p+1/q=1). \quad (3.3)$$

Here we used the well-known fact that multiplication by the function $a_{ij}^k\in C^{\beta_1}(\overline{G^\pm})$ is a pointwise multiplier in $W_q^{\beta_0}(G^\pm)$ for $\beta_1>\beta_0$, the inequality $\beta_0<\frac{1}{q}=1-\frac{1}{p}$, and the estimate

$$\|v_{x_ix_j}\|_{W_p^{-\beta_0}(G^\pm)} \le C\|v\|_{W_p^{2-\beta_0}(G^\pm)}. \quad (3.4)$$

This inequality is a well-known fact that follows from the following arguments: since $\partial_{x_ix_j} \in L(W_p^2(G^\pm), L_p(G^\pm))$ and $\partial_{x_ix_j} \in L(L_p(G^\pm), W_p^{-2}(G^\pm))$, where $W_p^{-2}(G^\pm) = (\overset{\circ}{W}{}_q^2(G^\pm))'$ $(1/p + 1/q = 1)$, then $\partial_{x_ix_j} \in L((W_p^2(G^\pm), L_p(G^\pm))_{\theta,p}, (L_p(G^\pm), W_p^{-2}(G^\pm))_{\theta,p})$ and, in particular for fractional s, $\partial_{x_ix_j} \in L(W_p^s(G^\pm), W_p^{s-2}(G^\pm))$ for $s \in [0, 2]$ (see[37]). Note that $v \in C([0, \tau]; W_p^{2-2/p}(G^\pm))$ for each τ (for example, this follows from Theorem 4.10.2, Chapter III in [4]). In this case $v \in C([0, \tau]; W_p^{2-\beta_0}(G^\pm))$ for $\beta_0 \geq 2/p$. From (3.3) we obtain the inequality

$$\|(a_{ij}^k v_{x_ix_l}, \varphi_j)\|_{C([0,\tau])} \leq C_1(\|v\|_{C([0,\tau];W_p^{2-\beta_0}(G^+))} + \|v\|_{C([0,\tau];W_p^{2-\beta_0}(G^-))}).$$

The further proof of the continuity of the expression $(a_{ij}^k v_{x_ix_j}, \varphi_l)$ is carried out using the definitions and continuity of the function a_{ij}^k. Consider the terms (a_0^k, v, φ_j),$(a_i^k v_{x_i}, \varphi_j)$. Due to embedding theorems $v \in C^{1-\frac{n+2}{2p}, 2-\frac{n+2}{p}}(\overline{Q^\pm})$. In this case $v_{x_i} \in C(\overline{Q^+}) \cap C(\overline{Q^-})$ and we have

$$|(a_i^k v_{x_i}, \varphi_l)| \leq \|\varphi_l\|_{L_q(G)} \|a_i^k\|_{L_p(G)} \|v_{x_i}\|_{L_\infty(G)} \leq C\|\nabla v\|_{L_\infty(G)}. \tag{3.5}$$

From this we obtain the estimate

$$\|(a_i^k v_{x_i}, \varphi_l)\|_{C([0,\tau])} \leq C\|v\|_{C([0,\tau];W_\infty^1(G))} \leq C_1(\|v\|_{L_\infty(0,\tau;W_p^{2-\beta_0}(G^+))} + \|v\|_{L_\infty(0,\tau;W_p^{2-\beta_0}(G^-))}), \tag{3.6}$$

where we used embedding theorems. Similar estimate is valid for $\|(a_0^k v, \varphi_l)\|_{C([0,\tau])}$. Using the definitions of continuity, the obtained estimates (3.5),(3.6) and the condition (2.6) it is easy to prove that $(a_{ij}^k v_{x_ix_j}, \varphi_l)$, $(a_i^k v_{x_i}, \varphi_l)$, $(a_0^k v, \varphi_l) \in C([0, \tau])$.

Theorem 3.2. *Let the conditions* (2.1)-(2.7) *hold. Then on some segment* $[0, \tau_0]$ $(\tau_0 \leq T)$ *there exists a unique solution* $(u, q_1, q_2, \ldots, q_s)$ *to the problem* (1.1)-(1.4) *such that* $u \in W_p^{1,2}(Q_{\tau_0}^\pm)$, $q_j \in C([0, \tau_0])$, $j = 1, 2, \ldots, s$.

Proof. Let $\vec{q} = (q_1, \ldots, q_s)^T$. Find a solution Φ to the problem (2.9), where instead of the function f we take the function $\tilde{f}_0 = f_0 + \sum_{i=r+1}^s f_i(t, x)q_{0i}$, and the functions g, u_0 are those in (1.1)-(1.3). By Theorem 1, there exists a solution to the problem (2.9) such that $\Phi \in W_p^{1,2}(Q^\pm)$. Making a change of variables $u = v + \Phi$, we arrive at the problem

$$Lv = v_t + S(\vec{\mu})v = (A^0 - A)\Phi + \sum_{i=r+1}^s f_i(t, x)\mu_i(t), \ (t, x) \in Q, \ S(\vec{\mu}) = A^0 + A(\vec{\mu}), \tag{3.7}$$

where $-A(\vec{\mu}) = \sum_{i=1}^r \mu_i(t)A_i(t, x, D_x)$, $\mu_i(t) = q_i(t) - q_{0i}$;

$$v|_{t=0} = 0, \quad Bv|_S = 0; \tag{3.8}$$

$$\frac{\partial v^+}{\partial N}(x, t) = \frac{\partial v^-}{\partial N}(t, x), \ v^+(t, x) = v^-(t, x), \ (t, x) \in S_0; \tag{3.9}$$

$$\int_G v(t, x)\varphi_j(x)\, dx = \psi_j(t) - \int_G \Phi(t, x)\varphi_j(x)\, dx = \tilde{\psi}_j, \quad j = 1, 2, \ldots, s. \tag{3.10}$$

Thus, we have reduced the problem (1.1)-(1.4) to an equivalent and simpler problem (3.7)-(3.10), which we will be studied below. Consider the expression $L(\vec{\xi}) = \sum_{ij=1}^n \tilde{a}_{ij}\xi_i\xi_j$, $\tilde{a}_{ij} = \sum_{k=1}^r a_{ij}^k \mu_k$ and find the value R_0 such that

$$|L(\vec{\xi})| \leq \delta_0|\xi|^2/2 \ \forall \xi \in \mathbb{R}^n, \forall (t, x) \in Q, \ \forall \vec{\mu} : \|\vec{\mu}\|_{C([0,T])} \leq R_0.$$

In this case the operator $S(\vec{\mu})$ is elliptic and Theorem 1 holds with the operator $S(\vec{\mu})$ instead of the operator A^0. We now have a mapping $\vec{\mu} \to v = v(\vec{\mu})$ ($\vec{\mu} = (\mu_1, \dots, \mu_s)$). Study its properties. Let $\vec{\mu} \in B_{R_0,\tau} = \{\vec{\mu} \in C([0,\tau]) : \|\vec{\mu}\|_{C([0,\tau])} \le R_0\}$. Theorem 1 yields

$$v = L^{-1} f, \quad f = (A^0 - A)\Phi + \sum_{i=r+1}^{s} f_i(t,x)\mu_i(t) \ (x \in G^{\pm}). \tag{3.11}$$

We have an estimate

$$\|v\|_{W_p^{1,2}(Q^+)} + \|v\|_{W_p^{1,2}(Q^-)} \le c\|f\|_{L_p(Q^\tau)}, \tag{3.12}$$

where

$$f = \sum_{i=1}^{r} \mu_i A_i \Phi(t,x) + \sum_{i=r+1}^{s} \mu_i f_i(t,x), \ (t,x) \in Q^{\pm}. \tag{3.13}$$

From this representation and the conditions on the coefficients, we conclude that

$$\|f\|_{L_p(Q^\tau)} \le c_2 \|\vec{\mu}\|_{C([0,\tau])}, \tag{3.14}$$

where the constant c_2 does not depend on τ and depends on the norms of the coefficients in Q and the quantities $\|f_i\|_{L_p(Q^\tau)}$, $\|A_i\Phi\|_{L_p(Q^\tau)}$. Assuming that $\vec{\mu}_i \in B_{R_0,\tau}$ $(i = 1,2)$, we consider two solutions v_1, v_2 to the problem (3.7)-(3.8) (or equation (3.11)), corresponding to two different vectors $\vec{\mu}_i$ ($\vec{\mu}_i = (\mu_{1i}, \mu_{2i}, \dots, \mu_{si})$ $(i = 1,2)$. Subtracting the second equation in (3.7) from the first one, we find that the difference $\omega = v_2 - v_1$, $v_i = v(\vec{\mu}_i)$, satisfies the equation

$$\omega_t + S(\frac{\mu_1+\mu_2}{2})\omega = \sum_{j=1}^{r} \big(\mu_{j2}(t) - \mu_{j1}(t)\big) A_j(t,x,D)(v_1+v_2)/2$$
$$+ \sum_{j=1}^{r} (\mu_{j2}(t) - \mu_{j1}(t)) A_j(t,x,D)\Phi + \sum_{j=r+1}^{s} f_j(t,x)\big(\mu_{j2}(t) - \mu_{j1}(t)\big). \tag{3.15}$$

We have that $(\mu_1 + \mu_2)/2 \in B_{R_0,\tau}$ and therefore the estimate is valid (see (**??**))

$$\|\omega\|_{W_p^{1,2}(Q^+)} + \|\omega\|_{W_p^{1,2}(Q^-)} \le c\|\tilde{f}\|_{L_p(Q^\tau)}, \tag{3.16}$$

$$\tilde{f} = \sum_{j=1}^{r} \big(\mu_{j2}(t) - \mu_{j1}(t)\big) A_j(t,x,D)(v_1+v_2)/2 + \sum_{j=r+1}^{s} f_j(t,x)\big(\mu_{j2}(t) - \mu_{j1}(t)\big) +$$
$$+ \sum_{j=1}^{r} (\mu_{j2}(t) - \mu_{j1}(t)) A_j(t,x,D)\Phi. \tag{3.17}$$

The estimates (3.16), (3.14) validate the inequality

$$\|\omega\|_{W_p^{1,2}(Q^+)} + \|\omega\|_{W_p^{1,2}(Q^-)} \le c\|\tilde{f}\|_{L_p(Q^\tau)} \le c_2 c \|\vec{\mu}_2 - \vec{\mu}_1\|_{C([0,\tau])}, \tag{3.18}$$

where, as before, the constant c_2 depends on the norms (as a linear function) $\|A_j(v_1+v_2)\|_{L_p(Q^\tau)}$, $\|f_i\|_{L_p(Q^\tau)}$,$\|A_j\Phi\|_{L_p(Q^\tau)}$. In view of our conditions on the coefficients and inequalities (3.12), (3.14), we obtain that $\|A_j(v_1+v_2)\|_{L_p(Q^\tau)} + \|A_j\Phi\|_{L_p(Q^\tau)} \le c(\|v_1+v_2\|_{W_p^{1,2}(Q^+)} + \|v_1+v_2\|_{W_p^{1,2}(Q^-)} + \|\Phi\|_{W_p^{1,2}(Q^+)} + \|\Phi\|_{W_p^{1,2}(Q^-)}) \le c(R_0)$. Let v, $\vec{\mu}$ be a solution to problem (3.7), (3.8) and, thus,

$v = v(\vec{\mu})$. Multiplying equation (3.7) scalarly in the space $L_2(G)$ by the function φ_j taking into account that $(v_t, \varphi_j) = \tilde{\psi}_j'$, we obtain the system

$$\tilde{\psi}_j' + (S(\vec{\mu})v(t,x), \varphi_j) = \sum_{j=1}^{r} \mu_j (A_j \Phi, \varphi_j) + \sum_{j=r+1}^{s} \mu_j (f_j, \varphi_j), \ j = 1, 2, \ldots, s. \tag{3.19}$$

The right-hand side of this equality can be represented in the form $B(t)\vec{\mu}$, where the matrix B has the rows

$$(A_1(t,x)\Phi(t,x), \varphi_j), \ldots, (A_r(t,x)\Phi(t,x), \varphi_j), (f_{r+1}(t,x), \varphi_j), \ldots, (f_s(t,x), \varphi_j), \ j \le s.$$

As it follows from Lemma 1 and condition (2.3), $(A_k\Phi, \varphi_j) \in C([0,\tau])$ $(k = 1, 2, \ldots, r)$, $(f_j, \varphi_j) \in C([0,\tau])$. Therefore, the elements of the matrix B are continuous in t and when $t = 0$ we have that $B(0) = B_0$. In this case there exist $\tau_0 \le T$ and a constant $\delta_3 > 0$ such that

$$|det\, B(t)| \ge \delta_3 > 0 \ \forall t \in [0, \tau_0]. \tag{3.20}$$

Thus, the system (3.19) can be written as

$$\vec{\mu}(t) = B^{-1} H(\vec{\mu})(t) = R(\vec{\mu}),$$
$$H(\vec{\mu}) = (\tilde{\psi}_1' + (S(\vec{\mu})v, \varphi_1), \tilde{\psi}_2' + (S(\vec{\mu})v, \varphi_2), \ldots, \tilde{\psi}_s' + (S(\vec{\mu})v, \varphi_s))^T. \tag{3.21}$$

Note that $S(\mu) = 0$ if $\vec{\mu} = 0$ due to the uniqueness of a solution to parabolic problems. The right-hand side of (3.21) involves an operator that associates the vector functions $\vec{\mu}$ with the vector $R(\vec{\mu})$ and contains the function v being a solution to the problem (3.7), (3.8). As shown earlier, this operator is defined for all vectors $\vec{\mu}$ such that $\vec{\mu} \in B_{R_0,\tau}$, $\tau \le \tau_0$. We have already investigated the properties of this mapping $\vec{\mu} \to v(\vec{\mu})$. We now show that there exists a number $\tau_1 \le \tau_0$ such that the operator $R(\vec{\mu}) = B^{-1}H(\vec{\mu})(t)$, $R : C([0,\tau_1]) \to C([0,\tau_1])$, is well-defined, maps the ball B_{R_0,τ_1} into itself and is a contraction. Consider equation (2.9) for Φ. Integrating it with the weight φ_j, we obtain

$$(\Phi_t, \varphi_j) + (A^0\Phi, \varphi_j) = (\tilde{f}_0, \varphi_j), \ j = 1, 2, \ldots, s. \tag{3.22}$$

By Lemma 1, $(A^0\Phi, \varphi_j) \in C([0,T])$. The conditions on the data also guarantee that $(\tilde{f}_0, \varphi_j) \in C([0,T])$ for all j. Therefore, we may assume that $(\Phi, \varphi_j) \in C^1([0,\tau])$ after a possible rearrangement on a set of zero measure. Next, we have

$$\tilde{\psi}_{jt}(0) = \psi_{jt}(0) - \int_G \Phi_t \varphi_j dx|_{t=0} = \psi_{jt} + (A^0\Phi, \varphi_j) - (f, \varphi_i) = 0, \tag{3.23}$$

due to the equality (2.8). Let $\vec{\psi}(t) = (\tilde{\psi}_{1t}, \tilde{\psi}_{2t}, \ldots, \tilde{\psi}_{st})$. In this case there exists a number $\tau_1 \le \tau_0$ such that $\|B^{-1}\vec{\psi}\|_{C([0,\tau])} \le R_0/2$. Note that $R(0) = B^{-1}(t)\vec{\psi}(t)$. Next, we obtain some estimates assuming that $\vec{\mu}_i \in B_{R_0,\tau}$ and $\tau \le \tau_1$. Estimate the difference $\|R(\vec{\mu}_1) - R(\vec{\mu}_2)\|_{C([0,\tau])}$. We have

$$\|R(\vec{\mu}_1) - R(\vec{\mu}_2)\|_{C([0,\tau])} \le c_0 (\sum_{i=1}^{s} \|(A_0 v_1, \varphi_i) - (A_0 v_2, \varphi_i)\|_{C([0,\tau])} +$$
$$\sum_{i=1}^{s} \sum_{k=1}^{r} \|\mu_{1k}(A_k v_1, \varphi_i) - \mu_{2k}(A_k v_2, \varphi_i)\|_{C([0,\tau])}). \tag{3.24}$$

Using Lemma 1 for the functions $v_1 - v_2$ and $v_1 + v_2$, we obtain

$$\|\mu_{1k}(A_k v_1, \varphi_i) - \mu_{2k}(A_k v_2, \varphi_i)\|_{C([0,\tau])} \leq \|(\mu_{1k} - \mu_{2k})((A_k v_1, \varphi_i) + (A_k v_2, \varphi))\|_{C([0,\tau])}/2$$
$$+ \|\frac{(\mu_{1k} + \mu_{2k})}{2}((A_k v_1, \varphi_i) - (Av_2, \varphi_i))\|_{C([0,\tau])} \leq$$
$$C\|\mu_{1k} - \mu_{2k}\|_{C([0,\tau])}(\|v_1 + v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^+))} + \|v_1 + v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^-))})$$
$$\|\mu_{1k} + \mu_{2k}\|_{C([0,\tau])}(\|v_1 - v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^+))} + \|v_1 - v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^-))}), \quad (3.25)$$

where the constants c_i do not depend on τ. Similarly, we obtain the estimate

$$\|(A_0 v_1, \varphi_i) - (A_0 v_2, \varphi_i)\|_{C([0,\tau])} \leq c_0(\|v_1 - v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^+))} + \|v_1 - v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^-))}), \quad (3.26)$$

The inequality [37] holds

$$\|u\|_{W_p^s(G)} \leq C\|u\|^{\theta}_{W_p^{s_1}(G)}\|u\|^{1-\theta}_{W_p^{s_2}(G)}, \quad (3.27)$$

$s_1\theta + s_2(1-\theta) = s,\ \theta \in (0,1),\ s_1 < s < s_2$. Using the interpolation inequality (3.27) we have

$$\|v_1 - v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^\pm))} \leq \|v_1 - v_2\|^{\theta}_{C([0,\tau];W_p^{2-\frac{2}{p}}(G^\pm))}\|v_1 - v_2\|^{1-\theta}_{C([0,\tau];L_p(G^\pm))}, \quad (3.28)$$

where $(2 - \frac{2}{p})\theta = 2 - \beta_0$. The inequality

$$\|v_1 - v_2\|_{C([0,\tau],L_p(G^\pm))} \leq \tau^{\frac{p-1}{p}}\|v_1 - v_2\|_{L_p(Q^\pm_\tau)} \quad (3.29)$$

results from the Newton-Leibniz formula $v = \int_0^t v_\xi(\xi)d\xi$ $(v(0,x) = 0)$. Futhermore, we have the inequality

$$\|v_1 - v_2\|_{C([0,\tau]),W_p^{2-\frac{2}{p}}(G^\pm))} \leq C\|v_1 - v_2\|_{W_p^{1,2}(Q^\pm_\tau)}, \quad (3.30)$$

where the constant C does not depend on τ. The estimate follows from [4, Theorem 4.10.2 Ch. III] and the possibility of extension of the function v by zero for $t < 0$ with the preservation of the class. In this case the inequalities (3.28),(3.29), and (3.30) imply that

$$\|v_1 - v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^\pm))} \leq C\|v_1 - v_2\|_{W_p^{1,2}(Q^\pm)_\tau}\tau^{\beta}, \quad (3.31)$$

where C does not depend on τ and $\beta = (1-\theta)(\frac{p-1}{p})$. Similarly, we obtain that

$$\|v_1 + v_2\|_{C([0,\tau];W_p^{2-\beta_0}(G^\pm))} \leq C\|v_1 + v_2\|_{W_p^{1,2}(Q^\pm_\tau)}\tau^{\beta}. \quad (3.32)$$

The inequalities (3.18), (3.25), (3.26), (3.31), (3.32) ensure the estimate

$$\|R(\vec{\mu}_1) - R(\vec{\mu}_2)\|_{C([0,\tau])} \leq c_{12}\tau^{\beta}\|\vec{\mu}_1 - \vec{\mu}_2\|_{C([0,\tau])}. \quad (3.33)$$

If we choose $\tau_2 \leq \tau_1$ such that $c_{12}\tau^{\beta} \leq \frac{1}{2}$ for $\tau \leq \tau_2$, then the operator R is a contraction and maps the ball $B_{R_0,\tau}$ into itself for all $\tau \leq \tau_2$. Applying the fixed-point theorem, we obtain the existence of a solution to system (3.21). Let $v = v(\vec{\mu})$. We now demonstrate that this solution satisfies the overdetermination conditions (3.10). Multiply (3.7) scalarly in $L_2(G)$ by φ_j. We obtain the system of equalities

$$(v_t, \varphi_j) + (S(\vec{\mu})v, \varphi_j) = \sum_{j=1}^{r}\mu_j(A_j\Phi, \varphi_j) + \sum_{j=r+1}^{s}(f_j, \varphi_j)\mu_j(t). \quad (3.34)$$

Subtracting these equalities from (3.19), we obtain $\int_G v_t(t,x)\varphi_j\,dx - \tilde{\psi}'_j = 0$ for all j, which means that the conditions (3.10) are satisfied. The uniqueness of solutions follows from the estimates presented in the proof of existence. Q.E.D.

We now present some results on the solvability of problem (1.1)-(1.4) in the case when the unknown functions $q_i(t)$ enter the equation only as lower-order coefficients. The formulation and the proof differ slightly. In this case, the unknown coefficients are sought not in the class of continuous functions but in the class of functions belonging to some Lebesgue space. The parabalic equitaion has the same form, but $a^k_{ij} = 0$ for $i,j = 1,\dots,n$, $k = 1,2,\dots,r$ and, thus,

$$A_i u = \sum_{k=1}^{n} a^i_k u_{x_i} + a^i_0 u,\ \ k = 1,2,\dots,r. \tag{3.35}$$

We preserve the notations of the previous section.

$$a^k_l \in L_\infty(0,T;L_p(G)),\ \ a_{ij} \in C(\overline{Q}) \cap L_\infty(0,T;C^{\beta_1}(\overline{G})),\ \ p > n+2, \tag{3.36}$$

$$\varphi_{i_1} \in W^{\beta_0}_q(G),\ \ f_0 \in L_p(Q),\ \ f_{i_2}(t,x) \in L_\infty(0,T;L_p(G)),\ \ \psi_{i_1}(0) = \int_G u_0(x)\varphi_{i_1}(x)dx, \tag{3.37}$$

where $i_1 = 1,\dots,s$, $l = 0,1,\dots,n$, $i,j = 1,\dots,n$, $k = 1,\dots,r$, $i_2 = r+1,\dots,s$, and we assume that $0 < \beta_0 < 1 - \frac{1}{p}$, $\beta_1 > \beta_0$. Consider the matrix B_0 of size $s \times s$ with the rows

$$((A_1(t,y,D)u_0(y),\varphi_i)\dots,(A_r(t,y,D)u_0(y),\varphi_i),(f_{r+1}(t,y),\varphi_i),\dots,(f_s(t,y),\varphi_i),$$

where $i = 1,\dots,s$. The following condition ensures well-posedness of the problem:

$$|det\, B_0| > \delta_2 > 0 \text{ a.e. on } (0,T), \tag{3.38}$$

where δ_2 is some constant.

Theorem 3.3. *Let conditions* (2.1), (2.2), (3.36)-(3.38) *be satisfied. Then on some segment* $[0,\tau_0]$ $(\tau_0 \le T)$ *there exists a unique solution* $(u,q_1,q_2,\dots,q_s)$ *to the problem* (1.1)-(1.4) *such that*

$$u \in W^{1,2m}_p(Q^{\tau_0}),\quad q_j \in L_p(0,\tau_0),\ \ j = 1,2,\dots,s.$$

Proof. First, we note that the above-introduced matrix A^0 coincides with the matrix A_0 in this case. The proof is in line with the proof of the previous theorem. Therefore, we only outline the main stages. Rewrite the problem as follows:

$$u_t - A_0(t,x,D)u = f_0 + \sum_{i=r+1}^{s} q_i A_i u + \sum_{i=1}^{r} b_i(t,x)q_i(t), \tag{3.39}$$

$$u|_{t=0} = u_0,\quad Bu|_S = g. \tag{3.40}$$

$$(u(t,y),\varphi_i) = \psi_i,\quad i = 1,2,\dots,s. \tag{3.41}$$

Let Φ be a solution to the problem (3.39)-(3.40), where $q_i \equiv 0$ for all i. A solution exists and by Theorem 1, $\Phi \in W^{1,2}_p(Q)$. Making the substitution $u = v + \Phi$, we reduce the problem to the problem

$$v_t - A_0(t,x,D)v = \sum_{i=r+1}^{s} q_i A_i v + \sum_{i=r+1}^{s} b_i(t,x)\mu_i(t) + \sum_{i=1}^{r} q_i A_i \Phi, \tag{3.42}$$

$$v|_{t=0} = 0, \quad Bv|_S = 0. \tag{3.43}$$

$$(v(t,x), \varphi_i) = \psi_i - (\Phi(t,x), \varphi_i) = \tilde{\psi}_i, \quad i = 1, 2, \ldots, s. \tag{3.44}$$

Let v be a solution to problem (3.42)-(3.43), so $v = v(\vec{q})$. Multiplying the equation by φ_j scalarly in $L_2(G)$, we obtain the system

$$\tilde{\psi}_{jt} - (A_0 v(t,y), \varphi_j) - \sum_{i=r+1}^{s} q_i(t)(A_i v(t,y), \varphi_j) =$$

$$\sum_{i=r+1}^{s} q_i(t)(A_i(t,y,D)\Phi(t,y), \varphi_j) + \sum_{i=1}^{r} q_i(b_i(t,y), \varphi_j), \tag{3.45}$$

where the brackets $(\cdot,\cdot)$ denote the inner product in $L_2(G)$. The left-hand side is a vector $H(\vec{q})$ with the coordinates

$$\tilde{\psi}_{jt} - ((A_0 v(t,y), \varphi_j) + \sum_{i=1}^{r} q_i(t)(A_i v(t,y), \varphi_j)). \tag{3.46}$$

The right-hand side of (3.45) is written as $B(t)\vec{q}$. As in the proof of Lemma 1, it is easy to see that the entries $(b_i(t,y), \varphi_j)$, $(A_i(t,y,D)\Phi(t,y), \varphi_j)$ of the matrix B belongs to $L_\infty(0,T)$. The last $s-r$ columns of the matrix B coincide with those of the matrix B_0. The remaining entries are written as

$$(A_i(t,y,D)\Phi(t,y), \varphi_j) = (A_i(t,y,D)u_0(y), \varphi_j) + (A_i(t,y,D)(\Phi(t,y) - u_0(y)), \varphi_j).$$

Due to the condition on the coefficients, the second summand on an arbitrary interval $[0,\tau]$ admits an estimate

$$|(A_i(t,y,D)(\Phi(t,y) - u_0(y)), \varphi_j)| \le c\|\Phi(t,y) - u_0(y)\|_{L_\infty(0,\tau,W^1_\infty(G))}. \tag{3.47}$$

Furthermore, using interpolation inequalities, embedding theorems [37], and Theorem 1, we have that the right-hand side is estimated by the quantity

$$c_2(\|\Phi(t,x) - u_0(x)\|_{L_\infty(0,\tau,W_p^{1+n/p+\varepsilon}(G^+))} + \|\Phi(t,x) - u_0(x)\|_{L_\infty(0,\tau,W_p^{1+n/p+\varepsilon}(G^-))} \le$$

$$c_3(\|\Phi(t,x) - u_0(x)\|_{L_\infty(0,\tau;W_p^{2-2/p}(G^+))} + \|\Phi(t,x) - u_0(x)\|_{L_\infty(0,\tau;W_p^{2-2/p}(G^-))})^\theta \cdot$$

$$\|\Phi - u_0\|^{1-\theta}_{L_\infty(0,\tau,L_p(G))} \le c_4(\|\Phi(t,y) - u_0\|_{L_\infty(0,\tau;W_p^{2-2/p}(G^+))} +$$

$$\|\Phi(t,y) - u_0\|_{L_\infty(0,\tau;W_p^{2-2/p}(G^-))})^\theta \|\Phi_t\|^{1-\theta}_{L_p(0,\tau,L_p(G))} \tau^{(1-\theta)(p-1)/p} \le c_5 \tau^{(1-\theta)(p-1)/p}, \tag{3.48}$$

where $(2-2/p)\theta = 1+n/p+\varepsilon$, the parameter $\varepsilon > 0$ is chosen such that $1+n/p+\varepsilon < 2-2/p$ and without loss of generality we may assume that the constant c_5 does not depend on τ. Thus, the elements of the matrix B differ from the elements of the matrix B_0 by $c\tau^\beta$ where the parameter $\beta = (1-\theta)(p-1)/p > 0$ and c is a constant. Therefore, there exists a number τ_0 such that $|det\, B(t)| > \delta_2/2 > 0$ a.e. on $(0,\tau_0)$. Hence, we obtain the system

$$\vec{q}(t) = B^{-1}H(\vec{q})(t) = R(\vec{q}) \tag{3.49}$$

for finding the vector-function $\vec{q}$. The operator on the right-hand side is defined for $\tau \le \tau_0$. Note that $R(0) = (\tilde{\psi}_1, \ldots, \tilde{\psi}_s)^T$ due to the uniqueness of solutions to the problem (**??**), (3.40). Proceeding as in the previous theorem, we show that there exists $\tau_1 \le \tau_0$ such that the operator

R is a contraction in some ball $B_\tau = \{\vec{q}:\ \|\vec{q}\|_{L_p(0,\tau)} \leq R_0\}$, where $\tau \leq \tau_1$, and maps it into itself. As the parameter R_0, we take the value: $R_0 = 2\|R(0)\|_{L_p(0,T)}$. Fixing the functions $q_j \in L_p(0,\tau)$ and solving the problem (3.39)-(3.40) on the interval $(0,\tau)$, we obtain the mapping $v = v(\vec{q})$. Study the properties of this mapping. We have the estimate

$$\|v\|_{W_p^{1,2}(Q_\tau^-)} + \|v\|_{W_p^{1,2}(Q_\tau^+)} \leq c_6\| \sum_{i=1}^{r} q_i A_i v + \sum_{i=r+1}^{s} b_i(t,x)q_i(t) + \sum_{i=1}^{r} q_i A_i \Phi\|_{L_p(Q^\tau)} \leq$$
$$\leq c_7\|v\|_{L_\infty(0,\tau;W_\infty^1(G))} + c_8\|\vec{q}\|_{L_p(0,\tau)}. \quad (3.50)$$

The first summand on the right-hand side (see the estimates (3.47), (3.48)), is estimated by $c_8\tau^\beta(\|v\|_{W_p^{1,2}(Q_\tau^+)} + \|v\|_{W_p^{1,2}(Q_\tau^-)})$, where the constant c_8 does not depend on τ. In this case we obtain the estimate

$$\|v\|_{W_p^{1,2}(Q_\tau^+)} + \|v\|_{W_p^{1,2}(Q_\tau^-)} \leq c_9\|\vec{q}\|_{L_p(0,\tau)} + c_8\tau^\beta(\|v\|_{W_p^{1,2}(Q_\tau^+)} + \|v\|_{W_p^{1,2}(Q_\tau^-)}). \quad (3.51)$$

Choosing $\tau_1 \leq \tau_0$ so that $c_8\tau^\beta \leq 1/2$ for $\tau \leq \tau_1$, we infer

$$\|v\|_{W_p^{1,2}(Q_\tau^+)} + \|v\|_{W_p^{1,2}(Q_\tau^-)} \leq c_{10}\|\vec{q}\|_{L_p(0,\tau)}, \quad (3.52)$$

where the constant c_{10} does not depend on $\tau \leq \tau_1$. Now consider two solutions v_1, v_2 to the problem (3.39), (3.40) corresponding to two different collections $\vec{q}^{\,i} = (q_1^i, q_2^i, \ldots, q_s^i)$ $(i = 1,2)$ in the right-hand side of the equation (3.39). Subtracting the second equation from the first, we obtain that the difference $\omega = v_1 - v_2$ satisfies the equation

$$\omega_t - A_0\omega = \sum_{j=1}^{r}(q_j^1 - q_j^2)A_j(t,x,D)(v_2+v_1)/2$$
$$+ \sum_{j=1}^{r}\frac{(q_j^1+q_j^2)}{2}A_j(t,x,D)\omega + \sum_{j=r+1}^{s} b_j(t,x)\big(q_j^1(t,x') - q_j^2(t,x')\big). \quad (3.53)$$

If $\vec{q}_i \in B_\tau$ $(i = 1,2)$, then repeating the proof of (3.52), we derive that

$$\|v_1 - v_2\|_{W_p^{1,2}(Q_\tau^+)} + \|v_1 - v_2\|_{W_p^{1,2}(Q_\tau^-)} \leq c_{11}\|\vec{q}^{\,1} - \vec{q}^{\,2}\|_{L_p(0,\tau)}. \quad (3.54)$$

Now estimate the norm $\|R(\vec{q}^{\,1}) - R(\vec{q}^{\,2})\|_{L_p(0,\tau)}$. We have that

$$(H(\vec{q}^{\,1}) - H(\vec{q}^{\,2}))_j = (A_0\omega(t,y), \varphi_j) + \frac{1}{2}\sum_{i=r+1}^{s}(q_i^1 - q_i^2)(A_i(v_1+v_2)(t,y), \varphi_j) +$$
$$\frac{1}{2}\sum_{i=r+1}^{s}(q_i^1 + q_i^2)(A_i w(t,y), \varphi_j),$$

where the subscript on the left side denotes the number of the coordinate of the vector $H(\vec{q}^{\,1}) - H(\vec{q}^{\,2})$. We can write down analogs of the inequalities (3.47), (3.48), in which instead of the function $\Phi - u_0$ we take the functions $v_1 - v_2$, $v_1 + v_2$. In this case the second and third summands as estimated as

$$\|\frac{1}{2}\sum_{i=1}^{r}(q_i^1 - q_i^2)(A_i(v_1+v_2)(t,y), \varphi_j) + \frac{1}{2}\sum_{i=1}^{r}(q_i^1 + q_i^2)(A_i(v_1-v_2)(t,y), \varphi_j)\|_{L_p(0,\tau)}$$
$$\leq c_{12}(\|\vec{q}^{\,1} - \vec{q}^{\,2}\|_{L_p(0,\tau)}(\|v_1+v_2\|_{W_p^{1,2}(Q_\tau^+)} + \|v_1+v_2\|_{W_p^{1,2}(Q_\tau^-)}) +$$
$$\|\vec{q}^{\,1} + \vec{q}^{\,2}\|_{L_p(0,\tau)}(\|v_1-v_2\|_{W_p^{1,2}(Q_\tau^+)} + \|v_1-v_2\|_{W_p^{1,2}(Q_\tau^-)}))\tau^\beta. \quad (3.55)$$

Note that $\|\vec{q}^1+\vec{q}^2\|_{L_p(0,\tau)} \le 2R_0$, $\|v_1+v_2\|_{W_p^{1,2}(Q_\tau^\pm)} \le 2c(R_0)$, upper bounds of the norms generally depend on R_0, the constant $c(R_0)$ coincides with that in the estimates (3.52). Using (3.54), we obtain the estimate

$$\|\frac{1}{2}\sum_{i=1}^{r}(q_i^1-q_i^2)(A_i(v_1+v_2)(t,y),\varphi_j)+\frac{1}{2}\sum_{i=1}^{r}(q_i^1+q_i^2)(A_i(v_1-v_2)(t,y),\varphi_j)\|_{L_p(0,\tau)} \le c_{13}\tau^\beta\|\vec{q}^1-\vec{q}^2\|_{L_p(0,\tau)}, \quad (3.56)$$

where the constant c_{13} depends on R_0 but does not depend on τ. Estimate the first summand $(A_0\omega(t,y),\varphi_j)$. It contains summands of the form $I_{ij}=(a_{ij}^0 w_{x_ix_j},\varphi_j)$, $I_j=(a_j^0 w_{x_j},\varphi_j)$ $i,j=1,2,\ldots,n)$, $I_0=(a_0^0 w_{x_j},\varphi_j)$.

$$|I_{ij}| \le (\|w_{x_ix_j}\|_{W_p^{-\beta_0}(G^+)}+\|w_{x_ix_j}\|_{W_p^{-\beta_0}(G^-)})(\|a_{ij}^0\varphi_j\|_{W_q^{\beta_0}(G^+)}+\|a_{ij}^0\varphi_j\|_{W_q^{\beta_0}(G^-)}) \le c_1(\|w\|_{W_p^{2-\beta_0}(G^+)}+\|w\|_{W_p^{2-\beta_0}(G^-)}) \le c_2(\|w\|_{W_p^2(G^+)}+\|w\|_{W_p^2(G^-)})^\theta\|w\|_{L_p(G)}^{1-\theta}.$$

From here we obtain the estimate

$$\|I_{ij}\|_{L_p(0,\tau)} \le c_2(\|w\|_{L_p(0,\tau;W_p^2(G^+))}+\|w\|_{L_p(0,\tau;W_p^2(G^-))})^\theta\|w\|_{L_p(Q^\tau)}^{1-\theta} \le c_3(\|w\|_{W_p^{1,2}(Q_\tau^+)}+\|w\|_{W_p^{1,2}(Q_\tau^-)})\tau^{1-\theta},\ \theta=(2-\beta_0)/2, \quad (3.57)$$

where we used the inequality $\|w\|_{L_p(Q^\tau)} \le \tau\|w_t\|_{L_p(Q^\tau)}$, which follows from the Newton-Leibnitz formula. The conditions on the data imply that

$$|I_j| \le c_1\|w\|_{W_\infty^1(G)} \le c_2(\|w\|_{W_p^{1+n/p+\varepsilon}(G^+)}+\|w\|_{W_p^{1+n/p+\varepsilon}(G^-)}) \le c_2(\|w\|_{W_p^2(G^+)}+\|w\|_{W_p^2(G^-)})^{\theta_1}\|w\|_{L_p(G)}^{1-\theta_1},\ \theta_1=(1+n/p+\varepsilon)/2,$$

where we choose $\varepsilon<1-n/p$. Thus, we obtain

$$|I_j|_{L_p(0,\tau)} \le c_3(\|w\|_{W_p^{1,2}(Q_\tau^-)}+\|w\|_{W_p^{1,2}(Q_\tau^+)})\tau^{1-\theta_1}. \quad (3.58)$$

Repeating the arguments, we similarly obtain that

$$\|I_0\|_{L_p(0,\tau)} \le c_4(\|w\|_{W_p^{1,2}(Q_\tau^-)}+\|w\|_{W_p^{1,2}(Q_\tau^+)})\tau^{1-\theta_1}. \quad (3.59)$$

The estimate we need follows from (3.54), (3.56)-(3.59) and has the form

$$\|R(\vec{q}^1)-R(\vec{q}^2)\|_{L_p(0,\tau)} \le c_{19}\tau^{\beta_1}\|\vec{q}^1-\vec{q}^2\|_{L_p(0,\tau)},\ \tau\le\tau_1, \quad (3.60)$$

where the constant c_{19} does not depend on τ. Find $\tau_2\le\tau_1$ so that $c_{19}\tau^{\beta_1}\le 1/2$ for $\tau\le\tau_2$. In this case the equation (3.49) has a unique solution in the ball B_{τ_2}. Recover the function v as a solution to problem (3.39), (3.40). Demonstrate that conditions (**??**) are satisfied. Multiplying equation (3.39) scalarly by φ_j, we obtain the system

$$(v_t(t,y),\varphi_j)-(A_0v(t,y),\varphi_j)-\sum_{i=r+1}^{s}q_i(t)(A_i(v(t,y),\varphi_j)= +\sum_{i=r+1}^{s}q_i(t)(A_i\Phi(t,y),\varphi_j)+\sum_{i=1}^{r}q_i(f_i(t,y),\varphi_j).$$

Subtracting the obtained equalities from (3.45), we obtain $(v_t(t,y),\varphi_j) - \tilde{\psi}_{jt} = 0$ for all j. Integrating with respect to time and using the consistency conditions, we arrive at equality (3.41). The uniqueness of solutions follows from the arguments and estimates of the proof of the theorem.

Acknowledgement. The research was carried out within the state assignment of Ministry of Science and Higher Education of the Russian Federation (theme No. FENG-2023-0004, "Analytical and numerical study of inverse problems on recovering parameters of atmosphere or water pollution sources and (or) parameters of media").

References

[1] *Agranovich, M. S., Vishik, M. I.* Elliptic problems with a parameter and parabolic problems of general type. Russian Math. Surv. 1964. V. 19, P. 53-157.

[2] *Alifanov O. M., Artyukhin E. A., Nenarokomov A. V.* Inverse problems in the study of complex heat transfer. Moscow: Janus-K, 2009.

[3] *Amann H.* Maximum regularity of parabolic transmission problems // J. Evol. Equ. 2021. V. 21. P. 33753420.

[4] Amann H. Linear and Quasilinear Parabolic Problems, I. Monographs in Mathematics, v. 89. Basel: Birkhauser Verlag, 1995.

[5] *Belov Ya. Ya.* Inverse problems for parabolic equations. Utrecht: VSP, 2002.

[6] *Bushuev A. Yu., Timofeev V. N.* Two-criteria approach to solving the problem identification of thermophysical characteristics of a multilayer plate // Engineering Journal: Science and Innovation. 2013. No. 9(21), http://engjournal.ru/articles/963/963.pdf.

[7] *Bykov N. Yu., Khvatov A. A., Kalyuzhnaya A. V., Bukhanovsky A. V.* Method for reconstructing heat and mass transfer models by spatio-temporal distributions of parameters // Letters to ZhTF. 2021, Vol. 47, Issue 24. Pp. 9-12.

[8] *Denk R., Hieber M., and Prüss J.* Optimal Lp-Lq-estimates for parabolic boundary value problems with inhomogeneous data // Math. Z. 2007. V. 257, 1. P. 93-224.

[9] *Denk R., Hieber M., Prüss J.* R-boundedness, Fourier multipliers, and problems of elliptic and parabolic type // Memoirs of the A.M.S. 2003. V. 166. No 788.

[10] *Frolenkov I. V., Romanenko G. V.* On the solution of one inverse problem for a multidimensional parabolic equation // Sib. Journal of Industrial Mathematics. 2012. Vol. 15. No. 2 (50). P. 139-146.

[11] *Huntul M. J., Lesnic D.* An inverse problem of finding the time-dependent thermal conductivity from boundary data. International Communications in Heat and Mass Transfer. 2017. V. 85. P. 147154.

[12] *Hussein M. S., Huntul M. J.* Simultaneous Identification of Thermal Conductivity and Heat Source in the Heat Equation // Iraqi Journal of Science. 2021. V. 62, no. 6, P. 1968-1978.

[13] *Hussein M. S., Lesnic D., Ivanchov M. I.* Simultaneous determination of time-dependent coefficients in the heat equation // Computers and Mathematics with Applications. 2014. V. 67, pp. 1065-1091.

[14] *Isakov V.* Inverse Problems for Partial Differential Equations. Appl. Math. Sci. v.127. 2006. Berlin: Springer.

[15] *Iskenderov A. D.* Multidimensional inverse problems for linear and quasilinear parabolic equations // Dokl. Academy of Sciences of the USSR. 1975. T. 225, No. 5, 10051008.

[16] *Iskenderov A. D., Akhundov A. Ya.* Inverse Problem for a Linear System of Parabolic Equations // Doklady Mathematics. 2009, V. 79, no. 1. P. 7375.

[17] *Ivanchov M.* Inverse problems for equations of parabolic type. Math. Studies. Monograph Series. V. 10. Lviv: WNTL Publishers, 2003.

[18] *Ivanchov N. I., Pabyrivska N. V.* On the determination of two time-dependent coefficients in a parabolic equation // Siberian Math. J. 2002, Vol. 43, No. 2, pp. 406413.

[19] *Kabanikhin S. I., Khasanov A. Kh., Penenko A. V.*, Gradient descent method for solving the inverse coefficient problem of heat conduction // Siberian Journal of Computational Mathematics. 2008. Vol. 11, No. 1, pp. 4151.

[20] *Kabanikhin S. I.* Inverse and Ill-Posed Problems. Theory and Applications. Boston/Berlin: Walter de Gruyter GmbH & Co. KG, 2012.

[21] *Kanca F.* Inverse Coefficient Problem of the Parabolic Equation with Periodic Boundary and Integral Overdetermination Condition. Generalized Differential and Integral Equations, 2013.

[22] *Klibanov M. V., Li J.* Inverse Problems and Carleman Estimates. Inverse and Ill-Posed Problems Series. v. 63. Berlin/Boston: Walter de Gruyter GmbH & Co. KG, 2021.

[23] *Ladyzhenskaya O. A., Solonnikov V. A., Uraltseva N. N.* Linear and quasilinear equations parabolic type. Moscow: Nauka, 1967.

[24] *Mustonen L.* Numerical study of a parametric parabolic equation and a related inverse boundary value problem // Inverse Problems. 2016. V. 32, 105008.

[25] *Ozisik M. N., Orlande H. R. B.* Inverse Heat Transfer. New York: Taylor and Francis, 2000.

[26] *Permyakov P. P.* Identification of parameters of a mathematical model of heat and moisture transfer in frozen soils. Novosibirsk: Nauka, 1989.

[27] *Prüss V., Simonett G.* Moving Interfaces and Quasilinear Parabolic Evolution Equations. Basel: Birkhauser Publishing, 2016.

[28] *Prilepko A. I., Orlovsky D. G., and Vasin I. A.* Methods for Solving Inverse Problems in Mathematical Physics. New York: Marcel Dekker, Inc., 1999.

[29] *Pyatkov S. G., Tsybikov B. N.* On some classes of inverse problems for parabolic and elliptic equations // J. Evol. Equat. 2011. V. 11, No. 1. P. 155186.

[30] *Pyatkov S. G.* On some classes of inverse problems for parabolic equations. // J. Inv. Ill-Posed problems. 2011. V. 18, No. 8, P. 917-934.

[31] *Pyatkov S. G.* On some classes of inverse problems with overdetermination data on spatial manifolds // Siberian Mathematical Journal, 2016. V. 57, No. 5. P. 870880.

[32] *Pyatkov S. G., Rotko V. V.* Inverse Problems with Pointwise Overdetermination for some Quasilinear Parabolic Systems // Siberian Advances in Mathematics 2020. V. 30, Iss. 2, pp. 124-142.

[33] *Pyatkov S. G., Rotko V. V.* On some parabolic inverse problems with the pointwise overdetermination // AIP Conference Proceedings. 2017. V. 1907, 020008.

[34] *Pyatkov S. G.* Identification of thermophysical parameters in mathematical models of heat and mass transfer // Journal of Computational and Engineering Mathematics. 2022, V. 9, No. 2, P. 52-66.

[35] *Samarskii A. A., Vabishchevich P. N.* Numerical Methods for Solving Inverse Problems of Mathematical Physics. Berlin/Boston: Walter de Gruyter GmbH & Co. KG, 2007.

[36] *Tkachenko V. N.* Mathematical modeling, identification and control of technological processes of heat treatment of materials. Kyiv: Naukova Dumka, 2008.

[37] *Triebel H.* Interpolation Theory. Function Spaces. Differential Operators. Berlin: VEB Deutscher Verlag der Wissenschaften, 1978.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.246-256
DOI: 10.29229/uzmj.2025-2-23

On Poisson type integral

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. The problem of finding the temperature on the upper border of a stripe under known conditions on the lower border is considered. The existence and uniqueness of the solution of the Cauchy problem for elliptic equation are proved.

Keywords: Poisson integral, elliptic equation, Cauchy problem, existence, uniqueness.
MSC (2020): 35K05, 35K15

1. Introduction

Consider in the halfspace $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \ : \ x \in \mathbb{R}, \ y > 0\}$ differential equation

$$\operatorname{div}[k(y)\operatorname{grad} u(x,y)] = 0 \tag{1.1}$$

with boundary condition

$$u(x,0) = \phi(x). \tag{1.2}$$

We assume that the function $k(y) > 0$ is defined for $y \geq 0$, two times continuously differentiable, and for large y is stabilized. In other words, $k \in C^2(\mathbb{R}_+)$, $k(y) > 0$ and there exists $R > 0$ such that the condition

$$k(y) = \text{const}, \quad y \geq R. \tag{1.3}$$

is fulfilled.

We say that operator $Q : L_2(\mathbb{R}) \to C^2(\mathbb{R}^2_+)$ is Poisson operator of the second type, if for any function $\psi \in L_2(\mathbb{R})$ the following function

$$u(x,y) = (Q\psi)(x,y) \tag{1.4}$$

satisfies equation (1.1) in $\mathbb{R}^2_+$ and boundary condition

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \psi(x). \tag{1.5}$$

Boundary condition (1.7) is understood in the following sense:

$$\lim_{y\to 0+0} \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial y}(x,y) - \psi(x) \right|^2 dx = 0. \tag{1.6}$$

Following theorem is true.

Theorem 1.1. For any positive function $k \in C^2(\mathbb{R}_+)$, satisfying condition (1.3), there exists second type Poisson operator.

2. Solution of the axuiliary integral equation

To find the solution to the equation (1.1) we use the Fourier transform :

$$u(x,y) \;=\; \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \widehat{u}(s,y)e^{isx}\,ds, \tag{2.1}$$

where function $\widehat{u}(s,y)$ satisfies the following differential equation

$$\frac{1}{k(y)}\frac{d}{dy}\left[k(y)\,\widehat{u}_y\right] \;=\; s^2\widehat{u}(s,y), \quad y>0. \tag{2.2}$$

By introducing new function

$$v(s,y) \;=\; \sqrt{k(y)}\;\widehat{u}(s,y) \tag{2.3}$$

we reduce equation (2.2) to the following:

$$\frac{d^2v}{dy^2} - q(y)v - s^2v \;=\; 0. \tag{2.4}$$

Potential $q(y)$ is a continuous function on half-line $y \geq 0$ and has the form

$$q(y) \;=\; \frac{1}{2}\,\frac{k''(y)}{k(y)} \;-\; \frac{1}{4}\left(\frac{k'(y)}{k(y)}\right)^2. \tag{2.4*}$$

According to (2.3), potential $q(y)$ satisfied following condition:

$$q(y)=0, \quad y\geq R. \tag{2.5}$$

Among the all solutions of the equation (2.4) the most important is $v=e(s,y)$, which satisfies for $s\in\mathbb{R}$, $s\neq 0$, the integral equation

$$e(s,y) \;=\; e^{-|s|y} \;-\; \int\limits_{y}^{\infty}\frac{\sinh s(y-\xi)}{s}q(\xi)e(s,\xi)\,d\xi. \tag{2.6}$$

(see. Naimark, §27, Theorem 1).

In order to obtain necessary estimates, we introduce the following function:

$$v(s,y) \;=\; e(s,y)e^{|s|y} \tag{2.7}$$

Multiplying by $e^{|s|y}$ each terms of equation (2.6), we get the integral equation

$$v(s,y) \;=\; 1 \;-\; \int\limits_{y}^{\infty} e^{|s|y}\,\frac{\sinh|s|(y-\xi)}{|s|}\,e^{-|s|\xi}q(\xi)v(s,\xi)\,d\xi. \tag{2.8}$$

Then, the equation (2.8) can be writtten as follows:

$$v(s,y) \;=\; 1 \;+\; \frac{1}{2|s|}\int\limits_{y}^{\infty}\left[1-e^{-2|s|(\xi-y)}\right]q(\xi)v(s,\xi)\,d\xi. \tag{2.9}$$

Lemma 2.1. The solution of the equation (2.9) exists and satisfies condition

$$v(s,y) \;=\; 1 \;+\; \frac{O(1)}{1+|s|}, \quad y \geq 0, \quad s \in \mathbb{R}. \tag{2.10}$$

Proof. Introduce the following kernel:

$$K(y,\xi) \;=\; K(y,\xi,s) \;=\; (\xi - y)\Phi(|s|(\xi - y))q(\xi), \quad \xi \geq y \geq 0, \tag{2.11}$$

where

$$\Phi(t) \;=\; \frac{1}{2t}\,(1 - e^{-2t}), \quad t \geq 0.$$

In this case, taking into account (2.5), equation (2.9) takes the form

$$v(s,y) \;=\; 1 \;+\; \int\limits_y^R K(y,\xi)v(s,\xi)\,d\xi, \quad 0 \leq y \leq R. \tag{2.12}$$

It also follows from condition (2.5) that $v(s,y) = 1$ for $y > R$.
Set

$$M \;=\; \sup_{y \geq 0}\, |q(y)|.$$

Since the function Φ satisfies the relations

$$\Phi(0) = 1, \quad 0 < \Phi(t) < 1, \quad \Phi(t) < \frac{1}{2t}, \quad t > 0,$$

we get for kernel (2.11) the estimates

$$|K(y,\xi)| \;\leq\; M \cdot (\xi - y), \quad 0 \leq y \leq \xi \tag{2.13}$$

and

$$|K(y,\xi)| \;\leq\; \frac{M}{2|s|}, \quad 0 \leq y \leq \xi. \tag{2.14}$$

Define the sequence $\{w_n(y)\}_{n=0}^{\infty}$ by the following recurrence relation:

$$w_{n+1}(y) \;=\; \int\limits_y^R K(y,\xi)w_n(\xi)\,d\xi, \quad w_0(y) \equiv 1, \quad 0 \leq y \leq R. \tag{2.15}$$

Let us prove the estimate

$$|w_n(y)| \;\leq\; M^n \frac{(R-y)^{2n}}{(2n-1)!!}. \tag{2.16}$$

We use method of mathematical induction. Inequality (2.16) is formally satisfied for $n = 0$, if set $(-1)!! = 1$. Further, according to (2.15) and (2.13),

$$|w_{n+1}(y)| \;\leq\; \int\limits_y^R |K(y,\xi)| \cdot |w_n(\xi)|\,d\xi \;\leq\; \int\limits_y^R (\xi - y)MM^n \frac{(R-\xi)^{2n}}{(2n-1)!!}\,d\xi \;\leq$$

$$\leq\; \frac{M^{n+1}(R-y)}{(2n-1)!!}\int\limits_y^R (R-\xi)^{2n}\,d\xi \;=\; M^{n+1}\frac{(R-y)^{2n+2}}{(2n+1)!!}.$$

Set

$$w(s,y) \;=\; \sum_{n=1}^{\infty} w_n(y). \tag{2.17}$$

Then function

$$v(s,y) \;=\; 1+w(s,y) \;=\; \sum_{n=0}^{\infty} w_n(y) \tag{2.18}$$

is solution to the equation (2.12).

Indeed,

$$\int_y^R K(y,\xi)v(s,\xi)\,d\xi \;=\; \sum_{n=0}^{\infty}\int_y^R K(y,\xi)w_n(s,\xi)\,d\xi \;=$$

$$= \sum_{n=0}^{\infty} w_{n+1}(y) \;=\; \sum_{n=1}^{\infty} w_n(y) \;=\; w(s,y) \;=\; v(s,y)-1.$$

We estimate series (2.17) by using inequality (2.16):

$$|w(s,y)| \;\le\; \sum_{n=1}^{\infty} M^n \frac{(R-y)^{2n}}{(2n-1)!!} \;\le\; \sum_{n=1}^{\infty} M^n \frac{R^{2n}}{(2n-1)!!} \;=\; C_R, \quad 0 \le y < R.$$

Besides, the condition (2.5) implies the following equality

$$w(s,y) \;=\; 0, \quad y \ge R.$$

Therefore,

$$|w(s,y)| \;\le\; C_R, \quad y \ge 0, \quad s \in \mathbb{R}. \tag{2.19}$$

According to (2.19) and (2.18) we get necessary estimate

$$|v(s,y)| \;\le\; C, \quad y \ge 0, \quad s \in \mathbb{R}. \tag{2.20}$$

Further, according to definition (2.11), (2.18) and integral equation (2.12), we obtain that the following equality is valid

$$w(s,y) \;=\; \frac{1}{2|s|}\int_y^R \left[1-e^{-2|s|(\xi-y)}\right] q(\xi)v(s,\xi)\,d\xi. \tag{2.21}$$

Moreover, the relation above and estimate (2.20) follow that

$$|w(s,y)| \;\le\; \frac{M}{2|s|}\int_0^R |v(s,\xi)|\,d\xi \;\le\; \frac{C}{|s|}, \quad s \ne 0, \quad y \ge 0, \tag{2.22}$$

which together with (2.19) imply the required estimate (2.10).

□

Lemma 2.2. The integral equation (2.6) has a solution of the form

$$e(s,y) \;=\; e^{-|s|y}[1+w(s,y)], \tag{2.23}$$

where the function $w(s,y)$ has derivatives which satisfies the following estimates:

$$\frac{d^k w}{dy^k} \;=\; O(1)(1+|s|)^{k-2}, \quad y \ge 0, \quad s \in \mathbb{R}, \quad k=1,2. \tag{2.24}$$

Proof. Differentiate (2.21).

$$w'(s,y) \;=\; -\int\limits_y^R e^{-2|s|(\xi-y)} q(\xi)v(s,\xi)\,d\xi.$$

Then by applying (2.10), we have

$$|w'(s,y)| \;\le\; C\int\limits_y^R e^{-2|s|(\xi-y)}\,d\xi \;=\; C\frac{1-e^{-2|s|(R-y)}}{2|s|} \;\le\; \frac{C}{|s|}.$$

Further,

$$w''(s,y) \;=\; -2|s|\int\limits_y^R e^{-2|s|(\xi-y)} q(\xi)v(s,\xi)\,d\xi \;+\; q(y)v(s,y)$$

Analogously,

$$|w''(s,y)| \;\le\; C|s|\int\limits_y^R e^{-2|s|(\xi-y)}\,d\xi \;+\; C \;=\; O(1).$$

□

3. Estimation of the solution of the main equation

Assume that the coefficients $q(y)$ and $k(y)$ are connected with the relation (2.4*). Set

$$V(s,y) \;=\; \frac{e(s,y)}{\sqrt{k(y)}}, \tag{3.1}$$

where the function $e(s,y)$ is defined as (2.23).

The function $e(s,y)$ satisfies the equation (2.4) and the function $V(s,y)$ is solution of the equation (2.2) (see. formula (2.3)). Therefore, the following representation is a true

$$V(s,y) \;=\; \frac{e^{-|s|y}}{\sqrt{k(y)}}[1+w(s,y)] \tag{3.2}$$

and the function $w(s,y)$ and its derivative satisfy the estimates (2.19) and (2.22).

The function $V(y)=V(s,y)$ is a solution to the differential equation (2.2), which satisfies the following equality:

$$V'(y) \;=\; \frac{k(0)}{k(y)}V'(0)+\frac{s^2}{k(y)}\int\limits_0^y k(t)V(t)\,dt. \tag{3.3}$$

Note that, for $s\neq 0$ the initial data $V(s,0)$ and $V'(s,0)$ are not identically zero at one time, which follows from the uniqueness of the solution to the Cauchy problem for the differential equation (2.2). Moreover, it follows directly from the equality (3.3) that each of the initial data taken by the solution $V(s,y)$ also cannot be equal to zero. Namely, the following statement is true.

Lemma 3.1. Let $s \neq 0$. Then the function $V(s,y)$ satisfies the following equalities:

$$V(s,0) \neq 0, \quad V'(s,0) \neq 0. \tag{3.4}$$

Proof. **1**. First of all, we proove that $V(s,0) \neq 0$. Assume that the $V(s,0) = 0$. Consider two possible case. If $V'(s,0) > 0$, then function $V(y) = V(s,y)$ increases at zero and strictly positive in some interval on the right side of zero. Let $(0, y^*) -$ be the maximum interval such that

$$V(y) > 0, \quad 0 < y < y^*.$$

We show that $y^* = +\infty$. If it is not, then the following relations have to satisfy

$$V(y^*) = 0, \quad V'(y^*) \leq 0.$$

According to (3.3), we get the following estimate

$$V'(y^*) \; > \; \frac{k(0)}{k(y^*)} V'(0) \; > \; 0.$$

Obtained contradiction shows that the $V'(y) > 0$ on the halfline $y > 0$. Hence, the function $V(y)$ strictly increases on the halfline.

Analogously, we consider the second case, which $V'(0) < 0$. Thus, the assumption that $V(s,0) \neq 0$ is incorrect.

2. Then we will prove that $V'(s,0) \neq 0$. If we assume that $V'(s,0) = 0$ then the equality (3.3) takes the form

$$V'(y) \; = \; \frac{s^2}{k(y)} \int_0^y k(t) V(t)\, dt. \tag{3.5}$$

Again we consider two possible case: $V(s,0) > 0$ and $V(s,0) < 0$. If $V(0) > 0$, then according to (3.5) we get $V'(y) > 0$ in some interval $(0, y^*)$. Hence, the function $V(y)$ increases and positive on the entire halfline. Therefore, the case of $V(0) > 0$ could not be. Analogously, we consider second case, which $V(0) < 0$. In this case, the initial assumption that $V'(s,0) = 0$ is incorrect.

□

Lemma 3.1 allows to introduce the following two functions which have a main role to construct kernel of Poisson type:

$$V_1(s,y) \; = \; \frac{V(s,y)}{V(s,0)}, \quad V_2(s,y) \; = \; \frac{V(s,y)}{V'(s,0)}. \tag{3.6}$$

Each functions in (3.6) are solutions of the equation (2.2) and the following boundary conditions are fulfilled:

$$V_1(s,0) = 1, \quad \frac{\partial V_2}{\partial y}(s,0) = 1, \quad s \neq 0. \tag{3.7}$$

To construct Poisson operator of second type for small $|s|$ as a main solution of the differential equation (2.4) we introduce a function $v = e_2(s,y)$ which satisfies the following integral equation for $s \in \mathbb{R}$, $s \neq 0$

$$e_2(s,y) \; = \; \frac{\sinh sy}{s} \; + \; \int_0^y \frac{\sinh s(y-t)}{s} q(t) e_2(s,t)\, dt. \tag{3.8}$$

Lemma 3.2. The integral equation (3.8) has a solution, for $s \neq 0$, which is a solution of the differenatial equation (2.4) and satisfies the following estimates

$$|e_2(s,y)| \;\le\; Cye^{|s|y}, \quad y>0, \quad s\in\mathbb{R}, \tag{3.9}$$

$$\frac{d^k e_2}{dy^k}(s,y) \;=\; O(1)(1+|s|)^{k-1}e^{|s|y}, \quad y>0, \quad s\in\mathbb{R}, \quad k=1,2. \tag{3.10}$$

Proof. First of all, we assume that the $y \le R$. Set

$$v_0(y) \;=\; \frac{\sinh sy}{s}, \quad v_{n+1}(y) \;=\; \int\limits_0^y \frac{\sinh s(y-t)}{s} q(t)v_n(t)\,dt, n=0,1,2,\dots. \tag{3.11}$$

Then the following inequalities hold

$$|v_n(y)| \;\le\; M^n R^n \frac{y^{n+1}}{(n+1)!}e^{|s|y}, \quad n=0,1,2,\dots. \tag{3.12}$$

Further,

$$|v_{n+1}(y)| \;\le\; \int\limits_0^y \frac{\sinh s(y-t)}{s}|q(t)|\cdot|v_n(t)|\,dt \;\le$$

$$\le\; M\int\limits_0^y (y-t)e^{|s|(y-t)}M^nR^n\frac{t^{n+1}}{(n+1)!}e^{|s|t}\,dt \;\le\; M^{n+1}R^{n+1}\frac{y^{n+2}}{(n+2)!}e^{|s|y}.$$

Then the solution takes the form

$$v(s,y) \;=\; \sum_{n=0}^{\infty} v_n(y).$$

Estimate of the solution will be in the following form

$$|v(s,y)| \;\le\; \sum_{n=0}^{\infty} M^nR^n\frac{y^{n+1}}{(n+1)!}e^{|s|y} \;\le C(R)ye^{|s|y}, \quad 0\le y\le R, \tag{3.13}$$

where

$$C(R) \;=\; \sum_{n=0}^{\infty} M^nR^n\frac{R^n}{(n+1)!}.$$

If $y \ge R$, then the integral equation (3.8) takes the form

$$e_2(s,y) \;=\; \frac{\sinh sy}{s} \;+\; \int\limits_0^R \frac{\sinh s(y-t)}{s}q(t)e_2(s,t)\,dt. \tag{3.14}$$

According to (3.13), we obtain

$$|e_2(s,y)| \;\le\; ye^{|s|y} \;+\; C(R)M\int\limits_0^R (y-t)e^{|s|(y-t)}e^{|s|t}\,dt \;\le\; Cye^{|s|y}. \tag{3.15}$$

Thus , the inequality (3.9) has been proved. The validity of the estimate (3.10) is proved by differentiating equation (3.8). We have

$$e_2'(s,y) \;=\; \cosh sy \;+\; \int_0^y \cosh s(y-t)q(t)e_2(s,t)\,dt. \tag{3.16}$$

According to (3.15) and (3.16), we obtain

$$|e_2'(s,y)| \;\leq\; e^{|s|y} \;+\; CM\int_0^R e^{|s|(y-t)}te^{|s|t}\,dt \;\leq\; C(R)e^{|s|y}. \tag{3.17}$$

Further, for $y<R$,

$$e_2''(s,y) \;=\; s\sinh sy \;+\; s\int_0^y \sinh s(y-t)q(t)e_2(s,t)\,dt \;+\; q(y)e_2(s,y).$$

Hence

$$\begin{aligned}|e_2''(s,y)| \;&\leq\; \frac{|s|}{2}e^{|s|y} \;+\; M\frac{|s|}{2}\int_0^y e^{|s|(y-t)}te^{|s|t}\,dt \;+\; Mye^{|s|y} \;\leq\\ &\leq\; C(R)(1+|s|)e^{|s|y}.\end{aligned} \tag{3.18}$$

Finally, according to (3.16) for $y>R$, we obtain

$$e_2'(s,y) \;=\; \cosh sy \;+\; \int_0^R \cosh s(y-t)q(t)e_2(s,t)\,dt.$$

Hence,

$$e_2''(s,y) \;=\; s\sinh sy \;+\; s\int_0^R \sinh s(y-t)q(t)e_2(s,t)\,dt.$$

Therefore, as estimating the first two terms of (3.18).

□

4. Proof of the Theorem 1.1

Set

$$U_2(s,y) \;=\; \sqrt{\frac{k(0)}{k(y)}}\,e_2(s,y). \tag{4.1}$$

Let $\omega\in C_0^\infty(\mathbb{R})$, $s\omega'(s)\leq 0$ for $s\in\mathbb{R}$,

$$\omega(s) \;=\; \begin{cases}1, & |s|\leq A,\\ 0, & |s|\geq A+1.\end{cases}$$

Set

$$W_2(s,y) \;=\; \omega(s)U_2(s,y)+\big(1-\omega(s)\big)V_2(s,y), \tag{4.2}$$

where V_2 is defined by equality (3.6).

Lemma 4.1. The function $W_2(s,y)$ is infinitely differentiable by $s \in \mathbb{R}$. For each $y \geq 0$, the function increases exponentially by s at infinity and two times continuously differentiable by $y \geq 0$. And the following relations

$$|W_2(s,y)| \ \leq \ B(y)e^{-|s|y}, \quad \lim_{y\to 0} \frac{\partial W_2(s,y)}{\partial y} \ = \ 1 \tag{4.3}$$

are valid

Proof. If $|s| \geq A+1$, then

$$|W_2(s,y)| = |V_2(s,y)| \leq B(y)e^{-|s|y}$$

also according to (3.7)

$$\lim_{y\to 0} \frac{\partial W_2(s,y)}{\partial y} = \lim_{y\to 0} \frac{\partial V_2(s,y)}{\partial y} \ = \ 1.$$

If $|s| \leq A+1$, then

$$|W_2(s,y)| \ \leq \ Ce^{|s|y} \ \leq Ce^{(2A+2-|s|)y} \ = \ B(y)e^{-|s|y}, \quad B(y) \ = \ Ce^{(2A+2)y}$$

and

$$\lim_{y\to 0} \frac{\partial W_2(s,y)}{\partial y} = \lim_{y\to 0} \frac{\partial U_2(s,y)}{\partial y} \ = \ 1.$$

□

Set

$$\widetilde{W}_2(x,y) \ = \ \frac{1}{2\pi}\int_{-\infty}^{\infty} W_2(s,y)e^{isx}\,ds.$$

Lemma 4.2. The function $\widetilde{W}_2(x,y)$ is infinitely differentiable by $x \in \mathbb{R}$ and two times differentiable by $y \in \mathbb{R}_+$. For any function $\psi \in L_2(\mathbb{R})$, the following function

$$v(x,y) \ = \ Q\psi(x,y) \ = \ \int_{-\infty}^{\infty} \widetilde{W}_2(x-x',y)\psi(x')\,dx'$$

is a solution of the differential equation (1.1) in the halfspace K, satisfying condition (1.5) in the sense (1.6).

Proof. We have

$$v(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_2(s,y)e^{is(x-x')}\psi(x')\,dx'ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} W_2(s,y)\widehat{\psi}(s)e^{isx}ds.$$

Hence,

$$\widehat{v}(s,y) \ = \ W_2(s,y)\widehat{\psi}(s).$$

Then

$$\int_{-\infty}^{\infty}\left|\frac{\partial v(x,y)}{\partial y} - \psi(x)\right|^2 dx \ = \ \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\partial W_2(s,y)}{\partial y} - 1\right|^2 |\widehat{\psi}(s)|^2 ds$$

and according to (4.3)

$$|\frac{\partial W_2(s,y)}{\partial y} - 1| \;\le\; C, \quad s \in \mathbb{R}, \quad 0 \le y \le y_0.$$

Then according to Lebesgue theorem lemma is proved.

□

5. Solvability conditions of the Cauchy problem

Consider the following Cauchy problem:

$$\operatorname{div}\left[k(y)\operatorname{grad} u(x,y)\right] \;=\; 0, \tag{5.1}$$

$$u(x,0) \;=\; \phi(x), \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} \;=\; \psi(x). \tag{5.2}$$

We will say that the function $u(x,y)$ is a solution of the Cauchy problem (5.1)-(5.2) in the stripe

$$\Omega_h \;=\; \{(x,y) \in \mathbb{R}^2 \;:\; x \in \mathbb{R}, \quad 0 \le y \le h\},$$

if this function satisfies the equation (5.1) in Ω_h and boundary condition (5.2) in the sense of (1.6) and

$$\lim_{y \to 0+0} \int\limits_{-\infty}^{\infty} |u(x,y) - \phi(x)|^2 \, dx \;=\; 0. \tag{5.3}$$

We prove the following theorem.

Theorem 5.1. Then a necessary and sufficient condition for the existence of a solution of the Cauchy problem (5.1)-(5.2) in the stripe Ω_h is that the function

$$f(x) \;=\; \phi(x) - (Q\psi)(x,0) \tag{5.4}$$

belongs to class A_σ, where $\sigma = h$.

Where class A_σ was introduced in (see [2], Theorem 3).

Proof. **Necessity.** Let the function $u(x,y)$ is the solution of the Cauchy problem (5.1)-(5.2). Then the function

$$v(x,y) \;=\; u(x,y) - Q\psi(x,y) \tag{5.5}$$

is the solution of the Cauchy problem for the equation (5.1) with the following boundary conditions

$$v(x,0) = f(x), \quad \left.\frac{\partial v}{\partial y}\right|_{y=0} = 0. \tag{5.6}$$

Further, the function $f(x)$ belongs to the class A_σ with $\sigma = h$ (see [2], Theorem 2)

Sufficiency. Let the function (5.4) belongs to the class A_σ with $\sigma = h$. Then the solution of the problem (5.1)-(5.2) exists. It is clear that the function $u(x,y) = v(x,y) + Q\psi(x,y)$ is a solution of the problem (5.1)-(5.2).

□

The author thanks academician S. A. Alimov for his attention to the work and valuable comments.

References

[1] Alimov Sh. A., Qudaybergenov A. K.; Determination of temperature at the outer boundary of a body. Journal of Mathematical Sciences. Vol. 274.–No. 2., –2023.–P.159-171.

[2] Alimov Sh. A., Qudaybergenov A. K.; On the determining of the stationar temperature in the unbounded stripe. Differential equations. Vol. 60. –No. 8. –2024. –P.1049-1062.

[3] Alimov Sh. A., Qudaybergenov A. K.; On the solvability of the Cauchy Problem for Laplace equation. Uzbek Mathematical Journal. 2022. –Volume 66., –Issue 3. –P.5-14.

[4] Hadamard J.; Lectures on Cauchy's problem in linear partial differential equations. New Haven., Yale University Press. London., Humphrey Milford. Oxford., University Press. 1923.– VIII u. –316 S.

[5] Lavrentyev M. M.; On the Cauchy problem for Laplace equation. Izv. Akad. Nauk SSSR. Ser. Mat., 1956.–P.819-842.

[6] KabanikhinS. I.; Inverse and Ill-posed Problems: Theory and Applications. Walter de Gruyter GmbH & Co. KG. Berlin-Boston., 2012.–Ser. 55.

[7] Naimark M. A.; Linear Differential Operators (Russian). 2nd Edition. Nauka., Moscow., 1969. –P.1-528.

[8] Tikhonov A. N., Goncharsky A. V., Stepanov V. V., Yagola A. G.; Numerical Methods for the Solution of Ill-Posed Problems. Kluwer Academic Publishers. 1995.

[9] Tikhonov A. N., Samarsky A.A.; Equations of Mathematical Physics. Nauka., Moscow (Russian)., 1966.–P.1–735.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.257-265
DOI: 10.29229/uzmj.2025-2-24

Nonlineroverdetermined systems of integral equation with super-singular kernels in cylindrical domains

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this work, we investigate one class of nonlinear overdetermined system integral equation with boundary super-singular kernels. For this type overdetermined system integral equation in depend of sign parameters present in kernels, integral representation manifold solution by one arbitrary constant obtained. Investigate the property of the obtained integral representation. In this basis formulation Caushy type boundary value problems and its solution found in explicit form.

Keywords: Volterra type singular integral equation; integral representation; singular and super-singular lines; non linear integral equation.
MSC(2020): 44A15, 35C10.

1. Introduction

The problem of studying an overdetermined linear system of Volterra integral equations with singular and super-singular kernels has been addressed in works [6], [11], [12]. The study of overdetermined linear systems of differential equations has been considered in [16], [5], [8].

The investigation of degenerate differential equations and the related integral equations, in particular, has been covered in works [14], [15], [3], [4], [2], [13], [18], [10], [7].

In this work, for system (2.1), where the right-hand sides are expanded into uniformly convergent functional series depending on the signs of the parameters λ, $\mu_j (j = 1, 2)$, a representation of the solution manifold of the general solution of the inhomogeneous system is obtained in the class of functions representable as uniformly convergent functional series satisfying the condition $\varphi(t, z) = \varphi(t, r)$. It is assumed that the functions present in the right-hand side of the system also satisfy the properties $f(t, z) = f(t, r)$, $g(t, z) = g(t, r)$.

Based on results previously obtained in [9] and depending on the signs of the parameters $\lambda, \mu_j (j = 1, 2)$, a representation of the solution manifold for the homogeneous system of integral equations (2.1) is obtained. By adding a particular solution of the inhomogeneous system in the form of functional series to the general solution of the homogeneous system, we obtain the general solution of the inhomogeneous system. The properties of the obtained solutions are studied. On this basis, the solution to Cauchy-type problems is formulated and found.

In [7], for the linear system (2.1) (when $\mu_2 = 0$), a representation of the solution manifold was obtained through an arbitrary analytic function of a complex variable defined on the lower base of the cylinder. Formulas for the inversion of the integral representation were derived. Based on this, the formulation of boundary value problems of the Schwartz type was clarified, and their reduction to the Schwartz boundary value problem in the theory of analytic functions was studied [1], [17]. Cases where system (2.1) has a unique solution were also examined.

2. Representation of the Solution Manifold

Let Ω denote the cylindrical domain $\Omega = \{(t, z) : a < t < b, |z| < R\}$, the lateral surface of this cylinder is denoted by $S = \{a < t < b, |z| = R\}$, and its lower base is denoted by $D = \{t = a, |z| < R\}$.

In the domain Ω, we consider the overdetermined system of integral equations:

$$\begin{cases} \varphi(t,z) + \lambda \int_a^t \dfrac{\varphi(\tau,z)}{(\tau-a)^\alpha} d\tau = f(t,z) \\ \varphi(t,z) + \frac{1}{\pi} \iint\limits_D \frac{\exp(i\theta)\left[\mu_1\varphi(t,\zeta)+\mu_2\overline{\varphi(t,\zeta)}\right]}{(R-\rho)^\beta(\zeta-z)} d\xi\, d\eta = g(t,z) \end{cases} \tag{2.1}$$

where λ, $\mu_j (j = 1, 2)$, are given real constants, $f(t, z)$, $g(t, z)$ are given functions in the domain Ω, $\varphi(t, z)$ is the unknown function, $\Theta = arg\zeta$, $\zeta = \xi + i\eta$, $z = x + iy$, $\alpha = const > 1$, $\beta = const > 1$.

We will seek the solution to the system of integral equations (2.1) in the class of functions $\varphi(t, z) \in C(\overline{\Omega})$, with the condition $\varphi(a, z) = 0$ and the following asymptotic behavior: $\varphi(t, z) = 0[(t - a)^{\delta_1}]$, $\delta_1 > \alpha - 1$ as $r \to a$,

and

$\varphi(t, Rexp[i\Theta]) = 0$ with the asymptotic behavior

$\varphi(t, z) = 0[(R - r)^{\delta_2}], \delta_2 > \beta - 1$ as $r \to R$.

Definition 2.1. We will say that in system (2.1), the first equation is the main equation if we first find the solution to the first equation and then subordinate it to the second equation.

1. Let the functions $f(t, z)$ and $g(t, z)$ in system (2.1) be expanded into the following uniformly convergent functional series:

$$f(t,z) = \sum_{k=0}^{\infty} \exp\left[-(k+\delta)\omega_R^\beta(r)\right] f_k(t) \tag{2.2}$$

$$g(t,z) = \sum_{k=0}^{\infty} \exp\left[-(k+\delta)\omega_R^\beta(r)\right] g_k(t) \tag{2.3}$$

where $\delta = const > 0, f_k(t), g_k(t)$are known functions, an $\omega_R^\beta(r) = \left[(\beta-1)(R-r)^{\beta-1}\right]^{-1}$

A particular solution of the inhomogeneous system (2.1) will be sought in the form:

$$\varphi(t,z) = \sum_{k=0}^{\infty} \exp\left[-(k+\delta)\omega_R^\beta(r)\right] \varphi_k(t) \tag{2.4}$$

where $\varphi_k(t)$ are unknown functions.

Let the first equation in system (2.1) be the main equation. Substituting the value of $\varphi(t, z)$ from equation (2.4) and the value of $f(t, z)$ from equation (2.2) into the first equation of system (2.1), we get:

$$\sum_{k=0}^{\infty} \exp\left[-(k+\delta)\omega_R^\beta(r)\right] \left[\varphi_k(t) + \lambda \int_a^t \frac{\varphi_k(\tau)}{(\tau-a)^\alpha} d\tau\right] = \sum_{k=0}^{\infty} \exp\left[-(k+\delta)\omega_R^\beta(r)\right] f_k(t).$$

In this equation, by equating the coefficients of $exp[-(k+\delta)\omega_R^\beta(r)]$ on the left and right-hand sides, we obtain the solution to the following infinite separated system of Volterra-type integral equations with super-singular kernels:

$$\varphi_k(t) + \lambda \int_a^t \frac{\varphi_k(\tau)}{(\tau-a)^\alpha} d\tau = f_k(t), \quad k = 0, 1, 2, \ldots \tag{2.5}$$

According to [9], when $\lambda < 0$, the solution to system (2.5) is given by the following formulas:

$$\varphi_k(t) = \exp\left[\lambda\omega_a^\alpha(t)\right] C_k + f_k(t) - \lambda \int_a^t \exp\left[\lambda\left(\omega_a^\alpha(t) - \omega_a^\alpha(\tau)\right)\right] \cdot$$

$$\cdot \frac{f_k(\tau)}{(\tau-a)^\alpha} d\tau \equiv T_k[C_k, f_k(t)], \quad k = 0, 1, 2, 3, \ldots \tag{2.6}$$

where $\omega_a^\alpha(t) = [(\alpha-1)(t-a)^{\alpha-1}]^{-1}$.

The solution of the form (2.6) exists if $\lambda < 0$, $f_k(a) = 0$, with the following asymptotic behavior:

$$f_k(t) = 0\left[\exp\left(\lambda\omega_a^\alpha(t)\right)(t-a)^{\delta_3}\right], \quad \delta_3 > \alpha - 1ast \to a. \tag{2.7}$$

Substituting the found solution of the separated system of integral equations (2.5) from formula (2.6) into formula (2.4), we obtain the solution to the first equation of system (2.1):

$$\varphi(t,z) = \sum_{k=0}^{\infty} \exp\left[-(k+\delta)\omega_R^\beta(r)\right] T_k[C_k, f_k(t)], \tag{2.8}$$

where $C_k (k = 0, 1, 2, 3, \dots)$-arbitrary constants.

The solution of the form (2.8) exists if the series of the form

$$\sum_{k=0}^{\infty} \exp\left[-(k+\delta)\omega_R^\beta(r)\right] C_k$$

converges.

A direct verification shows that if the series of the form (2.2) converges, then the series of the form (2.8) also converges.

Now, we substitute the solution of the form (2.8) into the second equation of system (2.1).

$$\sum_{k=0}^{\infty}\{T_k[C_k, f_k(t)] \exp\left[-(k+\delta)\omega_R^\beta(r)\right] +$$

$$+\frac{1}{\pi}\iint\limits_D \frac{\exp[i\theta]\left[\exp\left[-(k+\delta)\omega_R^\beta(\rho)\right]\left(\mu_1 T_k[C_k, f_k(t)] + \mu_2 T_k[\overline{C_k}, \overline{f_k(t)}]\right)\right]}{(R-\rho)^\beta(\zeta - z)} d\xi\, d\eta\} =$$

$$= \sum_{k=0}^{\infty} \exp\left[-(k+\delta)\omega_R^\beta(r)\right] g_k(t), \tag{2.9}$$

Noting that

$$\mu_1 T_k\left[C_k, f_k(t)\right] + \mu_2 T_k\overline{[C_k, f_k(t)]} =$$

$$= \exp\left[\lambda\omega_a^\alpha(t)\right](\mu_1+\mu_2)C_k + \left(\mu_1 f_k(t) + \mu_2\overline{f_k(t)}\right) -$$

$$-\lambda\int_a^t \frac{\exp\left[\lambda\left(\omega_a^\alpha(t) - \omega_a^\alpha(\tau)\right)\right]\left(\mu_1 f_k(\tau) + \mu_2\overline{f_k(\tau)}\right)}{(\tau-a)^\alpha} d\tau,$$

according to [9], we have:

$$\frac{1}{\pi}\iint_D \frac{\exp(i\theta)\left[\exp\left(-(k+\delta)\omega_R^\beta(\rho)\right)\left(\mu_1 T_k\left[C_k, f_k(t)\right] + \mu_2 T_k\overline{[C_k, f_k(t)]}\right)\right]}{(R-\rho)^\beta(\zeta-z)} d\xi d\eta =$$

$$= \left(\mu_1 T_k\left[C_k, f_k(t)\right] + \mu_2 T_k\overline{[C_k, f_k(t)]}\right)\frac{1}{\pi}\iint_D \frac{\exp[i\theta]\exp\left[-(k+\delta)\omega_R^\beta(\rho)\right]}{(R-\rho)^\beta(\zeta-z)} d\xi d\eta =$$

$$= \left(\mu_1\{T_k\left[C_k, f_k(t)\right] + \mu_2 T_k\overline{[C_k, f_k(t)]}\right)\left[-\frac{\exp\left[-(k+\delta)\omega_R^\beta(r)\right]}{k+\delta}\right].$$

Then, equation (2.9) takes the following form:

$$\sum_{k=0}^{\infty}\left\{T_k\left[C_k, f_k(t)\right]\exp\left[-(k+\delta)\omega_R^\beta(r)\right] + \mu_1 T_k\left[C_k, f_k(t)\right] + \mu_2\overline{T_k\left[C_k, f_k(t)\right]}\right\}\times$$

$$\times(-\frac{\exp\left[-(k+\delta)\omega_R^\beta(r)\right]}{k+\delta}) = \sum_{k=0}^{\infty}\exp\left[-(k+\delta)\omega_R^\beta(r)\right]g_k(t).$$

Now, by equating the coefficients of the same powers of the function $\exp\left[-(k+\delta)\omega_R^\beta(r)\right]g_k(t)$ for $k=0,1,2,3,\ldots,$ on the left and right-hand sides, we obtain the following equalities:

$$T_k\left[C_k, f_k(t)\right]\left(1-\frac{\mu_1}{k+\delta}\right) - \frac{\mu_2}{k+\delta}\overline{T_k\left[C_k, f_k(t)\right]} = g_k(t), \quad k=0,1,2,3,\ldots \tag{2.10}$$

or

$$\exp\left[\lambda\omega_a^\alpha(t)\right]C_k\left(\frac{k+\delta-(\mu_1+\mu_2)}{k+\delta}\right) + \left(\frac{k+\delta-\mu_1}{k+\delta}\right)f_k(t) - \frac{\mu_2}{k+\delta}\overline{f_k(t)} -$$

$$-\lambda\int_a^t \frac{\exp\left[\lambda\left(\omega_a^\alpha(t)-\omega_a^\alpha(\tau)\right)\right]\left((k+\delta-\mu_1)f_k(\tau)-\mu_2\overline{f_k(\tau)}\right)}{(k+\delta)(\tau-a)^\alpha}d\tau = g_k(t) \tag{2.11}$$

Multiplying both sides of the equalities (2.11) by $\exp[-\lambda\omega_a^\alpha(t)]$, we will get

$$C_k\left[\frac{k+\delta-(\mu_1+\mu_2)}{k+\delta}\right] + \exp\left[-\lambda\omega_a^\alpha(t)\right]\left[\frac{k+\delta-\mu_1}{k+\delta}f_k(t) - \frac{\mu_2}{k+\delta}\overline{f_k(t)}\right] -$$

$$-\int_a^t \exp\left[-\lambda\omega_a^\alpha(\tau)\right]\frac{\left((k+\delta-\mu_1)f_k(\tau)-\mu_2\overline{f_k(\tau)}\right)}{(k+\delta)(\tau-a)^\alpha}d\tau = \exp\left[-\lambda\omega_a^\alpha(t)\right]g_k(t). \tag{2.12}$$

Differentiating both sides of this equality with respect to the variable t, we obtain:

$$\frac{d}{dt}[\exp[-\lambda\omega_a^\alpha(t)]\{[\frac{k+\delta-\mu_1}{k+\delta}]f_k(t) - \frac{\mu_2}{k+\delta}\overline{f_k}(t) - g_k(t)\}] =$$

$$= \lambda\exp[-\lambda\omega_a^\alpha(t)]\frac{(k+\delta-\mu_1)f_k(t)-\mu_2\overline{f_k}(t)}{(k+\delta)(t-a)^\alpha}. \tag{2.13}$$

Taking into account equality (2.13), we rewrite expression (2.12) in the following form:

$$\exp[-\lambda\omega_a^\alpha(t)]\{[\frac{k+\delta-\mu_1}{k+\delta}]f_k(t) - \frac{\mu_2}{k+\delta}\overline{f_k}(t)\} -$$

$$-\int_a^t \frac{d}{d\tau}[\exp(-\lambda\omega_a^\alpha(\tau))\{[\frac{k+\delta-\mu_1}{k+\delta}]f_k(\tau) - \frac{\mu_2}{k+\delta}\overline{f_k(\tau)} - g_k(\tau)\}]d\tau =$$

$$= \exp[-\lambda\omega_a^\alpha(t)]g_k(t)$$

or

$$[C_k(\frac{k+\delta-(\mu_1+\mu_2)}{k+\delta}) - [\exp[-\lambda\omega_a^\alpha(t)]g_k(t)]]_{\tau=a} = 0.$$

From here,

$$C_k = \frac{k+\delta}{k+\delta-(\mu_1+\mu_2)}\left[\exp\left[-\lambda\omega_a^\alpha(t)\right]g_k(t)\right]_{t=a}, \quad k=0,1,2,3,\ldots$$

Substituting the found value of C_k into formula (2.8), we obtain the solution to system (2.1) in the form:

$$\varphi(t,z) = \sum_{k=0}^{\infty}\exp[-(k+\delta)\omega_R^\beta(r)]T_k\times$$

$$\times[\frac{k+\delta}{k+\delta-(\mu_1+\mu_2)}[\exp[-\lambda\omega_a^\alpha(t)]g_k(t)t]_{t=a}, f_k(t)] \equiv N_1[f(t,z), g(t,z)] \tag{2.14}$$

Thus, the following statement is proved:

Theorem 2.2. *Let in the system of integral equations (2.1), $\lambda < 0$, $\mu_j < 0$, $(j = 1, 2)$, the functions $f(t, z) = f(t, r)$, $g(t, z) = g(t, r)$ are expanded into uniformly convergent functional series of the forms (2.2), (2.3), where $f_k(a) = 0$ with asymptotic behavior (2.7). There exist limits of the forms (2.13), and the functions $f_k(t)$, $g_k(t)$ are related to each other by the equalities (2.10) or (2.11). The functions $g_k(t)$, $f_k(t)$ are such that the functional series of the form*

$$\sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_R^{\beta}(r)] \frac{k+\delta}{k+\delta-(\mu_1+\mu_2)} [\exp[-\lambda\omega_a^{\alpha}(t)]g_k(t)]_{t=a}$$

converges. Then, any particular solution of the inhomogeneous overdetermined system of integral equations (2.1) in the class of functions representable in the form (2.4) is given by formula (2.14)

2. Suppose that in system (2.1), the functions $f(t, z) = f(t, r)$, $g(t, z) = g(t, r)$ are expanded into the following uniformly convergent functional series:

$$f(t, z) = \sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)]F_k(r), \tag{2.15}$$

$$g(t, z) = \sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)]G_k(r), \tag{2.16}$$

where $F_k(r), G_k(r)$-known functions of the lower base of the cylinder.

In this case, we will seek the solution to the overdetermined system of integral equations (2.1) in the form:

$$\varphi(t, z) = \sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)]\Phi_k(r) \tag{2.17}$$

where $\Phi_k(r)$-unknown functions.

Let the first equation in system (2.1) be the main equation. Then, by substituting the values of $f(t, z)$ and $\varphi(t, z)$ from formulas (2.15) and (2.17) into the first equation of system (2.1), we obtain:

$$\sum_{k=0}^{\infty} [\exp[-(k+\delta)\omega_a^{\alpha}(t)]\Phi_k(r) + \lambda \int_a^t \frac{\exp[-(k+\delta)\omega_a^{\alpha}(\tau)]\Phi_k(r)}{(\tau-a)^{\alpha}} d\tau] =$$

$$= \sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)]F_k(r). \tag{2.18}$$

Noting that

$$\int_a^t \frac{\exp[-(k+\delta)\omega_a^{\alpha}(\tau)]}{(\tau-a)^{\alpha}} f_k(t) d\tau = \frac{1}{k+\delta} \exp[-(k+\delta)\omega_a^{\alpha}(t)].$$

Substituting these calculated values of the integrals into equality (2.18), we have:

$$\sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)](1 + \frac{\lambda}{k+\delta})\Phi_k(r) =$$

$$= \sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)]F_k(r).$$

Equating the coefficients of the function $exp[-(k+\delta)\omega_a^{\alpha}(t)]$ for $k = 0, 1, 2, \dots$ on the left and right-hand sides, we obtain:

$$[1 + \frac{\lambda}{k+\delta}]\Phi_k(r) = F_k(r), k = 0, 1, 2, 3, \dots$$

From here

$$\Phi_k(r) = \frac{k+\delta}{k+\delta+\lambda} F_k(r), k = 0, 1, 2, 3, \ldots.$$

Substituting the found value of $\Phi_k(r)$ into equality (2.17), we obtain the particular solution to the first equation in the overdetermined system of integral equations (2.1) in the form:

$$\varphi(t,z) = \sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)](\frac{k+\delta}{k+\delta+\lambda})F_k(r) = N_2[f(t,z)] \qquad (2.19)$$

Substituting the found value of $\varphi(t,z)$ from equality (2.19) into the second equation of system (2.1), we obtain:

$$\sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)](\frac{k+\delta}{k+\delta+\lambda})[F_k(r)+$$
$$+\frac{1}{\pi}\iint_D \frac{\exp(i\theta)[\mu_1 F_k(\rho) + \mu_2 \overline{F_k(\rho)}]}{(R-\rho)^{\beta}(\zeta - z)} d\xi d\eta] =$$
$$= \sum_{k=0}^{\infty} \exp[-(k+\delta)\omega_a^{\alpha}(t)] G_k(r).$$

Equating the coefficients of the same powers of the function $exp[-(k+\delta)\omega_a^{\alpha}(t)]$ on the left and right-hand sides, we obtain:

$$(\frac{k+\delta}{k+\delta+\lambda})[F_k(r)+$$
$$+\frac{1}{\pi}\iint_D \frac{\exp(i\theta)[\mu_1 F_k(\rho) + \mu_2 \overline{F_k(\rho)}]}{(R-\rho)^{\beta}(\zeta - z)} d\xi d\eta] = G_k(r), \quad k = 0, 1, 2, 3, \ldots. \qquad (2.20)$$

The integrals on the left-hand side of expression (2.20) converge if $F_k(R) = 0$ with the following asymptotic behavior:

$$F_k(r) = o[(R-r)^{\delta}], \quad \delta > \beta - 1, \quad k = 0, 1, 2, 3, \ldots, \quad \text{when} \quad r \to R. \qquad (2.21)$$

Thus, the following is proven:

Theorem 2.3. *Let in the system of integral equations (2.1), $\lambda < 0$, the functions $f(t,z) = f(t,r)$, $g(t,z) = g(t,r)$ are expanded into uniformly convergent functional series of the forms (2.15) and (2.16), where $F_k(R) = 0$ with asymptotic behavior (2.21), and $\lambda < 0$. The functions $F_k(r)$ and $G_k(r)$ are related to each other by the formula (2.20). Then, any solution of the inhomogeneous overdetermined system (2.1) in the class of functions representable in the form (2.17) is given by formula (2.19).*

As follows from [9], the solution to the first homogeneous equation in system (2.1) as $\lambda < 0$ is given by the formula:

$$\varphi_0(t,z) = \exp[\lambda\omega_a^{\alpha}(t)]C(z)$$

where $C(z)$ the general solution of the second homogeneous integral equation in system (2.1).

As is well known [9], the solution to the second homogeneous integral equation in system (2.1) at $\mu_1 < 0, \mu_2 < 0$ is given by the following formula:

$$C(z) = \exp[(\mu_1 + \mu_2)\omega_R^{\beta}(r)]C_1,$$

where C_1 – an arbitrary real constant.

For $\mu_1 > 0, \mu_2 < 0, \mu_1 + \mu_2 = \mu_1 - \lfloor \mu_2 \rfloor < 0$ the solution to the second homogeneous integral equation in system (2.1) is given by the formula:

$$C(z) = \exp[(\mu_1 - \lfloor \mu_2 \rfloor)\omega_R^\beta(r)]C_2$$

where C_1 – an arbitrary real constant.

For $\mu_1 > 0$, $\mu_2 > 0$ where [2] $-\mu_1 + \mu_2 = \mu_2 - \mu_1 < 0$ when $\mu_2 < \mu_1$. The general solution of the second homogeneous integral equation in system (2.1) is given by the formula:

$$C(z) = iexp[(\mu_2 - \mu_1)\omega_R^\beta(r)]C_3,$$

where C_3 – an arbitrary real constant.

For $\mu_1 < 0$, $\mu_2 > 0$, $-\mu_1 + \mu_2 = \mu_2 + \lfloor \mu_2 \rfloor > 0$. In this case, the second homogeneous equation in system (2.1) has no solution other than the trivial one.

From the above reasoning, it follows that for $\mu_1 < 0$, $\mu_2 < 0$, the following statement is true:

Theorem 2.4. *Let in the overdetermined system of integral equations (2.1), the functions $f(t,z) = f(t,r)$, $g(t,z) = g(t,r)$, and the parameter λ satisfy all the conditions of Theorem 4.1. Additionally, let $\mu_1 < 0, \mu_2 < 0$. Then, any solution of the overdetermined system (2.1) in the class of functions, which is the particular solution of the inhomogeneous system representable in the form (2.4), is given by the formula:*

$$\varphi(t,z) = i\exp[\lambda\omega_a^\alpha(t)]\exp[(\mu_1 + \mu_2)\omega_R^\beta(r)]C_2 + N_2[f(t,z)], \tag{2.22}$$

where C_1 – an arbitrary real constant.

For $\mu_1 > 0, \mu_2 > 0$ for the over determined system (2.1), the following statement holds:

Theorem 2.5. *Let in the overdetermined system of integral equations (2.1), the functions $f(t,z) = f(t,r)$, $g(t,z) = g(t,r)$, and the parameter λ satisfy all the conditions of Theorem 2.3. Additionally, let $\mu_1 > 0$, and $\mu_2 > 0$, $\mu_2 < \mu_1$. Then, any solution of the overdetermined system (2.1) in the class of functions, which is the particular solution of the inhomogeneous system representable in the form (2.17), is given by the formula:*

$$\varphi(t,z) = iexp[\lambda\omega_a^\alpha(t)]exp[(\mu_2 - \mu_1)\omega_R^\beta(r)]C_2 + N_2[f(t,z)], \tag{2.23}$$

where C_2 – an arbitrary real constant.

At $\mu_1 < 0$, $\mu_2 > 0$ for the overdetermined system (2.1), the following statement holds:

Theorem 2.6. *Let in the overdetermined system of integral equations (2.1), the functions $f(t,z) = f(t,r)$, $g(t,z) = g(t,r)$ and the parameter λ, satisfy all the conditions of Theorem 2.5, except for the condition $\mu_1 > 0$, $\mu_2 > 0$. Suppose that $\mu_1 < 0$, $\mu_2 > 0$, $\mu_2 < \mu_1$. Then, any solution of the overdetermined system (2.1) in the class of functions, which is the particular solution of the inhomogeneous system representable in the form (2.17), is unique and is given by the formula:*

$$\varphi(t,z) = N_2[f(t,z)].$$

For $\mu_1 > 0$, $\mu_2 < 0$ the system (2.1), the following statement holds:

Theorem 2.7. *Let in the overdetermined system of integral equations (2.1), the functions $f(t,z) = f(t,r)$, $g(t,z) = g(t,r)$, and the parameter λ satisfy all the conditions of Theorem 2.4, except for the condition $\mu_1 < 0$, $\mu_2 < 0$. Suppose that $\mu_1 > 0, \mu_2 < 0$. Then, any solution of the overdetermined system (2.1) in the class of functions, which is the particular solution of the inhomogeneous system representable in the form (2.4), is unique and is given by formula (2.14).*

3. Properties of Solutions and Boundary Problems

Let all the conditions of Theorem 1 be satisfied. Then, multiplying both sides of the representation (2.22) by $exp[-\lambda\omega_a^\alpha(t)]$, and taking the limit as $t \to a$, we obtain:

$$[\exp[-\lambda\omega_a^\alpha(t)]\varphi(t,z)]_{t=a} = \exp[(\mu_1 + \mu_2)\omega_R^\beta(r)]C_1. \tag{3.1}$$

Multiplying both sides of the representation (3.1) by $exp[-(\mu_1+\mu_2)\omega_R^\beta(r)]$, after taking the limit as $r \to R$, we obtain:

$$\{\exp[-(\mu_1+\mu_2)\omega_R^\beta(r)][\exp[-\lambda\omega_a^\alpha(t)]\varphi(t,z)]_{t=a}\}_{r=R} = C_1. \tag{3.2}$$

The representation of the form (2.22) and its properties (3.2) provide the foundation for the overdetermined system of integral equations (2.1) to formulate and investigate the following Cauchy-type problem:

Problem K_1**.** It is required to find the solution of the overdetermined system integral equations (2.1) for $\mu_1 < 0$, $\mu_2 < 0$, and $\lambda < 0$, where the particular solution of the inhomogeneous system is representable in the form (2.4), subject to the boundary condition:

$$\{\exp[-(\mu_1+\mu_2)\omega_R^\beta(r)][\exp[-\lambda\omega_a^\alpha(t)]\varphi(t,z)]_{t=a}\}_{r=R} = E_1 \tag{3.3}$$

where E_1 a given real constant.

Solution to Problem K_1**.** Let in the system of integral equations (2.1), the parameters and right-hand sides satisfy all the conditions of Theorem 2.4. Then, using the representation of the solution manifold of the form (2.22), its properties (3.2), and the condition (3.3), we find $C_1 = E_1$. Substituting this value of C_1 into the representation (2.22), we obtain the solution to problem K_1:

$$\varphi(t,z) = \exp[\lambda\omega_a^\alpha(t)]\exp[(\mu_1+\mu_2)\omega_R^\beta(r)]E_1 + N_1[f(t,z),g(t,z)]. \tag{3.4}$$

Thus, it is proven:

Theorem 3.1. *Let in the overdetermined system of integral equations (2.1), the functions* $f(t,z) = f(t,r)$, $g(t,z) = g(t,r)$, *and the parameters* λ, μ_j $(j=1,2)$ *satisfy all the conditions of Theorem 2.4. Then, the problem* K_1 *has a unique solution, which is given by formula (3.4).*

Let all the conditions of Theorem 2.5 be satisfied. Then, the solution to the overdetermined system (2.1) is representable in the form (2.23). Multiplying both sides of equality (2.23) by $exp[-\lambda\omega_a^\alpha(t)]$ and taking the limit as $t \to a$, we obtain:

$$[\exp[-\lambda\omega_a^\alpha(t)]\varphi(t,z)]_{t=a} = i\exp[(\mu_2-\mu_1)\omega_R^\beta(r)]C_2.$$

Multiplying both sides by $exp[(\mu_1-\mu_2)\omega_R^\beta(r)]$, after taking the limit as $r \to R$, we obtain:

$$\{exp[(\mu_1-\mu_2)\omega_R^\beta(r)][\exp[-\lambda\omega_a^\alpha(t)]\varphi(t,z)]_{t=a}\}_{r=R} = iC_2. \tag{3.5}$$

The representation (2.23) and its properties (3.5) provide the foundation for the system of integral equations (2.1) to formulate and solve the following Cauchy-type problem in this case:

Problem K_2**.** It is required to find the solution of the overdetermined system (2.1) for $\mu_1 > 0$, $\mu_2 > 0$, $\lambda < 0$, where the particular solution of the inhomogeneous system is representable in the form (2.17), subject to the boundary conditions:

$$\{\exp[(\mu_1-\mu_2)\omega_R^\beta(r)][\exp[-\lambda\omega_a^\alpha(t)]\varphi(t,z)]_{t=a}\}_{r=R} = iE_2 \tag{3.6}$$

where E_2 a given real constant.

Solution to Problem K_2**.** Let in the overdetermined system of integral equations (2.1), the parameters and right-hand sides satisfy all the conditions of Theorem 2.5. Then, using the representation (2.23) and its properties (3.5), as well as condition (3.6), we find $C_2 = E_2$. Substituting this value of C_2 into representation (2.23), we obtain the solution to problem K_2:

$$\varphi(t,z) = i\exp[\lambda\omega_a^\alpha(t)]\exp[(\mu_2-\mu_1)\omega_R^\beta(r)]E_2 + N_2[f(t,z)]. \tag{3.7}$$

Thus, it is proven

Theorem 3.2. *Let in the overdetermined system of integral equations (2.1), the functions* $f(t,z) = f(t,r)$, $g(t,z) = g(t,r)$, *and the parameters* λ, $\mu_j(j=1,2)$ *satisfy all the conditions of Theorem 2.5. Then, the problem* K_2 *has a unique solution, which is given by formula (3.7).*

References

References

[1] Gakhov F. D.; Boundary Value Problems. Nauka, Moscow, –1977. –640 p.

[2] Juraev T. D.; Boundary Value Problems for Mixed and Mixed-Composite Type Equations. FAN, Tashkent, –1979. –240 p.

[3] Mikhaylov L. G.; Integral Equations with Homogeneous Kernels of Degree -1. Donish, Dushanbe, –1966. –47 p.

[4] Mikhaylov L. G.; A New Class of Singular Integral Equations and Its Application to Differential Equations with Singular Coefficients. Academy of Sciences of the Tajik SSR, Dushanbe, –1963.

[5] Rajabov N., Mohamed El-Sayed Abdel Aal; Overdetermined Second-Order Linear System with Singular and Supersingular Lines. LAP LAMBERT Academic Publishing, Germany, –2011. –234 p.

[6] Rajabov N.; Overdetermined Linear System of Integral Equations and Singular, Supersingular Volterra Integral Equations of the Third Kind with Logarithmic and Supersingular Kernels and Their Applications. TNU Press, Dushanbe, –2021. –317 p.

[7] Rajabov N.; Overdetermined System of Integral Equations in Cylindrical Domains, Where the Main Equation Is the Second Equation. DAN of Tajikistan, –2023. –66. –(78). –P. 408416.

[8] Rajabov N.; An Introduction to the Theory of Partial Differential Equations with Super-Singular Coefficients. Tehran University Press, Tehran, –1997. –228 p.

[9] Rajabov N.; Volterra Type Integral Equation with Boundary and Interior Fixed Singularity and Super-Singularity Kernels and Their Application. Irfon, Tajik State University, Dushanbe, –2010. –295 p.

[10] Rajabov N., Rajabova L. N.; Volterra Integral Equations with Two Boundary and One Internal Singular Points. Differential Equations, –2023. –50. –(9). –P. 12471259.

[11] Rajabov N., Rajabova L., Mamadaliev M. I.; A Class of Nonlinear Overdetermined Systems of Integral Equations with Singular and Supersingular Kernels in Cylindrical Domains. Journal of Mathematical Sciences, –2025. –287. –(1). –P. 1524.

[12] Rajabova L., Rajabov N.; Explicit Solution of an Overdetermined System of Volterra-Type Integral Equations with Strongly Singular Lines. Journal of Mathematical Sciences, –2025. –287. –(1). –P. 2534.

[13] Salahiddinov M. S., Mirsaburov M.; Non-local Problems for Mixed-Type Equations with Singular Coefficients. University Press, Tashkent, –2005. –224 p.

[14] Sabitov K. B.; Direct and Inverse Problems for Mixed Parabolic-Hyperbolic Type Equations. Nauka, –2016. –271 p.

[15] Sabitov K. B.; Functional, Differential, and Integral Equations. Vysshaya Shkola, Moscow, –2005. –670 p.

[16] Tasmambetov Z. N.; Construction of Normal and Normally Regular Solutions for Special Systems of Second-Order Partial Differential Equations. Aktobe, –2018. –263 p.

[17] Vekua I. N.; Generalized Analytical Functions. Fizmatgiz, –1959. –628 p.

[18] Zikirov O. S., Kholikov D. K.; Solvability of Non-local Problems for Loaded Pseudo-Parabolic Third-Order Equations. Abstracts of the International Scientific Conference "Non-Classical Equations of Mathematical Physics and Their Applications", Tashkent, –2024. –P. 159.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.266-272
DOI: 10.29229/uzmj.2025-2-25

Boundary value problem for a fractional-order diffusion equation and a degenerate hyperbolic equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. This paper investigates a boundary value problem for a fractional order diffusion equation and a degenerate hyperbolic equation with singular coefficients in an unbounded domain. The unique solvability of the problem considered is proven.

Keywords: fractional differential equation; boundary value problem; singular coefficient; unbounded domain; special functions.
MSC (2020): 35M10, 35M12, 35Q05, 35R11.

1. Introduction and Problem Statement

Fractional-order differential equations arise in various applications, such as classical mechanics (inverse problems), heat conduction (thermal flux dynamics), diffusion (electrochemical analysis of electrode surfaces), and the study of stochastic transport processes. Many problems involving fluid filtration in highly porous (fractal) media also necessitate the study of boundary value problems for partial differential equations of fractional order. Boundary value problems for fractional diffusion equations have been studied in [1, 2, 5, 8]. Fractal theory is used to describe the structure of disordered media, such as porous materials, and the processes occurring within them. The transport of an impurity in a homogeneous fluid flow is modeled using fractional-order differential equations [7]. Fractional diffusion is also the subject of the study in [6]. In [14], a certain family of generalized Riemann–Liouville derivative operators $D_{a+}^{\alpha,\beta}$ of orders α and β was studied. The applications of this operator are presented in [3]. In [4], the unique solvability of a boundary value problem with an integral conjugation condition in a mixed domain was proven for a mixed-type equation involving the Hilfer derivative.

Consider the fractional-order diffusion equation and the degenerate hyperbolic equation with singular coefficients

$$\begin{cases} u_{xx} - D_{0+,y}^{\gamma} u = 0, \ \ y > 0, 0 < \gamma < 1, \\ -(-y)^m u_{xx} + u_{yy} + \dfrac{\alpha_0}{(-y)^{1-\frac{m}{2}}} u_x + \dfrac{\beta_0}{y} u_y = 0, \ \ y < 0, \end{cases} \tag{1.1}$$

where

$D_{0+,y}^{\gamma}$ is the partial fractional Riemann–Liouville derivative of order $\gamma (0 < \gamma < 1)$ of function $u(x, y)$ for the second variable [13] in domain D, which is the union of the upper half-plane $D^+ = \{(x, y) : -\infty < x < \infty, y > 0\}$ and the domain D^-, located in the lower half-plane ($y < 0$) and bounded by characteristics $OC : x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$, $BC : x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1$, and segment [0,1] of the straight line $y = 0$. In equation (1.1) m, α_0, β_0 are some real numbers satisfying conditions $m > 0$, $|\alpha_0| < \frac{m+2}{2}$, $-\frac{m}{2} < \beta_0 < 1$.

The solution properties of equation (1.1) essentially depend on the coefficients α_0 and β_0 for the lowest terms of equation (1.1).

In plane $\alpha_0 O \beta_0$ of parameters α_0, β_0 the triangle $A_0B_0C_0$, is bounded by straight lines A_0C_0 : $\beta_0 + \alpha_0 = -m/2$, $B_0C_0 : \beta_0 - \alpha_0 = -m/2$, $A_0B_0 : \beta_0 = 1$, and depends on the location of point $P(\alpha_0, \beta_0)$ in this triangle, boundary value problems are formulated and investigated for equation (1.1).

Note that boundary value problems for equation (1.1), in both bounded and unbounded domains, where the point $P(\alpha_0, \beta_0)$ lies inside the triangle $A_0B_0C_0$, were studied in [11]. Boundary value problems for equation (1.1) in the case where $\alpha_0 = 0$, $\beta_0 = 0$ were studied in [1, 5], while the shifted problem for equation (1.1) with $-\frac{m}{2} < \beta_0 < 1$ in domain D was studied in [9].

In this work, a boundary value problem for the equation (1.1) is studied in the case of $P(\alpha_0, \beta_0) \in A_0C_0$.

Problem. Find in domain D function $u(x, y)$, which:

1) function $u(x, y)$ tends to zero as $(x^2 + y^2) \to \infty$;

2) satisfies the boundary conditions

$$y^{1-\gamma} u|_{y=0} = 0, \ (-\infty < x \leq 0, \ 1 \leq x < \infty), \tag{1.2}$$

$$u|_{OC} = \psi(x), \ x \in \left[0, \frac{1}{2}\right] \tag{1.3}$$

as well as the conjugation conditions

$$\lim_{y \to +0} y^{1-\gamma} u(x, y) = \lim_{y \to -0} u(x, y), \ x \in \bar{I} = [0, 1], \tag{1.4}$$

$$\lim_{y \to +0} y^{1-\gamma} (y^{1-\gamma} u(x, y))_y = \lim_{y \to -0} (-y)^{\beta_0} u(x, y)_y, \ x \in I = (0, 1).$$

where $\psi(x)$ is the given function, $\psi(0) = 0$.

We seek a solution $u(x, y)$ of the problem in the class of twice differentiable functions in domain D such that

$y^{1-\gamma} u \in C(\bar{D^+})$, $u(x, y) \in C(\bar{D^-})$, $y^{1-\gamma}(y^{1-\gamma} u)_y \in C(D^+ \cup \{(x, y) : 0 < x < 1, y = 0\})$, $u_{xx} \in C(D^+ \cup D^-)$, $u_{yy} \in C(D^-)$.

2. Main results

Let $P(\alpha_0, \beta_0) \in A_0C_0$.

It is known that in domain D^-, the solution of the modified Cauchy problem with initial data

$u(x, 0) = \tau(x), x \in \bar{I}$ $\lim\limits_{y \to -0} (-y)^{\beta_0} u(x, y)_y = \nu(x)$, $x \in I$, in the case of $P(\alpha_0, \beta_0) \in A_0C_0$, has the form [10, 12]

$$\begin{gathered} u(x, y) = \tau\left(x + \tfrac{2}{m+2}(-y)^{\frac{m+2}{2}}\right) \\ -\tfrac{2}{m+2}(-y)^{1-\beta_0} \int\limits_0^1 \nu\left(x + \tfrac{2}{m+2}(2t-1)(-y)^{\frac{m+2}{2}}\right)(1-t)^{-\beta} dt, \end{gathered} \tag{2.1}$$

where $\beta = \frac{m+2\beta_0}{(m+2)}$.

From formula (2.1) by virtue of boundary condition (1.3), we have

$$\psi(x) = \tau(2x) - \frac{2}{m+2}\left(\frac{m+2}{2}\right)^{1-\beta} x^{1-\beta} \int\limits_0^1 \nu(2tx)(1-t)^{-\beta} dt. \tag{2.2}$$

In equality (2.2), we make a change of variable integration $z = 2tx$ and have

$$\psi(x) = \tau(2x) - \frac{1}{2}\left(\frac{4}{m+2}\right)^{\beta} \int\limits_0^{2x} \nu(z)(2x - z)^{-\beta} dz. \tag{2.3}$$

In equality (2.3) we replace $2x$ with X and get

$$\psi(X) = \tau(X) - \frac{1}{2}\left(\frac{4}{m+2}\right)^{\beta} \int\limits_0^{X} \nu(z)(X - z)^{-\beta} dz, X \in [0, 1]. \tag{2.4}$$

Next, applying the Riemann-Liouville operator

$$D^l_{a,x}f(x)=\frac{1}{\Gamma(-l)}\int\limits_a^x\frac{f(t)dt}{(x-t)^{1+l}},l<0,$$

from equality (2.4), we obtain

$$\psi\left(\frac{X}{2}\right)=\tau(X)-\frac{1}{2}\left(\frac{4}{m+2}\right)^{\beta}\Gamma(1-\beta)D^{\beta-1}_{0X}\nu(X),X\in[0,1]. \tag{2.5}$$

Dividing both sides of the equality (2.5) by $\frac{1}{2}\left(\frac{4}{m+2}\right)^{\beta}\Gamma(1-\beta)$, we obtain

$$\gamma_0\tau(X)-D^{\beta-1}_{0,X}\nu(X)=\Psi_1(X),\ X\in[0,1], \tag{2.6}$$

where $\gamma_0=\frac{1}{\frac{1}{2}\left(\frac{4}{m+2}\right)^{\beta}\Gamma(1-\beta)}$, $\Psi_1(x)=\frac{\psi\left(\frac{X}{2}\right)}{\frac{1}{2}\left(\frac{4}{m+2}\right)^{\beta}\Gamma(1-\beta)}$.

Equality (2.6) is a functional relation between $\tau(x)$ and $\nu(x)$ brought to the line $y=0$ in the hyperbolic part D^- of domain D.

We introduce the notation $\lim\limits_{y\to+0}y^{1-\gamma}u(x,y)=\tau(x)$, $\lim\limits_{y\to+0}y^{1-\gamma}(y^{1-\gamma}u(x,y))_y=\nu(x)$.

It is known that the solution to equation (1.1) in the half-plane $y>0$, satisfying condition (1.2) and condition $\lim\limits_{y\to+0}y^{1-\gamma}u(x,y)=\tau(x)$, $x\in\bar{I}$, has the following form [2]

$$u(x,y)=\int\limits_0^1 G(x,y,t)\tau(t)dt, \tag{2.7}$$

where

$$G(x,y,t)=\frac{\Gamma(\gamma)}{2}y^{\gamma-1}e^{1,\frac{\gamma}{2}}_{1,\frac{\gamma}{2}}\left(-|x-t|y^{-\frac{\gamma}{2}}\right),$$
$$e^{1,\frac{\gamma}{2}}_{1,\frac{\gamma}{2}}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\frac{(1-k)\gamma}{2}\right)k!}. \tag{2.8}$$

Remark 2.1. The solution $u(x,y)$ can be expressed in terms of the special Wright functions $\varphi(\alpha,\delta,z)$ defined for real α,δ and complex z by means of a power series [7]

$$\varphi(\alpha,\delta;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\alpha k+\delta)}.$$

By (2.8), we have $e^{1,\frac{\gamma}{2}}_{1,\frac{\gamma}{2}}(z)=\varphi\left(-\frac{\gamma}{2},\frac{\gamma}{2};z\right)$, and therefore,

$$u(x,y)=\frac{\Gamma(\gamma)}{2}y^{\frac{\gamma}{2}-1}\int\limits_0^1\varphi\left(-\frac{\gamma}{2},\frac{\gamma}{2};-|x-t|y^{-\frac{\gamma}{2}}\right)\tau(t)dt.$$

It is also known [1] that the functional relation between $\tau(x)$ and $\nu(x)$, brought from the parabolic part of D^+ to the line $y=0$ has the following form:

$$\nu(x)=\frac{1}{\Gamma(1+\gamma)}\tau''(x). \tag{2.9}$$

Differentiating both sides of equality (2.6) with respect to x, we have

$$\gamma_0\tau''(x)-\frac{d^2}{dx^2}\left(D^{\beta-1}_{0,x}\nu(x)\right)=\Psi_1''(x),\ x\in[0,1]. \tag{2.10}$$

By virtue of the formula

$$(D_{0+}^{\alpha}f)(x)=\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int\limits_{0}^{x}(x-t)^{n-a-1}f(t)dt,(n=[\alpha]+1,\alpha>0),$$

from (2.10), we get

$$\gamma_0\tau''(x)-D_{0,x}^{\beta+1}\nu(x)=\Psi_1''(x),\ x\in[0,1]. \tag{2.11}$$

According to (1.4), (2.9), from (2.11), we arrive at the fractional differential equation $1+\beta$

$$D_{0,x}^{\beta+1}\nu(x)-\lambda\nu(x)=\Phi(x),\ x\in[0,1], \tag{2.12}$$

where

$\lambda=\gamma_0\Gamma(1+\gamma)$, $\Phi(x)=-\Psi_1''(x)$.

It is known [13] that the general solution of the fractional differential equation $\gamma_0>0$

$$(D_{0+}^{\gamma_0}y(t))(x)-\lambda_0 y(x)=h(x),\gamma_0>0,n=-[-\gamma_0] \tag{2.13}$$

is given by the formula

$$y(x)=\sum_{k=0}^{n}c_k x^{\gamma_0-k}E_{\gamma_0,1+\gamma_0-k}(\lambda_0 x^{\gamma_0})+\int\limits_{0}^{x}(x-t)^{\gamma_0-1}E_{\gamma_0,\gamma_0}(\lambda_0(x-t)^{\gamma_0})h(t)dt. \tag{2.14}$$

Here $c_1,c_2,...,c_n$ are arbitrary constants, and functions $E_{\gamma_0,1+\gamma_0-k}(\lambda_0 x^{\gamma_0})$ and $E_{\gamma_0,\gamma_0}(\lambda_0(x-t)^{\gamma_0})$ are special cases of the Mittag-Leffler function $E_{\gamma_0,\delta}(z)$,

$$E_{\gamma_0,\delta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\gamma_0 m+\delta)},\gamma_0>0,\delta>0,E_{\gamma_0}(z)\equiv E_{\gamma_0,1}(z)$$

which is an entire function of z [13].

Equation (2.12) is an equation of the form (2.13) with $y(x)=\nu(x)$, $\gamma_0=1+\beta$, $\lambda_0=\lambda$, and $h(x)=\Phi(x)$. Since $1<\beta+1<2$, the general solution of the form (2.14) for equation (2.12) is given by

$$\nu(x)=c_1x^{\beta}E_{1+\beta,1+\beta}(\lambda x^{1+\beta})+c_2x^{\beta-1}E_{1+\beta,\beta}(\lambda x^{1+\beta})+$$
$$\int\limits_{0}^{x}(x-t)^{\beta}E_{1+\beta,\beta}\left(\lambda(x-t)^{1+\beta}\right)\Phi(t)dt, \tag{2.15}$$

where c_1,c_2 are arbitrary constants. Substituting (2.15) into (2.6), we obtain an expression for $\tau(x)$:

$$\gamma_0\tau(x)+c_1\left(D_{0,x}^{\beta-1}t^{\beta}E_{1+\beta,1+\beta}(\lambda t^{1+\beta})\right)(x)+c_2\left(D_{0,x}^{\beta-1}t^{\beta-1}E_{1+\beta,\beta}(\lambda t^{1+\beta})\right)(x)+$$
$$+\left(D_{0,x}^{\beta-1}\int\limits_{0}^{t}(t-s)^{\beta}E_{1+\beta,1+\beta}\left(\lambda(t-s)^{1+\beta}\right)\Phi(s)ds\right)(x)=\Psi_1(x). \tag{2.16}$$

From (2.16), we get

$$\tau(x)=c_1^*\left(D_{0,x}^{\beta-1}t^{\beta}E_{1+\beta,1+\beta}(\lambda t^{1+\beta})\right)(x)+c_2^*\left(D_{0,x}^{\beta-1}t^{\beta-1}E_{1+\beta,\beta}(\lambda t^{1+\beta})\right)(x)-$$
$$-\frac{1}{\gamma_0}\left(D_{0,x}^{\beta-1}\int\limits_{0}^{t}(t-s)^{\beta}E_{1+\beta,1+\beta}\left(\lambda(t-s)^{1+\beta}\right)\Phi(s)ds\right)(x)+\frac{1}{\gamma_0}\Psi_1(x). \tag{2.17}$$

where

$c_1^*=-\frac{c_1}{\gamma_0}$, $c_2^*=-\frac{c_2}{\gamma_0}$.

In (2.17), we simplify the expression

$$\left(D_{0,x}^{\beta-1} t^{\beta} E_{1+\beta,1+\beta}(\lambda t^{1+\beta})\right)(x).$$

According to the formula

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m+\beta)}, \quad (\alpha>0,\ \beta>0) \tag{2.18}$$

we have

$$\left(D_{0,x}^{\beta-1} t^{\beta} E_{1+\beta,1+\beta}(\lambda t^{1+\beta})\right)(x) = \frac{1}{\Gamma(1-\beta)} \int\limits_0^x (x-t)^{\beta} t^{\beta} \times$$
$$\times \sum_{m=0}^{\infty} \frac{(\lambda t^{1+\beta})^m}{\Gamma((1+\beta)m+1+\beta)} dt.$$

Now, rearranging the orders of integration and summation, we have

$$\left(D_{0,x}^{\beta-1} t^{\beta} E_{1+\beta,1+\beta}(\lambda t^{1+\beta})\right)(x) = \sum_{m=0}^{\infty} \frac{\lambda^m}{\Gamma((1+\beta)m+1+\beta)} \frac{1}{\Gamma(1-\beta)} \times$$
$$\times \int\limits_0^x (x-t)^{\beta} t^{(1+\beta)m+\beta} dt. \tag{2.19}$$

In (2.19) we calculate the integral by changing the integration variable $t = x\sigma$:

$$\int\limits_0^x (x-t)^{-\beta} t^{(1+\beta)m+\beta} dt = \int\limits_0^1 x^{-\beta}(1-\sigma)^{-\beta} x^{(1+\beta)m+\beta} \sigma^{(1+\beta)m+\beta} x d\sigma =$$
$$= x x^{(1+\beta)m} \int\limits_0^1 \sigma^{(1+\beta)m+\beta}(1-\sigma)^{-\beta} d\sigma = x x^{(1+\beta)m} B\left((1+\beta)m+1+\beta, 1-\beta\right) =$$
$$= x x^{(1+\beta)m} \frac{\Gamma\left((1+\beta)m+1+\beta\right)\Gamma(1-\beta)}{\Gamma\left((1+\beta)m+2\right)}. \tag{2.20}$$

Substituting (2.20) into (2.19), we get

$$\left(D_{0,x}^{\beta-1} t^{\beta} E_{1+\beta,1+\beta}(\lambda t^{1+\beta})\right)(x) =$$
$$= \sum_{m=0}^{\infty} \frac{\lambda^m}{\Gamma((1+\beta)m+1+\beta)} \frac{1}{\Gamma(1-\beta)} \times$$
$$\times x x^{(1+\beta)m} \frac{\Gamma\left((1+\beta)m+1+\beta\right)\Gamma(1-\beta)}{\Gamma\left((1+\beta)m+2\right)} =$$
$$= x \sum_{m=0}^{\infty} \frac{\left(\lambda x^{1+\beta}\right)^m}{\Gamma((1+\beta)m+2)} = x E_{1+\beta,2}(\lambda x^{1+\beta}).$$

Thus,

$$\left(D_{0,x}^{\beta-1} t^{\beta} E_{1+\beta,1+\beta}(\lambda t^{1+\beta})\right)(x) = x E_{1+\beta,2}(\lambda x^{1+\beta}). \tag{2.21}$$

Similarly, it is easy to show that

$$\left(D_{0,x}^{\beta-1} t^{\beta-1} E_{1+\beta,\beta}(\lambda t^{1+\beta})\right)(x) = E_{1+\beta}(\lambda x^{1+\beta}), \tag{2.22}$$

$$\left(D_{0,x}^{\beta-1} \int_0^t (t-s)^{\beta} E_{1+\beta,1+\beta}\left(\lambda (t-s)^{1+\beta}\right) \Phi(s) ds \right)(x) =$$

$$= \int_0^x (x-s) E_{1+\beta,2}\left(\lambda (x-s)^{1+\beta}\right) \Phi(s) ds. \tag{2.23}$$

By virtue of (2.21)-(2.23) from (2.17), we obtain

$$\tau(x) = c_1^* x E_{1+\beta,2}(\lambda x^{1+\beta}) + c_2^* E_{1+\beta}(\lambda x^{1+\beta}) -$$

$$- \frac{1}{\gamma_0} \int_0^x (x-t) E_{1+\beta,2}\left(\lambda (x-t)^{1+\beta}\right) \Phi(t) dt + \frac{1}{\gamma_0} \Psi_1(x). \tag{2.24}$$

To find constants c_1^* and c_2^* we can use relation $\tau(0) = \tau(1) = 0$ following from condition (1.2).

Substituting $x = 0$ into the formula (2.24) and taking into account the equality $E_{1+\beta}(0) = 1$ that follows from the condition (2.18), we obtain $\tau(0) = c_2^* + \frac{1}{\gamma_0}\Psi_1(0)$, hence

$$c_2^* = -\frac{\Psi_1(0)}{\gamma_0}. \tag{2.25}$$

Substituting $x = 1$ into formula (2.24) and taking into account equality (2.25), we find

$$0 = c_1^* E_{1+\beta,2}(\lambda) - \frac{\Psi_1(0)}{\gamma_0} E_{1+\beta}(\lambda) -$$

$$-\frac{1}{\gamma_0} \int_0^1 (1-t) E_{1+\beta,2}\left(\lambda (1-t)^{1+\beta}\right) \Phi(t) dt + \frac{1}{\gamma_0} \Psi_1(1).$$

From here

$$c_1^* = \frac{1}{E_{1+\beta,2}(\lambda)} \times$$

$$\times \left(\frac{\Psi_1(0)}{\gamma_0} E_{1+\beta}(\lambda) - \frac{1}{\gamma_0}\Psi_1(1) + \frac{1}{\gamma_0} \int_0^1 (1-t) E_{1+\beta,2}\left(\lambda (1-t)^{1+\beta}\right) \Phi(t) dt \right). \tag{2.26}$$

Taking into account (2.25) and (2.26), substituting equality (2.24) into formula (2.7), we obtain an explicit solution $u(x,y)$ of the problem under study.

Let $P(\alpha_0, \beta_0) \in B_0 C_0$.

In this case, the solution of the modified Cauchy problem with initial data

$u(x,0) = \tau(x), x \in \bar{I}$ $\lim\limits_{y \to -0} (-y)^{\beta_0} u(x,y)_y = \nu(x),\ x \in I$, has the form [10, 12]

$$u(x,y) = \tau\left(x - \tfrac{2}{m+2}(-y)^{\frac{m+2}{2}}\right)$$
$$-\tfrac{2}{m+2}(-y)^{1-\beta_0} \int_0^1 \nu\left(x - \tfrac{2}{m+2}(2t-1)(-y)^{\frac{m+2}{2}}\right)(1-t)^{-\alpha} dt,$$

where $\alpha = \frac{m+2\beta_0}{(m+2)}$.

Further research is conducted as in the case of $P(\alpha_0, \beta_0) \in A_0 C_0$.

References

[1] Gekkieva S.Kh.; On an analogue of the Tricomi problem for a mixed-type equation with a fractional derivative. Reports of the AMAN., –2001. –5. –No.2. –P.18-22.

[2] Gekkieva S.Kh.; The Cauchy problem for a generalized transport equation with a fractional derivative in time. Reports of the AMAN., –2000. –5. –No.1. –P.16-19.

[3] Hilfer R.; Experimental evidence for fractional time evolution in glass forming materials. Chemical Phys., –2002. –284. –No.(1-2). –P.399-408.

[4] Karimov E. T.; Tricomi type boundary value problem with integral conjugation condition for a mixed type equation with Hilfer fractional operator. Bulletin of the Institute of Mathematics., –2019. –No.1. –P.19-26.

[5] Kilbas A.A., Repin O.A.; An analogue of the Bitsadze-Samarskii problem for a mixed-type equation with a fractional derivative. Differential Equations., –2003. –39. –No.5. –P.638-644.

[6] Kochubey A.N.; Fractional diffusion. Differential Equations., –1990. –26. –No.4. –P.660-670.

[7] Nigmatullin R.R.; The realization of generalized transfer equation in a medium with fractal geometry. Phys. Status solidi., –1986. –133. –P.425-430.

[8] Pskhu A.V.; Solution of boundary value problems for the fractional diffusion equation by the Green's function method. Differential Equations., –2003. –39. –No.10. –P.1430-1433.

[9] Ruziev M.Kh.; A boundary value problem for a partial differential equation with fractional derivative. Fractional Calculus and Applied Analysis., –2021. –24. –No.2. –P.509-517.

[10] Ruziev M.Kh.; On a problem with shift on pieces of boundary characteristics for the Gellerstedt equation with singular coefficients. Lobachevskii Journal of Mathematics., –2022. –43. –No.2. –P.484-495.

[11] Ruziev M.Kh., Rakhimova G.B.; On a boundary value problem for a partial differential equation with fractional derivative. Bulletin of the Institute of Mathematics., –2023. –6. –No.2. –P.114-121.

[12] Salakhitdinov M.S., Mirsaburov M.; Nonlocal problems for equations of mixed type with singular coefficients. Tashkent, University., –2005. –P.224.

[13] Samko S.G., Kilbas A.A., Marichev O.I.; Integrals and derivatives of fractional order and some of their applications. Minsk: Science and Technology., –1987.

[14] Tomovski Z., Hilfer R., Srivastava H.M.; Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. Integ. Trans. and Special functions., –2010. –21. –No.11. –P.797-814.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.273-280
DOI: 10.29229/uzmj.2025-2-26

Inverse problem of determining source of the diffusion equation on a metric star graph

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this paper, we solve the inverse source problem that aims to determine the right-hand sides of the diffusion equation, in each of the bonds, under specific integral overdetermination conditions on a metric star graph in Sobolev space. We reformulate the problem as an operator equation and show that the corresponding resolvent operator is well-defined.

Keywords: inverse problem, metric graph, generalized solution, uniqueness of the solution.
MSC (2020): 34B45, 35D35, 35R11

1. Introduction

Differential equations on metric graphs effectively model various physical phenomena, including oscillations in elastic networks and diffusion processes. A significant body of research has concentrated on differential operators and equations within this framework, exploring their implications and applications in diverse contexts. These studies provide valuable information on the behavior of systems represented by metric graphs, improving our understanding of complex physical interactions [13, 17, 16, 19, 11].

The most significant challenge involves solving an inverse problem in differential equations. In [14], the primary focus is on an inverse problem related to recovering the right-hand side of a parabolic equation with variable coefficients, along with an integral overdetermination condition. The authors demonstrate the Fredholm property of this inverse problem, prove the existence and uniqueness of its solution under certain conditions, and provide stability estimates. A similar issue is addressed in [9]. In addition, several studies have examined inverse problems for fractional differential equations, such as references [2, 4, 21]. Ashurov et al. [3] investigate the inverse problem of determining the density of a heat source for a subdiffusion equation. They utilized the Fourier method to establish the existence, uniqueness, and stability of solutions.

The manuscript is organized as follows: The essential definitions and notations are presented in Section 2. Section 3 outlines the precise formulations of the problems investigated. In Section 4, the focus is on studying the inverse problem.

2. Preliminaries

In this section, we define key notations and function spaces essential for our problem.

Definition 2.1. (see [10], p. 69) The the Riemann-Liouville fractional integral of order $0<\alpha<1$ for a function $\varphi \in L_1(0,T)$ is defined by

$$\mathrm{I}_{0,t}^{\alpha}\varphi(t) \equiv \mathrm{D}_{0,t}^{-\alpha}\varphi(t) := \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau,$$

provided that, the integral, in the right-hand side of this expression, exists.

Definition 2.2. (see [10], p. 70) The Riemann-Liouville fractional derivative of order $0<\alpha<1$ for a function $\varphi(t)$ on $[0,T]$ is defined by

$$\mathrm{D}_{0,t}^{\alpha}\varphi(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\varphi(\tau)}{(t-\tau)^{\alpha}}d\tau,$$

provided that, the integral, in the right-hand side of this expression, exists.

Definition 2.3. (see [10]) The Caputo fractional derivative of order $0 < \alpha < 1$ for a function $\varphi(t)$ on $[0, T]$ is defined by

$$\partial_{0,t}^{\alpha}\varphi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\varphi^{'}(\tau)}{(t-\tau)^{\alpha}} d\tau,$$

provided that, the integral, in the right-hand side of this expression, exists.

We present functional spaces that help solve the problem. The previously defined fractional derivatives are pointwise, so we need a definition for the generalized fractional derivative, which is well-defined in a subspace of a fractional-order Sobolev space, as discussed in reference [8].

Following to ([8], p. 12), by $H^{\alpha}(0,T),\ 0 < \alpha < 1$, we denote fractional Sobolev - Slobodeskii space governed by the norm (see [7], [8])

$$\|u\|_{H^{\alpha}(0,T)} := \left(\|u\|^2_{L_2(0,T)} + \int_0^T \int_0^T \frac{|u(t) - u(s)|^2}{(t-s)^{1+2\alpha}} dsdt \right)^{\frac{1}{2}}.$$

We put

${}_0H^{\alpha}(0,T) = \{u \in H^{\alpha}(0,T) : u(0) = 0\}$ for $\frac{1}{2} < \alpha \leq 1$,

$$H_{\alpha}(0,T) = \begin{cases} H^{\alpha}(0,T), & 0 \leq \alpha < \frac{1}{2}, \\ \{v \in H^{\frac{1}{2}}(0,T) : \int_0^T \frac{|v(t)|^2}{t} dt < \infty\}, & \alpha = \frac{1}{2}, \\ {}_0H^{\alpha}(0,T), & \frac{1}{2} < \alpha \leq 1. \end{cases}$$

The space $H_{\alpha}(0,T)$ is a Banach space with the norm (see [8], p. 12)

$$\|v\|_{H_{\alpha}(0,T)} = \begin{cases} \|v\|_{H^{\alpha}(0,T)}, & 0 < \alpha < 1,\ \alpha \neq \frac{1}{2}, \\ \left(\|v\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|v(t)|^2}{t} dt \right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}. \end{cases}$$

According to ([8], p. 15) the space ${}_0C^1[0,T] = \{v \in C^1[0,T] : v(0) = 0\}$ is dense in $H_{\alpha}(0,T)$. In the same work, it is shown that $I_{0,t}^{\alpha} : L_2(0,T) \to H_{\alpha}(0,T)$ is injective and surjective, and so, the weak fractional derivative can be defined as $\partial_{0,t}^{\alpha} = (I_{0,t}^{\alpha})^{-1}$.

Furthermore, the following norms are equivalent in $H_{\alpha}(0,T)$ (see [8], p. 17)

$$\|\partial_{0,t}^{\alpha} v\|_{L_2(0,T)} \sim \|v\|_{H_{\alpha}(0,T)}.$$

We notice, that in the case $\frac{1}{2} < \alpha < 1$ for any $v(t) \in H^{\alpha}(0,T)$ the weak Caputo derivative can be defined by the equality $\partial_{0,t}^{\alpha} v(t) = \partial_{0,t}^{\alpha}(v(t) - v(0))$ (see [8]).

The star metric graph $\mathcal{G}$ is a graph with n bonds, consisting of a finite set of vertices $V = \{\nu_k\}_0^n$ and a finite set of edges $E = \{e_k\}_1^n$, where e_k connects the vertices ν_0 and ν_k, $k = \overline{1,n}$ [6]. Each bond e_k is assigned the interval $(0, l_k)$, and coordinates x_k are defined on each bond. The vertex ν_0 of the graph has a coordinate of 0 on each bond. Further, without loss of generality, we will use x instead of x_k. For the function, $u : \mathcal{G} \to R$, defined on the graph, we put $u|_{e_k} = u_k$.

We establish functional spaces within the metric star graph. For the functions defined on the graph, we use vector-type notations $u = (u_1, ..., u_n)$, $u_x = \left(\frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}\right)$, $u_{xx} = \left(\frac{\partial^2 u_1}{\partial x^2}, \ldots, \frac{\partial^2 u_n}{\partial x^2}\right)$, $\int\limits_{\mathcal{G}} u d\mathcal{G} = \sum\limits_{k=1}^{n} \int\limits_0^{l_k} u_k dx$. For $u : \mathcal{G} \to R$, $v : \mathcal{G} \to R$, we put $uv = (u_1v_1, u_2v_2, ..., u_nv_n)$.

Let $G_{\tau} = \{(x,t) : x \in \mathcal{G}, t \in (0, \tau]\}, 0 < \tau \leq T$.

Definition 2.4. (see [6], p. 10) The space $L_2(\mathcal{G})$ on $\mathcal{G}$ consists of functions that are measurable and square-integrable on each edge $e_k,\ k = \overline{1,n}$ with the scalar product and the norm:

$$(u(x), v(x))_{L_2(\mathcal{G})} = \int\limits_{\mathcal{G}} u(x) \cdot v(x) d\mathcal{G},$$

$$\|u\|^2_{L_2(\mathcal{G})} = \sum_k \|u\|^2_{L_2(e_k)}.$$

In other words, $L_2(\mathcal{G})$ is the orthogonal direct sum of spaces $L_2(e_k),\ k = \overline{1,n}$.

Definition 2.5. The space $\mathbb{L}_2(0,t)$ on G_T consists of functions that are measurable and square-integrable on each edge $e_k,\ k=\overline{1,n}$ with the norm:

$$\|u\|^2_{\mathbb{L}_2(0,t)} = \sum_{k=1}^{n} \int_0^t u_k^2(\tau)d\tau.$$

In other words, $\mathbb{L}_2(0,t)$ is the orthogonal direct sum of spaces $L_2(0,t)$, $k=\overline{1,n}$.

Definition 2.6. The Hilbert space $W_2^l(\mathcal{G}),\ l=1,2$ defined by

$$W_2^l(\mathcal{G}) = \bigoplus_{k=1}^{n} W_2^l(e_k),\ l=1,2$$

and with the scalar products

$$(u,v)_{W_2^1(\mathcal{G})} = \int\limits_{\mathcal{G}} (uv + u_x v_x)\, d\mathcal{G},$$

$$(u,v)_{W_2^2(\mathcal{G})} = \int\limits_{\mathcal{G}} (uv + u_x v_x + u_{xx}v_{xx})\, d\mathcal{G}.$$

Definition 2.7. Let the space $Q = \{u \in C(\mathcal{G}) : u_k(x) \in C^\infty(\bar{e}_k), u_k|_{x=l_k} = 0\}$. $\overset{\circ}{W}{}_2^1(\mathcal{G})$ is a subspace of the space $W_2^1(\mathcal{G})$ that is the closure of Q with respect to the norm $\|u\|_{W_2^1(\mathcal{G})} = \sqrt{(u,u)_{W_2^1(\mathcal{G})}}$.

Definition 2.8. Let $L_2(G_\tau) = L_2(0,\tau; L_2(\mathcal{G}))$. $W_2^{1,\alpha}(G_\tau) = \{u : u(t,\cdot) \in W_2^1(\mathcal{G}), u, u_x, \partial_{0,t}^\alpha u \in L_2(G_\tau)\}$ is a subspace of $L_2(G_\tau)$ with the scalar product

$$(u,v)_{W_2^{1,\alpha}(G_\tau)} = \int_0^\tau \int_{\mathcal{G}} \left(uv + u_x v_x + \partial_{0,t}^\alpha u \partial_{0,t}^\alpha v\right) d\mathcal{G}dt,$$

and with the norm

$$\|u\|_{W_2^{1,\alpha}(G_\tau)} = \left(\int_0^\tau \int_{\mathcal{G}} \left(u^2 + u_x^2 + (\partial_{0,t}^\alpha u)^2\right) d\mathcal{G}dt\right)^{\frac{1}{2}}.$$

$W_{2,0}^{1,\alpha}(G_\tau) = \{u \in W_2^{1,\alpha}(G_\tau) : u_k|_{x=l_k} = 0\}$ is a subset of $W_2^{1,\alpha}(G_\tau)$.

Definition 2.9. $W_2^{2,\alpha}(G_\tau)$ is the Hilbert space consisting of all elements of $L_2(G_\tau)$ that have generalized derivatives $\partial_{0,t}^\alpha u$, u_x and u_{xx} from $L_2(G_\tau)$. The scalar product in it is defined by the equality

$$(u,v)_{W_2^{2,\alpha}(G_\tau)} = \int_0^\tau \int_{\mathcal{G}} \left(uv + u_x v_x + \partial_{0,t}^\alpha u \partial_{0,t}^\alpha v + u_{xx}v_{xx}\right) d\mathcal{G}dt$$

and the norm is denoted as follows: $\|\cdot\|_{W_2^{2,\alpha}(G_\tau)}$.

Definition 2.10. $W_{2,0}^{2,\alpha}(G_\tau)$ is a subspace of $W_2^{2,\alpha}(G_\tau)$, which is the intersection of $W_2^{2,\alpha}(G_\tau)$ with $W_{2,0}^{1,\alpha}(G_\tau)$.

Lemma 2.11. [1] *For any function $v(t)$ absolutely continuous on $[0,T]$, one has the inequality*

$$v(t)\partial_{0,t}^\alpha v(t) \geq \frac{1}{2}\partial_{0,t}^\alpha v^2(t),\ \ 0<\alpha<1. \tag{2.1}$$

And also we need arithmetic inequality

$$|ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2,\ \ \textit{for all } \varepsilon > 0. \tag{2.2}$$

3. Problem statement

We consider the following time-fractional diffusion equation on the metric star graph $\mathcal{G}$

$$\partial_{0,t}^{\alpha} u_k(x,t) - u_{k,xx}(x,t) = f_k(t) g_k(x,t) + h_k(x,t),\ x \in e_k,\ t \in (0,T],\ k = \overline{1,n}, \tag{3.1}$$

where $\partial_{0,t}^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $g_k(x,t)$, $h_k(x,t)$ are given functions, $f_k(t)$ are unknown functions. We need to solve both the direct and inverse problems to determine the pair of functions $\{u(x,t), f(t)\}$ in the equation (3.1) within a bounded domain $G_T = \mathcal{G} \times (0,T]$. This solution must satisfy the following initial conditions

$$u_k(x,0) = 0, \quad x \in \bar{e}_k, \quad k = \overline{1,n}, \tag{3.2}$$

the vertex conditions

$$\sum_{k=1}^{n} u_{k,x}(0,t) = 0, \quad u_k(0,t) = u_j(0,t),\ k \neq j,\ k,j = \overline{1,n}, \quad t \in [0,T], \tag{3.3}$$

the boundary conditions

$$u_k(l_k,t) = 0,\ t \in [0,T],\ k = \overline{1,n}, \tag{3.4}$$

and the overdetermination conditions

$$\int_0^{l_k} \eta_k(x) u_k(x,t) dx = \psi_k(t),\ k = \overline{1,n}, \tag{3.5}$$

where $\eta_k(x)$, $\psi_k(t)$ are known functions.

In this problem, we are searching for a solution in the form $\{u_k(x,t), f_k(t)\} = \{y_k(x,t), 0\} + \{w_k(x,t), f_k(t)\}$, where $y_k(x,t)$ is a solution of the direct problem

$$\partial_{0,t}^{\alpha} y_k(x,t) - y_{k,xx}(x,t) = h_k(x,t), \quad x \in \bar{e}_k,\ t \in [0,T],\ k = \overline{1,n}, \tag{3.6}$$

and the pair $\{w_k(x,t), f_k(t)\}$ is a solution of the inverse problem

$$\partial_{0,t}^{\alpha} w_k(x,t) - w_{k,xx}(x,t) = f_k(t) g_k(x,t), \quad x \in \bar{e}_k,\ t \in [0,T],\ k = \overline{1,n}, \tag{3.7}$$

with (3.2)-(3.4) conditions and the following overdetermination conditions

$$\int_0^{l_k} \eta_k(x) w_k(x,t) dx = E_k(t),\ k = \overline{1,n}, \tag{3.8}$$

where $E_k(t) = \psi_k(t) - \int\limits_0^{l_k} \eta_k(x) y_k(x,t) dx$.

We express the problem of defining the function, as noted in reference ([12], p. 153), as $Ay = h$ over the domain

$$V(G_T) = \left\{ \begin{array}{c} y(x,t) \in W_{2,0}^{2,\alpha}(G_T),\ y_k(x,t)|_{t=0} = 0, \\ \sum\limits_{k=1}^{n} y_{k,x}(0,t) = 0,\ y_k(0,t) = y_j(0,t),\ k \neq j,\ k,j = \overline{1,n} \end{array} \right\}.$$

Theorem 3.1. *Let $g(x,t) = 0$, $h(x,t) \in L_2(G_T)$. Then the problem (3.1)-(3.4) has a unique strong solution in $V(G_T)$.*

Problems that were solved in a similar way were examined in [5, 18, 20]. In these studies, the approach involved reducing the given equation to an operator equation and solving it using a priori estimates, which can be classified as a functional method. This method was adapted and improved, originally drawing inspiration from Ladyzhenskaya [12], to address fractional order equations on metric graphs. In [20], unique solvability of inverse problem was presented for such a diffusion equation, where the unknown function on the right side of the equation depended solely on the variable x and was consistent with the overdetermination condition.

The uniqueness and existence of the direct problem are investigated in [15], so we will focus on solving the inverse problem.

4. Unique solvability of the inverse problem

We obtain the operator equation for the unknown functions $f_k(t)$, assuming the functions in the problem are measurable and meet the given conditions:

$$g(x,t) \in L_2(0,T,\overset{\circ}{W}{}^1_2(\mathcal{G})), \tag{K1}$$

$$E_k(t) \in H^\alpha(0,t), \quad \eta(x) \in L_2(\mathcal{G}), \quad g^*_k(t) \neq 0, \tag{K2}$$

where

$$g^*_k(t) = \int_0^{l_k} \eta_k(x) g_k(x,t)dx, \;\; k = \overline{1,n}.$$

By multiplying both sides of equation (3.7) with the function $\eta_k(x)$ and integrating over the interval $[0, l_k]$, we get

$$\int_0^{l_k} \eta_k(x)\partial^\alpha_{0,t} w_k(x,t)dx - \int_0^{l_k} \eta_k(x) w_{k,xx}(x,t)dx = f_k(t)\int_0^{l_k} \eta_k(x) g_k(x,t)dx.$$

Using conditions (3.8), we obtain the relation

$$f_k(t) = Bf_k(t) + \frac{\partial^\alpha_{0,t} E_k(t)}{g^*_k(t)}, \tag{4.1}$$

where

$$B_k f_k(t) = -\frac{1}{g^*_k(t)}\int_0^{l_k} \eta_k(x) w_{k,xx}(x,t)dx, \quad k = \overline{1,n}, \tag{4.2}$$

$$Bf = (Bf_1, Bf_2, \ldots, Bf_n), \quad Bf_k : \mathbb{L}_2(0,T) \to \mathbb{L}_2(0,T).$$

Theorem 4.1. *Let conditions (K1), (K2) hold. If $h(x,t) \in L_2(G_T)$, then there exists a generalized solution $\{u(x,t), f(t)\}$ of the inverse problem (3.1)-(3.5) and $u(x,t) \in V(G_T)$, $f(t) \in \mathbb{L}_2(0,T)$.*

Proof. It suffices to prove the unique solvability of the inverse problem (3.1)-(3.5) by showing that the operator B, defined by relation (4.2), is a continuous operator into $L_2(0,t)$.

By multiplying both sides of equation (3.7) by $\partial^\alpha_{0,t} w_{k,xx}(x,t)$ and integrate over $\mathcal{G}$

$$\sum_{k=1}^n \int_0^{l_k} \partial^\alpha_{0,t} w_k(x,t)\partial^\alpha_{0,t} w_{k,xx}(x,t)dx - \sum_{k=1}^n \int_0^{l_k} w_{xx}(x,t)\partial^\alpha_{0,t} w_{k,xx}(x,t)dx$$
$$= \sum_{k=1}^n \int_0^{l_k} f_k(t) g_k(x,t)\partial^\alpha_{0,t} w_{k,xx}(x,t)dx.$$

Then, using integration by parts and (2.1) for the second sum, we obtain

$$\sum_{k=1}^n \partial^\alpha_{0,t} w_k(x,t)\cdot\partial^\alpha_{0,t}\; w_{k,x}(x,t)|_{x=0}^{x=l_k} - \sum_{k=1}^n \int_0^{l_k} (\partial^\alpha_{0,t} w_k(x,t))^2 dx - \frac{1}{2}\sum_{k=1}^n \int_0^{l_k} \partial^\alpha_{0,t} w^2_{k,xx}(x,t)dx$$
$$\geq \sum_{k=1}^n f_k(t)\Big(g_k(x,t)\cdot\partial^\alpha_{0,t}\; w_{k,x}(x,t)|_{x=0}^{x=l_k} - \int_0^{l_k} g_{k,x}(x,t)\partial^\alpha_{0,t} w_{k,x}(x,t)dx\Big).$$

After multiplying both sides of the inequality by -1 and also taking into account conditions (3.2)-(3.4) and $(K1)$ we get

$$\frac{1}{2}\partial^\alpha_{0,t}||w_{k,xx}(x,t)||^2_{L_2(\mathcal{G})} + ||\partial^\alpha_{0,t} w_{k,x}(x,t)||^2_{L_2(\mathcal{G})} \leq$$
$$\leq \sum_{k=1}^n f_k(t)\int_0^{l_k} g_{k,x}(x,t)\partial^\alpha_{0,t} w_{k,x}(x,t)dx \tag{4.3}$$

By using Cauchy-Bunyakovski inequality and (2.2), also (K1), we can get:

$$\sum_{k=1}^{n} f_k(t) \int_0^{l_k} g_{k,x}(x,t)\partial_{0,t}^{\alpha} w_{k,x}(x,t)dx$$

$$\leq \sum_{k=1}^{n} f_k(t) \Big(\int_0^{l_k} g_{k,x}^2(x,t)dx \cdot \int_0^{l_k} \partial_{0,t}^{\alpha} w_{k,x}^2(x,t)dx \Big)^{\frac{1}{2}}$$

$$\leq \sum_{k=1}^{n} |f_k(t)| \cdot ||g_{k,x}(x,t)||_{L_2(0,l_k)} \cdot ||\partial_{0,t}^{\alpha} w_{k,x}(x,t)||_{L_2(0,l_k)}$$

$$\leq c \sum_{k=1}^{n} |f_k(t)| \cdot ||\partial_{0,t}^{\alpha} w_{k,x}(x,t)||_{L_2(0,l_k)} \leq \frac{c}{4\varepsilon} \sum_{k=1}^{n} |f_k(t)|^2 + \varepsilon ||\partial_{0,t}^{\alpha} w_x(x,t)||_{L_2(\mathcal{G})}^2$$

where $c = \underset{1\leq k\leq n}{max} \|g_{k,x}(\cdot,t)\|_{L_2(0,l_k)}$. Replacing the last result to (4.3) we can get:

$$\frac{1}{2}\partial_{0,t}^{\alpha} ||w_{xx}(x,t)||_{L_2(\mathcal{G})}^2 + ||\partial_{0,t}^{\alpha} w_x(x,t)||_{L_2(\mathcal{G})}^2 \leq \frac{c}{4\varepsilon} \sum_{k=1}^{n} |f_k(t)|^2 + \varepsilon ||\partial_{0,t}^{\alpha} w_x(x,t)||_{L_2(\mathcal{G})}^2$$

Choosing $\varepsilon = 1$ and simplifying the inequality, it will be :

$$\partial_{0,t}^{\alpha} ||w_{xx}(x,t)||_{L_2(\mathcal{G})}^2 \leq \frac{c}{2} \sum_{k=1}^{n} |f_k(t)|^2.$$

By fractional integrating and using initial conditions (3.2), we get

$$\|w_{xx}(x,t)\|_{L_2(\mathcal{G})}^2 \leq \frac{c}{2} I_{0,t}^{\alpha} \sum_{k=1}^{n} |f_k(t)|^2. \tag{4.4}$$

Using condition (K2) and Cauchy-Bunyakovsky inequality, we estimate the norm of the operator Bf

$$\|Bf\|_{\mathbb{L}_2(0,t)}^2 = \sum_{k=1}^{n} \int_0^t (Bf_k)^2 d\tau = \sum_{k=1}^{n} \int_0^t \left(\frac{1}{g_k^*(t)} \int_0^{l_k} \eta_k(x) w_{xx} dx \right)^2 d\tau$$

$$\leq \frac{1}{q^2} \sum_{k=1}^{n} \int_0^t \int_0^{l_k} \eta_k^2(x)dx \int_0^{l_k} w_{xx}^2 dx d\tau \leq \frac{m^2}{q^2} \int_0^t ||w_{xx}||_{L_2(\mathcal{G})}^2 d\tau.$$

where $q = \underset{1\leq k\leq n}{max} |g_k^*(t)|$, $m = \|\eta\|_{L_2(\mathcal{G})}$. Then, taking into account (4.4), we get

$$\|Bf\|_{\mathbb{L}_2(0,t)}^2 \leq \frac{cm^2}{2q^2} I_{0,t}^{\alpha} \sum_{k=1}^{n} \int_0^t |f_k(\tau)|^2 d\tau = C I_{0,t}^{\alpha} \|f\|_{\mathbb{L}_2(0,t)}^2, \tag{4.5}$$

where $C = \frac{cm^2}{2q^2}$. Now, iterating the inequality (4.5) i times, we get

$$\|(B^i f)(t)\|_{\mathbb{L}_2(0,t)}^2 \leq C^i I_{0,t}^{\alpha} \|B^{i-1} f\|_{\mathbb{L}_2(0,t)}^2 \leq C^i I_{0,t}^{i\alpha} \|f\|_{\mathbb{L}_2(0,t)}^2, \; i = 1,2,3,....$$

From the last inequality, taking into account that the function $\tilde{f}(t) = \|f\|_{\mathbb{L}_2(0,t)}^2$ is a nonnegative and non-decreasing function on $t \in [0,T]$, we have

$$\|(B^i f)(t)\|_{\mathbb{L}_2(0,t)}^2 \leq C^i \|f\|_{\mathbb{L}_2(0,t)}^2 \cdot I_{0,t}^{i\alpha} 1 = \frac{C^i t^{i\alpha}}{\Gamma(i\alpha+1)} \|f\|_{\mathbb{L}_2(0,t)}^2.$$

Consequently, we have

$$\|(I-B)^{-1}F\|_{\mathbb{L}_2(0,T)} \leq \sum_{i=0}^{+\infty} \frac{\left(\sqrt{CT^\alpha}\right)^i}{\sqrt{\Gamma(i\alpha+1)}} \|f\|_{\mathbb{L}_2(0,t)},$$

where $F = \frac{\partial_{0,t}^\alpha E_k(t)}{g_k^*(t)}$.

So, it follows that the resolvent operator $(I-B)^{-1} : \mathbb{L}_2(0,T) \to \mathbb{L}_2(0,T)$ is bounded and continuous mapping. □

References

[1] Alikhanov A. A.; A priori estimate for solutions of boundary value problems for fractional-order equations. *Differential Equations*, –2010. –46. –(5). –P. 660666.

[2] Alimov Sh., Ashurov R.; Inverse problem of determining an order of the Riemann-Liouville time-fractional derivative. *Progress in Fractional Differentiation and Applications*, –2022. –(4). –P. 467474.

[3] Ashurov R. R., Mukhitdinova A. T.; Inverse problem of determining the heat source density for the subdiffusion equation. *Differential Equations*, –2020. –56. –(12). –P. 15501563.

[4] Ashurov R., Umarov S.; An inverse problem of determining orders of systems of fractional pseudo-differential equations. *Fractional Calculus and Applied Analysis*, –2022. –25. –P. 109127.

[5] Ashurov R. R., Sobirov Z. A., Turemuratova A. A.; Inverse source problem for the space-time fractional parabolic equation on a metric star graph with an integral overdetermination condition. *Mathematical Notes*, –2024. –116. –(5). –P. 892904.

[6] Berkolaiko G., Kuchment P.; Introduction to quantum graphs. *Mathematical Surveys and Monographs, AMS*, –2013. –186.

[7] Gorenflo R., Luchko Y., Yamamoto M.; Time-fractional diffusion equation in the fractional Sobolev spaces. *Fractional Calculus and Applied Analysis*, –2015. –18. –P. 799820.

[8] Kubica A., Ryszewska K., Yamamoto M.; Time-Fractional Differential Equations: A Theoretical Introduction. *Springer*, Singapore, –2020.

[9] Kamiynin V. L.; On the inverse problem of determining the right-hand side of a parabolic equation under an integral overdetermination condition. *Mathematical Notes*, –2005. –77. –(4). –P. 482493.

[10] Kilbas A. A., Srivastava H. M., Trujillo J. J.; Theory and Applications of Fractional Differential Equations. *North-Holland Mathematics Studies*, –2006. –204. Elsevier.

[11] Kulaev R. Ch., Karkuzaev S. A.; Lower estimates for the leading eigenvalue of the Laplacian on a graph. *Mathematical Notes*, –2025. –117. –(2). –P. 240284.

[12] Ladyzhenskaya O. A.; Kraevye zadachi matematicheskoi fiziki. *Nauka*, Moscow, –1973.

[13] Mehandiratta V., Mehra M., Leugering G.; Existence and Uniqueness of Time-Fractional Diffusion Equation on a Metric Star Graph. *Indian Institute of Technology Delhi, University of Erlangen-Nuremberg*, (Springer, Singapore), –2020. –P. 2541.

[14] Prilepko A., Tkachenko D.; Inverse problem for a parabolic equation with integral overdetermination. *Journal of Inverse and Ill-Posed Problems*, –2003. –11. –(2). –P. 191218.

[15] Sobirov Z. A.; Inverse source problem for the subdiffusion equation on a metric star graph with integral overdetermination condition. *Lobachevskii Journal of Mathematics*, –2023. –44. –(12). –P. 54265436.

[16] Sobirov Z. A., Khujakulov J. R., Turemuratova A. A.; Unique solvability of IBVP for pseudo-subdiffusion equation with Hilfer fractional derivative on a metric graph. *Chelyabinsk Physical and Mathematical Journal*, –2023. –8. –(3). –P. 351370.

[17] Sobirov Z. A., Rakhimov K. U., Ergashov R. E.; Green's function method for time-fractional diffusion equation on the star graph with equal bonds. *Nanosystems: Physics, Chemistry, Mathematics*, –2021. –12. –(3). –P. 271278.

[18] Sobirov Z. A., Turemuratova A. A.; Inverse source problem for the subdiffusion equation with edge-dependent order of time-fractional derivative on the metric star graph. *Nanosystems: Physics, Chemistry, Mathematics*, –2024. –15. –(5). –P. 586596.

[19] Sobirov Z. A., Turemuratova A. A.; Inverse source problem for the heat equation on a metric star graph with integral over-determination condition. *Bulletin of National University of Uzbekistan Mathematics and Natural Sciences*, –2023. –6. –(1). –P. 115.

[20] Turemuratova A. A.; Inverse source problem for the subdiffusion equation on a metric graph. *Uzbek Mathematical Journal*, –2024. –68. –(3). –P. 158166.

[21] Yuldashev T. K., Karimov E. T.; Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters. *Axioms*, –2020. –9. –P. 121.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.281-286
DOI: 10.29229/uzmj.2025-2-27

Determining the order of time and spatial fractional derivatives in a subdiffusion equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. The paper considers the initial-boundary value problem for equation $D_t^\rho u(x,t) + (-\Delta)^\sigma u(x,t) = 0$, $\rho, \sigma \in (0,1)$, in an N-dimensional domain Ω with a homogeneous Dirichlet condition. The fractional derivative is taken in the sense of Caputo. The main goal of the work is to solve the inverse problem of simultaneously determining two parameters: the order of the fractional derivative ρ and the degree of the Laplace operator σ. A new formulation and solution method for this inverse problem are proposed.

Keywords: The Caputo fractional derivative, monotonicity in the parameter of the Mittag-Leffler function, inverse problems.

MSC (2020): 35R11

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain, with sufficiently smooth boundary $\partial\Omega$, $\rho, \sigma \in (0,1)$. Consider the following problem

$$\begin{cases} D_t^\rho u(x,t) + (-\Delta)^\sigma u(x,t) = 0, \quad x \in \Omega,\ 0 < t \le T, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ 0 < t \le T, \\ u(x,0) = \varphi(x), \quad x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\varphi(x) \in L_2(\Omega)$, Δ is the Laplace operator and D_t^ρ is the fractional Caputo derivative of order $0 < \rho < 1$, which for any continuous function $h(t)$ is defined as (see e.g. [12], p. 71)

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dt} \int_0^t \frac{h(\xi) - h(0)}{(t-\xi)^\rho}\, d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\rho)$ is Euler's gamma function. We call problem (1.1) the forward problem.

Fractional diffusion equations are widely applied in fields such as anomalous diffusion, elasticity, finance, and biology [8, 10, 14]. Determining the fractional order is difficult due to the lack of direct measurement tools, necessitating the solution of inverse problems based on indirect data [13].

Z. Li et al. [13] emphasized solving inverse problems using solution values over fixed time intervals. Subsequent works ([1, 2, 3, 4, 5]) introduced additional conditions, such as time-integrals of the solution, to prove uniqueness and existence, relying on the monotonicity of Mittag-Leffler functions.

The inverse problem involving a RiemannLiouville fractional derivative in equation (1.1) remains unstudied, though many results exist for the Caputo case. For example, Tatar and Ulusoy [15] (2013) proved uniqueness for a two-parameter inverse problem on $(-1,1)$, but under strict initial conditions, requiring all Fourier coefficients to be positive.

In contrast, Yamamoto [16] (2020) addressed a similar problem on a bounded domain $\Omega \subset \mathbb{R}^N$, proving uniqueness under weaker assumptions ($\varphi(x) \ge 0$) with an additional condition $u(x_0,t) = \psi(t)$.

In [7], the problem was studied on $\mathbb{R}^N$ with a general elliptic operator (not just Laplacian). The authors established both existence and uniqueness without restrictive assumptions on the initial function, requiring only smoothness ($\varphi \in L_2^\tau(\mathbb{R}^N), \tau > N/2$). Their approach leveraged properties of the continuous spectrum, particularly when $\lambda = 1$ is in the spectrum or an eigenvalue.

In this paper, we investigate the initial-boundary value problem (1.1), in which the parameters ρ and σ are assumed to be unknown. The primary objective of this study is the simultaneous determination of these two parameters. This type of problem is commonly referred to as a *two-parameter inverse problem*, and it arises naturally in various physical and engineering contexts where model coefficients are not directly measurable.

We use the following additional conditions for simultaneously determining the parameters ρ and σ:

$$|(u(x,t_0),v_k(x))| = d_0, \quad |(u(x,t_1),v_k(x))| = d_1, \tag{1.2}$$

where $t_0 \neq t_1,\ t_0,\ t_1 \in (0,T]$ and $d_0,\ d_1$ are given positive numbers. Here, the symbol $v_k(x)$ denotes the k-th eigenfunction of the Laplace operator with the Dirichlet boundary condition, corresponding to the eigenvalue $\lambda_k > 1$.

2. Preliminaries

Let Ω be a bounded N-dimentional domain with a sufficiently smooth boundary $\partial\Omega$ and $\{v_k(x)\}$ denote the complete system of orthonormal in $L_2(\Omega)$ eigenfunctions and $\{\lambda_k\}$ the set of positive eigenvalues of the spectral problem:

$$\begin{cases} -\Delta v(x) = \lambda v(x), & x \in \Omega, \\ v(x)\big|_{\partial\Omega} = 0. \end{cases}$$

Let us present some assertions about eigenfunctions $v_k(x)$ and eigenvalues λ_k proved by V.A.Ilyin [11].

Lemma 2.1. *The series* $\sum\limits_{k=1}^{\infty} \lambda_k^{-\left([\frac{N}{2}]+1\right)} v_k^2(x)$ *converges uniformly in a closed domain* $\overline{\Omega}$.

Lemma 2.2. *Let the function* $g(x)$ *satisfy the conditions*

(1) $g(x) \in C^p(\overline{\Omega}),\ \frac{\partial^{p+1} g(x)}{\partial x_1^{p_1} \dots \partial x_n^{p_n}} \in L_2(\Omega),\ p+1 = p_1 + p_2 + \dots p_n, p \geq 1,$

(2) $g(x)|_{\partial\Omega} = \Delta g(x)|_{\partial\Omega} = \dots = \Delta^{\left[\frac{p}{2}\right]} g(x)|_{\partial\Omega} = 0.$

Then the number series $\sum\limits_{k=1}^{\infty} g_k^2 \lambda_k^{p+1}$ *converges, where* $g_k = (g, v_k)$.

Definition 2.3. For $0 < \rho < 1$, let $E_{\rho,\mu}(z)$ denote the Mittag-Leffler function defined as:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad \mu, z \in \mathbb{C}.$$

If $\mu = 1$, then the Mittag-Leffler function is called the one-parameter or classical Mittag-Leffler function and is denoted by $E_\rho(z) = E_{\rho,1}(z)$.

Recall the following estimate of the Mittag-Leffler functions (see, e.g. [9], p. 29).

Lemma 2.4. *For any* $t \geq 0$ *one has*

$$|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad \mu \in \mathbb{C},$$

where constant C *does not depend on* t.

Let us represent by $\delta(1;\beta)$ a contour oriented by non-decreasing $\arg\xi$ consisting of the following parts: the ray $\arg\xi = -\beta$ with $|\xi| \geq 1$, the arc $-\beta \leq \arg\xi \leq \beta$, $|\xi| = 1$, and the ray $\arg\xi = \beta$, $|\xi| \geq 1$. The counter $\delta(1;\beta)$ is called Hankel path.

Let $\beta = \frac{3\pi}{4}\rho$, $\rho \in [\rho_0, 1)$ and $\sigma > 0$. Then, we write function $E_\rho(-\lambda^\sigma t^\rho)$ in the following form (see [9], p. 27):

$$E_\rho(-\lambda^\sigma t^\rho) = p(\rho,\sigma,t) + q(\rho,\sigma,t),$$

where

$$p(\rho,\sigma,t) = \frac{1}{\Gamma(1-\rho)\lambda^\sigma t^\rho},$$

$$q(\rho,\sigma,t) = -\frac{1}{2\pi i \lambda^\sigma \rho t^\rho} \int_{\delta(1;\beta)} \frac{e^{\xi^{1/\rho}}\xi}{\xi + \lambda^\sigma t^\rho} d\xi.$$

Lemma 2.5. *Let $\sigma \in (0,1)$ and $0 < \rho_0 < 1$. Then there exists a number $T_0 = T_0(\lambda, \rho_0)$ such that for all $t \geq T_0$ and $\lambda \geq \lambda_k$, function $E_\rho(-\lambda^\sigma t^\rho)$ is positive and monotonically decreasing for all $\rho \in [\rho_0, 1)$ and the following estimates hold:*

$$\frac{\partial}{\partial \rho} p(\rho, \sigma, t) \leq -\frac{1}{\lambda^\sigma t^\rho}, \quad t > 1, \quad \lambda > 0,$$

$$\left| \frac{\partial}{\partial \rho} q(\rho, \sigma, t) \right| \leq C \frac{1/\rho + \ln t}{(\lambda^\sigma t^\rho)^2}, \quad t > 1, \quad \lambda > 0.$$

This Lemma was proven in [6].

Lemma 2.6. *There is a constant $C > 0$, such that*

$$\left| \frac{\partial}{\partial \sigma} q(\rho, \sigma, t) \right| < \frac{C |\ln \lambda|}{\lambda^{2\sigma} t^\rho}, \quad t > 1, \ \lambda > 0 \ (\lambda \neq 1).$$

This Lemma was proven in [6].

3. Solution of the forward problem

Firstly, we present the definition of the solution of the problem (1.1).

Definition 3.1. A function $u(x,t)$ with the properties

(1) $u(x,t) \in C(\overline{\Omega} \times [0,T])$,

(2) $D_t^\rho u(x,t), (-\Delta)^\sigma u(x,t) \in C(\overline{\Omega} \times (0,T])$,

and satisfying conditions (1.1) is called **the solution** of the problem (1.1).

Now we present theorem about the solution of problem (1.1).

Theorem 3.2. *Let function $\varphi(x)$ satisfy conditions of Lemma 2.2 with exponent $p = \left[\frac{n}{2}\right]$. Then problem (1.1) has a unique solution:*

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k E_\rho(-\lambda_k^\sigma t^\rho) v_k(x), \tag{3.1}$$

where φ_k are the Fourier coefficients of function $\varphi(x)$.

Proof. Assume that a solution of the forward problem exists. Since system $\{v_k(x)\}$ is complete in $L_2(\Omega)$, then this solution has the form:

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) v_k(x),$$

where $T_k(t) = (u(x,t), v_k(x))$ are the Fourier coefficients of the function $u(x,t)$ and are unknown.

Multiply the orthonormal eigenfunctions $v_k(x)$ to the equation in the problem (1.1), to get

$$(D_t^\rho u(x,t), v_k(x)) + ((-\Delta)^\sigma u(x,t), v_k(x)) = 0.$$

We have $(D_t^\rho u(x,t), v_k(x)) = D_t^\rho T_k(t)$. Since the Laplace operator is self-adjoint, it follows that $((-\Delta)^\sigma u(x,t), v_k(x)) = \lambda_k^\sigma T_k(t)$. Therefore, to determine $T_k(t)$ we obtain the following Cauchy problem

$$D_t^\rho T_k(t) + \lambda_k^\sigma T_k(t) = 0, \qquad T_k(0) = \varphi_k.$$

This problem has a unique solution (see, for example, [9], p. 174):

$$T_k(t) = \varphi_k E_\rho(-\lambda_k^\sigma t^\rho). \tag{3.2}$$

From this, in particular, it follows that if a solution to the forward problem exists, then it is unique. Indeed, for this it is sufficient to prove that the solution $u(x,t)$ to the forward problem with the homogeneous condition (1.1) is identically zero. But from (3.2) it follows that $T_k(t) \equiv 0$ for all $k \geq 1$. Taking into account the definition of $T_k(t)$ and the completeness of the system $\{v_k(x)\}$, we obtain $u(x,t) \equiv 0$.

Let us show that the operators $(-\Delta)^\sigma$ and D_t^ρ can be applied term-by-term to series (3.1) and the resulting series converges uniformly in $(x,t) \in (\overline{\Omega} \times (0,T])$:

$$(-\Delta)^\sigma u(x,t) = \sum_{k=1}^{\infty} \lambda_k^\sigma \varphi_k E_\rho(-\lambda_k^\sigma t^\rho) v_k(x).$$

Using Lemma 2.4 and applying the Cauchy-Bunyakovsky inequality we get

$$|(-\Delta)^\sigma u(x,t)| \leq \sum_{k=1}^{\infty} |\lambda_k^\sigma \varphi_k E_\rho(-\lambda_k^\sigma t^\rho) v_k(x)| \leq \sum_{k=1}^{\infty} \lambda_k^\sigma |\varphi_k||v_k(x)| \frac{C}{1+\lambda_k^\sigma t^\rho} \leq$$

$$\leq Ct^{-\rho} \sum_{k=1}^{\infty} |\varphi_k||v_k(x)| = Ct^{-\rho} \sum_{k=1}^{\infty} |\varphi_k| (\sqrt{\lambda_k})^{\left(\left[\frac{n}{2}\right]+1\right)} (\sqrt{\lambda_k})^{-\left(\left[\frac{n}{2}\right]+1\right)} |v_k(x)| \leq$$

$$\leq Ct^{-\rho} \left(\sum_{k=1}^{\infty} |\varphi_k|^2 \lambda_k^{\left(\left[\frac{n}{2}\right]+1\right)} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \lambda_k^{-\left(\left[\frac{n}{2}\right]+1\right)} v_k^2(x) \right)^{\frac{1}{2}}$$

Therefore if $\varphi(x)$ satisfies conditions of Lemma 2.2, with the exponent $p = \left[\frac{n}{2}\right]$, then series $\sum\limits_{k=1}^{\infty} |\varphi_k|^2 \lambda_k^{\left(\left[\frac{n}{2}\right]+1\right)}$ is converge. Additionally, according to Lemma 2.1 the series $\sum\limits_{k=1}^{\infty} \lambda_k^{-\left(\left[\frac{n}{2}\right]+1\right)} v_k^2(x)$ is converge uniformly in a closed domain $\overline{\Omega}$. Hence, $(-\Delta)^\sigma u(x,t) \in C(\overline{\Omega} \times (0,T])$.

From equation (1.1) one has $D_t^\rho u(x,t) = -(-\Delta)^\sigma u(x,t)$, $t > 0$, and hence we get $D_t^\rho u(x,t) \in C(\overline{\Omega} \times (0,T])$.

The uniqueness of the solution is proved in the standard way (see, for example, [3]). $\square$

4. Solution of the two-parameter inverse problem

Now we turn to the study of the inverse problem of simultaneous determination of the orders of the fractional derivative ρ and the degree of the Laplace operator σ.

Using the additional conditions (1.2) and the solution (3.1) we derive the following system of equations:

$$\begin{cases} F_1(\rho,\sigma) = E_\rho(-\lambda_k^\sigma t_0^\rho) - \dfrac{d_0}{|\varphi_1|} = 0, \\ F_2(\rho,\sigma) = E_\rho(-\lambda_k^\sigma t_1^\rho) - \dfrac{d_1}{|\varphi_1|} = 0. \end{cases} \tag{4.1}$$

We introduce the following notation:

$$F(\rho,\sigma) = \begin{bmatrix} E_\rho\left(-\lambda_k^\sigma t_0^\rho\right) \\ E_\rho\left(-\lambda_k^\sigma t_1^\rho\right) \end{bmatrix},$$

where $t_1, t_2 > 0$, $t_1 \neq t_2$, $\sigma \in (0,1)$, and $\rho \in [\rho_0, 1]$ with $\rho_0 > 0$.

Suppose that $\frac{d_0}{|\varphi_1|}$ and $\frac{d_1}{|\varphi_1|}$ belong to the range of the function $F(\rho,\sigma)$. Then, the following theorem holds:

Theorem 4.1. *Let $\lambda_k > 1$ be the k-th eigenvalue of the Laplace operator. Then, there exists a positive number $T_1 = T_1(\lambda_k, \rho_0) > 0$ such that for all $t_1 > T_1$ and $t_2 > T_1$, the two-parameter inverse problem* (1.1)(1.2) *has a unique solution $\{u(x,t), \rho, \sigma\}$.*

Proof. First, we consider the partial derivative $\frac{\partial F_1}{\partial \sigma}$ of the function $F_1(\rho,\sigma)$ in the system of equations (4.8). It holds that $\frac{\partial F_1}{\partial \sigma} \neq 0$ for all $\rho \in [\rho_0, 1)$ and $\sigma \in (0,1)$ (see [6]). Then, by the implicit function theorem, there exists a function $\sigma = g(d_0,\rho)$ such that the equation

$$F_1(\rho, g(d_0,\rho)) = E_\rho\left(-\lambda_k^{g(d_0,\rho)} t_0^\rho\right) - \frac{d_0}{|\varphi_1|} = 0,$$

is satisfied.

From this, we obtain the equation

$$f(\rho) = F_2(\rho, g(d_0,\rho)) = E_\rho\left(-\lambda_k^{g(d_0,\rho)} t_1^\rho\right) - \frac{d_1}{|\varphi_1|} = 0.$$

We now compute the derivative $\frac{\partial f}{\partial \rho}$:

$$\frac{\partial f}{\partial \rho} = \frac{\partial}{\partial \rho} E_\rho\left(-\lambda_k^{g(d_0,\rho)} t_1^\rho\right) = \frac{\partial}{\partial \rho} E_\rho\left(-(q(d_0,\rho))^\rho\right),$$

where $q(d_0,\rho) = \lambda_k^{\frac{g(d_0,\rho)}{\rho}} t_1$. Then, using the chain rule, we have

$$\frac{\partial f}{\partial \rho} = A\frac{\partial q(d_0,\rho)}{\partial \rho},$$

where

$$A = \frac{\partial}{\partial \rho} E_\rho\left(-\lambda_k^\sigma t_1^\rho\right)\Big|_{\sigma=g(d_0,\rho)},$$

and

$$\frac{\partial q(d_0,\rho)}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\lambda_k^{\frac{g(d_0,\rho)}{\rho}} t_1\right) = t_1 \lambda_k^{\frac{g(d_0,\rho)}{\rho}} \frac{\ln \lambda_k}{\rho}\left(\frac{\partial g(d_0,\rho)}{\partial \rho} - \frac{g(d_0,\rho)}{\rho}\right).$$

Thus, we obtain

$$\frac{\partial f}{\partial \rho} = t_1 \lambda_k^{\frac{g(d_0,\rho)}{\rho}} \frac{\ln \lambda_k}{\rho} A\left(\frac{\partial g(d_0,\rho)}{\partial \rho} - \frac{g(d_0,\rho)}{\rho}\right).$$

The derivative of the implicit function $g(d_0,\rho)$ is given by

$$\frac{\partial g(d_0,\rho)}{\partial \rho} = -\frac{\frac{\partial}{\partial \rho} E_\rho\left(-\lambda_k^\sigma t_0^\rho\right)}{\frac{\partial}{\partial \sigma} E_\rho\left(-\lambda_k^\sigma t_0^\rho\right)}\Bigg|_{\sigma=g(d_0,\rho)}.$$

Since the following inequalities hold for all $\rho \in [\rho_0,1)$ and $\sigma \in (0,1)$ (see [6]):

$$\frac{\partial}{\partial \rho} E_\rho\left(-\lambda_k^\sigma t_1^\rho\right) < 0, \quad \frac{\partial}{\partial \sigma} E_\rho\left(-\lambda_k^\sigma t_0^\rho\right) < 0, \quad \frac{\partial}{\partial \rho} E_\rho\left(-\lambda_k^\sigma t_0^\rho\right) < 0,$$

it follows that $\frac{\partial f}{\partial \rho} > 0$ for all $\rho \in [\rho_0,1)$ and $\sigma \in (0,1)$. Therefore, the function $F_2(\rho, g(d_0,\rho))$ is strictly increasing with respect to ρ.

As a result, there exists a unique value of ρ that satisfies the equation $F_2(\rho, g(d_0,\rho)) = 0$.

□

Acknowledgement

The author are grateful to R. R. Ashurov and Z.A. Sobirov for discussions of these results.

The author acknowledge financial support from the Innovative Development Agency under the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan, Grant No F-FA-2021-424.

References

[1] Alimov Sh.A., Ashurov R.R.; Inverse problem of determining an order of the Riemann-Liouville time-fractional derivative. Progr. Fract. Differ. Appl.–2022.-8.–P.1-8.

[2] Alimov Sh.A., Ashurov R.R.; Inverse problem of determining an order of the Caputo time- fractional derivative for a subdiffusion equation. Journal Inverse Ill-Posed Problem.–2020.-28.-No5.–P.651-658.

[3] Ashurov R.R., Umarov S.R.; Determination of the order of farctional derivative for subdiffusion equations. Fractional Calculus and Applied Analysis.–2020.-23.-No6.–P.1647-1662.

[4] Ashurov R.R., Fayziev Yu. E.; Determination of fractional order and source term in a fractional subdiffusion equation. Eurasian Mathematical Journal. European Mathematical Journal,–2022.-13.-No1.–P.19-31.

[5] Ashurov R.R., Fayziev Yu. E.; Determination of fractional order and source term in a fractional subdiffusion equation. Lobachevski Journal of Mathematics,–2021.-42.-No3.–P.508-516.

[6] Ashurov R.R., Sulaymonov I.A.; Determining the order of time and spatial fractional derivatives. Math. Meth. Appl. Sci.–2024.-48.-No2.–P.1503-1518.

[7] Ashurov R.R., Zunnunov R. T.; Initial-boundary value and inverse problems for subdiffusion equation in R^N. Fractional differential calculus,–2020.-10.–P.291-306.

[8] Dipierro S., Palatucci G., Valdinoci E.; Dislocation dynamics in crystals: a macroscopic theory in a fractional laplace setting. Communications in Mathematical Physics.–2015.-333.-No2.–P.10611105.

[9] Gorenflo R., Kilbas A.A., Mainardi F., Rogozin S.V.; Mittag-Leffler functions, related topics and applications. Springer.–2014.

[10] Hilfer R.; Applications of Fractional Calculus in Physics. World Scientific.–2000.

[11] Ilyin V.; On the solvability of mixed problems for hyperbolic and parabolic equations. Russian Math. Surveys.–1960.-15.-No2.–P.97-154.

[12] Kilbas A., Srivastava H., Trujillo J., Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam.–2006.

[13] Li Z., Liu Y., Yamamoto M.; Inverse problems of determining parameters of the fractional partial differential equations, ed. by A. Kochubei and Yu. Luchko. Handbook of fractional calculus with applications, DeGruyter, Berlin.–2019.-vol2.–P.519.

[14] Meerschaert M., Benson D., Scheffler H., Baeumer B.; Stochastic solution of space-time fractional diffusion equations. Phys. Rev. E.–2002.-65.

[15] Tatar S., Ulusoy S.; A uniqueness result for an inverse problem in a space-time fractional diffusion equation. Electronic Journal of Differential Equations,–2013.-258.–P.1-9.

[16] Yamamoto M.; Uniqueness determining the order of time and spatial fractional derivatives. arXiv:2006, 15046v1[math. AP],–2020.–P.1-10.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.287-299
DOI: 10.29229/uzmj.2025-2-28

Nonadditive entropies and statistical mechanics - some open mathematical problems

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. Textbook statistical mechanics is grounded on the additive Boltzmann-Gibbs-von Neumann-Shannon entropic functional $S_{BG} = k\sum_{i=1}^{W} p_i \ln\frac{1}{p_i}$. The possibility of generalizing the entire BG theory was advanced in 1988 by generalizing S_{BG} into nonadditive entropic functionals such as $S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}$ $(q \in \mathbb{R})$ and, later on, such as $S_\delta = k\sum_{i=1}^{W} p_i\left[\ln\frac{1}{p_i}\right]^\delta$ $(\delta \in \mathbb{R}_+)$. A whole saga followed through successful applications in physics, mathematics, chemistry, engineering, medicine, computational sciences, economics, linguistics, theory of networks, to name but a few. Along this evolution, more than fifty nonadditive entropic functionals have been introduced and studied in the literature. Very few, however, have proved to be neatly useful in natural sciences. The present status of the various entropic functionals that have been advanced for complex systems is briefly reviewed, a function generalizing the q-logarithmic one is defined, and some open mathematical problems are focused on.

Keywords: Entropy, Thermodynamics, Statistical Mechanics, Nonlinear Dynamical Systems, Cosmology, Quantum entanglement, Optimization.
MSC (2020): 28D20, 82B03.

1. Introduction

Boltzmann-Gibbs (BG) statistical mechanics constitutes, together with Maxwell electromagnetism and Newtonian, Einstein and quantum mechanics, a pillar of contemporary theoretical physics. This theory was introduced in the 19th century [10, 20] and it is based on mechanics, electromagnetism, and theory of probabilities. It satisfactorily applies to myriads of physical systems. It is based on the Boltzmann-Gibbs-von Neumann-Shannon (BG) entropic functional, whose simplest (discrete) expression is the following one:

$$S_{BG} = -k\sum_{i=1}^{W} p_i \ln p_i\,, \tag{1.1}$$

with

$$\sum_{i=1}^{W} p_i = 1\,. \tag{1.2}$$

W is the total number of nonvanishing-probability microscopic possibilities consistent with the available macroscopic information on the system, and k is a positive conventional unit chosen once for ever (it is usually chosen $k = k_B \equiv$ *Boltzmann constant* in physics, and $k=1$ in computational sciences). For equal probabilities (currently referred to as the *microcanonical ensemble*) we verify the celebrated Boltzmann formula

$$S_{BG} = k\ln W\,, \tag{1.3}$$

a genius relation connecting the microscopic and the macroscopic worlds, graved on stone at Boltzmann grave in Vienna!

However, BG statistical mechanics exhibits failures for complex systems such as long-range-interacting Hamiltonians. A more powerful, i.e. more general, theory becomes then a must. In 1988 [45], a generalization of the BG theory was proposed grounded on the following entropic functional

$$S_q^T = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \quad (q \in \mathbb{R};\ S_1 = S_{BG}) \tag{1.4}$$

For equal probabilities we verify

$$S_q^T = k \ln_q^T W \,, \tag{1.5}$$

where the q-logarithmic function is defined as follows:

$$\ln_q^T z = \frac{z^{1-q} - 1}{1-q} \quad (\ln_1^T z = \ln z)\,. \tag{1.6}$$

The validity of this proposal has, by now, been legitimated in uncountable theoretical, computational, experimental and observational instances (see, e.g., the Bibliography available at [43], as well as [13, 46, 49, 50]). Among those, let us mention the following ones: cold atoms in dissipative optical lattices [16, 27], granular matter [15], high-energy collisions of elementary particles at LHC/CERN and elsewhere [63], quantum tunneling in ion-molecule chemical reaction [62, 7], cosmology [54].

2. Entropic Nonadditivity and nonextensivity

An entropic functional $S(\{p_i\})$ is said *additive*[31] if, for probabilistically independent subsystems A and B (i.e., $p_{ij}^{A+B} = p_i^A p_j^B$), it satisfies

$$S(A+B) = S(A) + S(B)\,. \tag{2.1}$$

Therefore, S_{BG} is additive, whereas S_q^T is nonadditive for any $q \neq 1$. Indeed, we straightforwardly verify

$$\frac{S_q^T(A+B)}{k} = \frac{S_q^T(A)}{k} + \frac{S_q^T(B)}{k} + (1-q)\frac{S_q^T(A)}{k}\frac{S_q^T(B)}{k}\,. \tag{2.2}$$

Let us also mention that S_q^T is related with the additive Renyi entropic functional defined as follows [32]:

$$S_q^R = k\frac{\ln \sum_{i=1}^W p_i^q}{1-q}\,. \tag{2.3}$$

Its maximal value (i.e., corresponding to equal probabilities) is given by

$$S_q^R = k \ln W \quad (\forall q)\,, \tag{2.4}$$

hence, a whole family of different entropic functionals share the same maximal value.

The Renyi entropic functional appears to be (though no proof is available as far as we know) the most general additive one. Indeed, it satisfies

$$S_q^R(A+B) = S_q^R(A) + S_q^R(B) \quad (\forall q)\,. \tag{2.5}$$

It is related to Eq. (1.4) as follows:

$$\frac{S_q^R}{k} = \frac{\ln\Big[1 + (1-q)S_q^T/k\Big]}{1-q} \tag{2.6}$$

The asymptotic functional form of $W(N)$, (N is the number of elements in the system) in the thermodynamical limit $N \to \infty$ basically determines (not necessarily univocally) the specific entropic form to be used in order to have a thermodynamically *extensive* entropy, i.e., $S(N) \propto N$ ($N \to \infty$), hence $\lim_{N\to\infty} \frac{S(N}{N} < \infty$, a property which is mandated by the Legendre structure of classical thermodynamics.

As a first example, let us consider a system satisfying, for equal probabilities, $W(N) \propto \mu^N$ ($\mu > 1$) (*exponential* family). It is straightforward to verify that Eq. (1.3) implies $S_{BG}(N) \propto N$, thus being extensive for such systems. The entropy S_q^R also satisfies, $\forall q$, $S_q^R(N) \propto N$ for the exponential family.

As a second example, let us consider now a system satisfying, for equal probabilities, $W(N) \propto N^\rho$ ($\rho > 0$) (*power-law* family), which is the case of very many so-called complex systems. It is straightforward to verify that Eq. (1.4) implies, for $q = 1 - 1/\rho$, $S_q^T(N) \propto N$, thus being extensive for such systems. We can see that, for this family, $S_{BG}(N) \propto \ln N \neq N$, being therefore thermodynamically inadmissible.

Let us consider now a third example, namely $W(N) \propto \nu^{N^\gamma}$ $(\nu > 1; \gamma < 1)$ (*stretched-exponential* family). We can easily verify that no value of q exists for which we could have $S_q^T(N) \propto N$. We need consequently a new entropic functional. Around 2007 (published in [49]), we introduced a new form of (nonadditive) functional, namely

$$S_\delta^T = k\sum_{i=1}^{W} p_i \Big[\ln \frac{1}{p_i}\Big]^\delta = k\sum_{i=1}^{W} p_i \ln_\delta \frac{1}{p_i} \;\; (\delta \in \mathbb{R}_+)\,, \tag{2.7}$$

where the δ-logarithmic function is defined as follows:

$$\ln_\delta^T z = (\ln z)^\delta\,. \tag{2.8}$$

For equal probabilities we verify

$$S_\delta^T = k \ln_\delta^T W = k(\ln W)^\delta\,. \tag{2.9}$$

For the stretched-exponential family we can verify that, for $\delta = 1/\gamma$, $S_\delta(N) \propto N$, thus satisfying classical thermodynamics. This same form was also proposed by Ubriaco [57] as a mathematical possibility. The form defined in Eq. (2.7) is not unique: other entropic functionals have been introduced in the literature which also are appropriate for this same family [2, 21, 22]. The Hanel-Thurner entropic functional $S_{c,d}^{HT}$ unifies in fact a variety of functionals available in the literature. In addition to these, one more functional is also available in the literature which works out correctly for the equal probability stretched-exponential family, namely [42, 54]

$$S_{\alpha,\delta}^{TTJ} = k(S_\alpha^R/k)^\delta = k\Big(\frac{\ln\sum_{i=1}^{W} p_i^\alpha}{1-\alpha}\Big)^\delta\,. \tag{2.10}$$

For equal probabilities, this functional yields

$$S_{\alpha,\delta}^{TTJ} = k(\ln W)^\delta\,, \tag{2.11}$$

which, interestingly enough, coincides with Eq. (2.9)! Functionals (2.7) and (2.10) are definitively different but they share the same maximal value. To review some interesting connections of $S_{\alpha,\delta}^{TTJ}$, we refer to Fig. 6.

To give some overall view of these various entropic functionals, let us mention that Sharma and Mittal [37] introduced, as a mathematical possibility, the following one:

$$S_{q,r}^{SM} = \frac{k}{1-r}\Big[\Big(\sum_{i=1}^{W} p_i^q\Big)^{\frac{1-q}{1-r}} - 1\Big] = \frac{1}{1-r}\Big[\Big(1+(1-q)S_q^T/k\Big)^{\frac{1-q}{1-r}} - 1\Big],\;\; [(q,r)\in\mathbb{R}^2]\,. \tag{2.12}$$

It unifies S_q^R and S_q^T as indicated in Fig. 7. Another unification was introduced in [53], namely

$$S_{q,\delta}^{TC} = k\sum_{i=1}^{W} p_i\Big[\ln_q \frac{1}{p_i}\Big]^\delta\,, \tag{2.13}$$

which can be seen in Fig. 8.

There are other interesting unifications in the literature, such as the Borges-Roditi entropic functional $S_{q,q'}^{BR}$ [11], which unifies the Kaniadakis S_κ^K [26] and the Abe S_q^A [3] ones.

We have reviewed the above functionals and connections to illustrate the present plethoric scenario. More details can be found in [50]. The reader should nevertheless keep in mind that very few (e.g., S_q^T, S_δ^T, S_κ^K, S_q^R) of those many functionals have already been shown to be useful for the concrete advancement of natural sciences.

3. Trace-form and Composability

Existing entropic functionals exhibit a variety of properties (positivity, concavity or convexity, additivity or nonadditivity, extensivity or nonextensivity for specific classes of systems, among others). We discuss here two basic ones, namely *trace-form* and *composability*.

$$
\begin{tikzcd}[column sep=small]
 & \boxed{\begin{array}{l} S^{TTJ}_{\alpha,\delta} = k(S^R_\alpha/k)^\delta = k\left(\frac{\ln\sum_{i=1}^W p_i^\alpha}{1-\alpha}\right)^\delta \\ S^{TTJ}_{\alpha,\delta} = k\Big[\ln W\Big]^\delta \ (\text{if } p_i = 1/W) \end{array}} \arrow[dl, "\alpha = 1"'] \arrow[dr, "\delta = 1"] & \\
\boxed{\begin{array}{lcl} k[S_{BG}/k]^\delta & = & k\Big[\sum_{i=1}^W p_i \ln\frac{1}{p_i}\Big]^\delta \\ k[S_{BG}/k]^\delta & = & k\Big[\ln W\Big]^\delta \ (\text{if } p_i = 1/W) \end{array}} \arrow[dr, "\delta = 1"'] & & \boxed{\begin{array}{lcl} S^R_\alpha & = & k\frac{\ln\sum_{i=1}^W p_i^\alpha}{1-\alpha} \\ S^R_\alpha & = & k\ln W \ (\text{if } p_i = 1/W) \end{array}} \arrow[dl, "\alpha = 1"] \\
 & \boxed{\begin{array}{lcl} S_{BG} & = & -k\sum_{i=1}^W p_i \ln p_i \\ S_{BG} & = & k\ln W \ (\text{if } p_i = 1/W) \end{array}} &
\end{tikzcd}
$$

FIGURE 6. The (nonadditive) entropic functional $S^{TTJ}_{\alpha,\delta}$ unifies the (additive) S^R_α and the (nonadditive) $k[S_{BG}/k]^\delta$ functionals.

$$
\begin{tikzcd}[column sep=small]
 & \boxed{\begin{array}{l} S^{SM}_{q,r} = \frac{k}{1-r}\left[\left(\sum_{i=1}^W p_i^q\right)^{\frac{1-q}{1-r}} - 1\right] = \frac{1}{1-r}\left[\left(1 + (1-q)S^T_q/k\right)^{\frac{1-q}{1-r}} - 1\right] \\ S^{SM}_{q,r} = \frac{k}{1-r}\left[\left(1 + (1-q)\ln_q W\right)^{\frac{1-q}{1-r}} - 1\right] \ (\text{if } p_i = 1/W) \end{array}} \arrow[dl, "q = r"'] \arrow[dr, "r = 1"] & \\
\boxed{\begin{array}{l} S^T_q = k\frac{1-\sum_{i=1}^W p_i^q}{q-1} = k\sum_{i=1}^W p_i \ln_q \frac{1}{p_i} \\ S^T_q = k\ln_q W \ (\text{if } p_i = 1/W) \end{array}} \arrow[dr, "q = 1"'] & & \boxed{\begin{array}{l} S^R_q = k\frac{\ln\sum_{i=1}^W p_i^q}{1-q} \\ S^R_q = k\ln W \ (\text{if } p_i = 1/W) \end{array}} \arrow[dl, "q = 1"] \\
 & \boxed{\begin{array}{lcl} S_{BG} & = & -k\sum_{i=1}^W p_i \ln p_i \\ S_{BG} & = & k\ln W \ (\text{if } p_i = 1/W) \end{array}} &
\end{tikzcd}
$$

FIGURE 7. The Sharma-Mittal (nonadditive) entropic functional unifies the (additive) S^R_q and the (nonadditive) S^T_q.

3.1. **Trace-form.** A generalized entropic functional is said of the trace-form if it can be expressed as

$$S_G(\{p_i\}) = k\sum_{i=1}^W f(p_i) = k\sum_{i=1}^W p_i \frac{f(p_i)}{p_i} = k\sum_{i=1}^W p_i \ln_G \frac{1}{p_i} = k\Big\langle \ln_G \frac{1}{p_i}\Big\rangle, \tag{3.1}$$

where $f(z)$ is some appropriate smooth positive function which enables the definition of the function $\ln_G(z)$ through $\ln_G \frac{1}{z} = \frac{f(z)}{z}$. The function $\ln_G(z)$ is expected to be a monotonically increasing one between zero for $z = 1$ and its maximal value for $z \to \infty$. For S_{BG} we have $\ln_G(z) = \ln z$; for S^T_q we have $\ln_G(z) = \ln^T_q(z)$; for S^T_δ we have $\ln_G(z) = \ln^T_\delta(z)$, and so on.

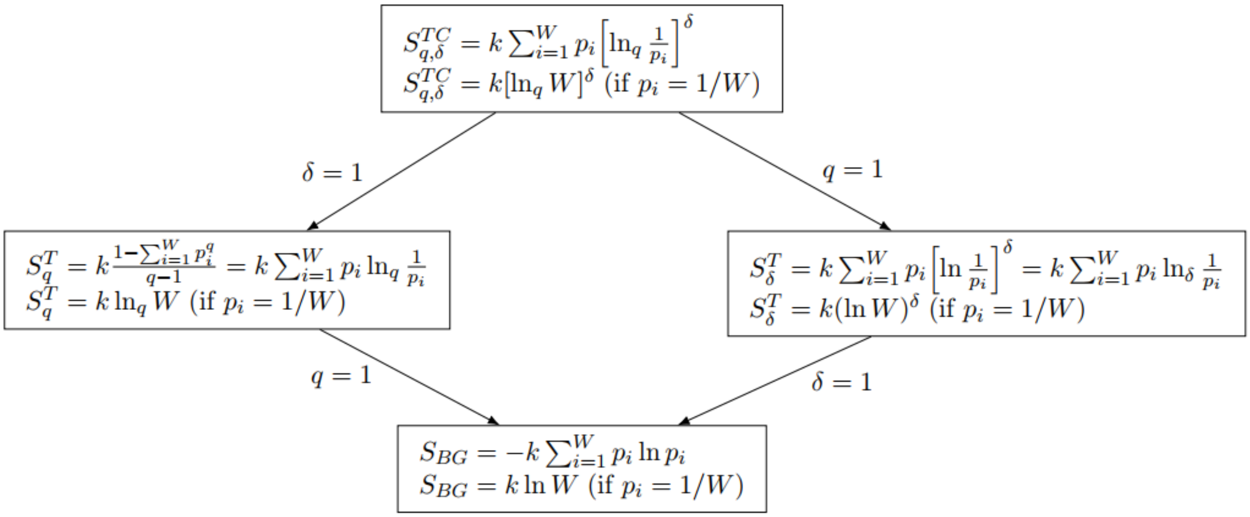

FIGURE 8. The $\ln_q z$ and $\ln_\delta z$ functions are respectively defined in Eqs. (1.6) and (2.8). Notice that, for $p_i = 1/W$, both S^T_δ and $S^{TTJ}_{1,\delta}$ lead to one and the same expression, namely $k(\ln W)^\delta$.

.

As we see, the trace-form guarantees that the entropic functional can be seen as the standard mean value of some simple function, namely of $\ln_G \frac{1}{p_i}$, referred to as *G-surprise* or *G-unexpectedness* [61, 4]. S_{BG}, S^T_q and S^T_δ are of the trace-fom; S^R_q and $S^{TTJ}_{\alpha,\delta}$ are not.

3.2. **Composability.** Composability is inspired by thermodynamics. A generalized entropic functional $S(\{p_i\}])$ is said composable [47, 48, 50] if $S(A+B)/k$ corresponding to a system composed of two *probabilistically independent* subsystems A and B (i.e., $p^{A+B}_{ij} = p^A_i p^B_j$) can be expressed in the form

$$S(A+B)/k = \Phi\Big(S(A)/k, S(B)/k; \{\eta\}\Big)\,, \tag{3.2}$$

where $\Phi(x,y;\{\eta\})$ is a smooth function of (x,y) which depends on a (typically small) set of universal entropic indices $\{\eta\}$ defined in such a way that $\Phi(x,y;\{0\}) = x+y$ (*additivity*), and which satisfies

(i) $\Phi(x,0;\{\eta\}) = x$ (*null-composability*),
(ii) $\Phi(x,y;\{\eta\}) = \Phi(y,x;\{\eta\})$ (*symmetry*),
(iii) $\Phi(x,\Phi(y,z;\{\eta\});\{\eta\}) = \Phi(\Phi(x,y;\{\eta\}),z;\{\eta\})$ (*associativity*).

For example, for S^T_q, we have that $\Phi(x,y;\{\eta\}) = x+y+(1-q)xy$ and, for $S^{TTJ}_{\alpha,\delta}$, we have that $\Phi(x,y;\{\eta\}) = \left[x^{\frac{1}{\delta}} + y^{\frac{1}{\delta}}\right]^{\delta}$.

For thermodynamical systems, associativity appears to be consistent with the $0th$ Principle of Thermodynamics. Ultimately, the whole concept of composability is constructed upon the requirement that the entropic functional $S(A+B)$ does *not* depend on the microscopic configurations of A and of B. Equivalently, we are able to macroscopically calculate the entropic functional of the composed system without any need of entering into the knowledge of the microscopic states of the subsystems. This property appears to be a natural one for an entropic form if we desire to use it as a basis for a statistical mechanics which would naturally connect to thermodynamics.

Let us mention at this point the important Enciso-Tempesta theorem [17], namely that S^T_q is the *unique* entropic functional which simultaneously is trace-form and composable; see Fig. 9. S^T_δ generically is trace-form but not composable, whereas $S^{TTJ}_{\alpha,\delta}$ and S^R_q generically are composable but not trace-form.

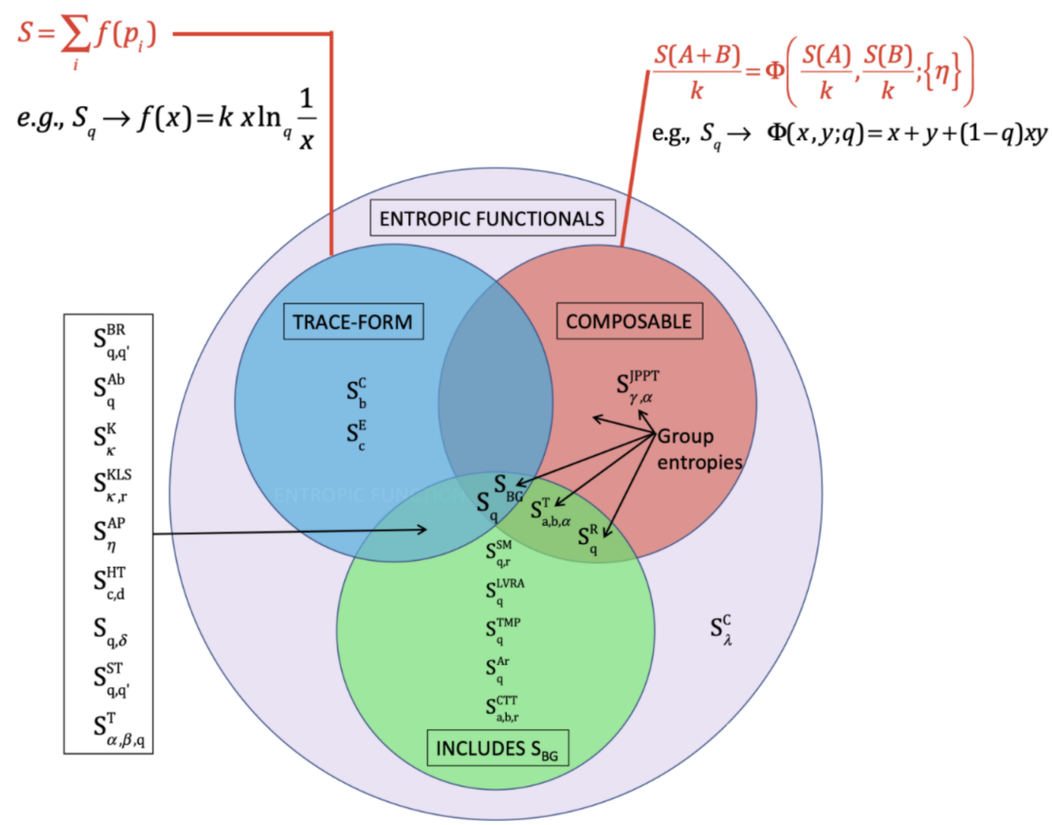

FIGURE 9. Enciso-Tempesta theorem [17]. For more details, see [50].

4. Some remarks and clarifications

4.1. **Generalizing the function** $\ln_q z$**.** The function $\ln_q z$ as defined in Eq. (1.6) places the value $q=1$ in a central position. It might be interesting to generalize it by putting an arbitrary value q^* in that central position. We define then the following monotonic function:

$$f_{q^*}(z,q) \equiv \frac{z^{q^*-q}-1}{q^*-q} \quad [z>0,\ (q^*,q)\in\mathbb{R}^2]\,, \tag{4.1}$$

which indeed recovers $\ln_q z$ for $q^*=1$. In fact, $f_{q^*}(z,q)=\ln_{q+1-q^*} z$ in general. See Fig. 10. For $z\to\infty$ we asymptotically have that

$$f_{q^*}(z,q) \to \begin{cases} \dfrac{1}{q-q^*} & \text{if} \quad q>q^*\,; \\[2ex] \ln z & \text{if} \quad q=q^*\,; \\[2ex] \infty & \text{if} \quad q<q^*\,. \end{cases} \tag{4.2}$$

It also follows that

$$\frac{df_{q^*}(z,q)}{d(\ln z)} = z\frac{f_{q^*}(z,q)}{dz} = z^{q^*-q}\,, \tag{4.3}$$

hence, for $z\to\infty$, we have that

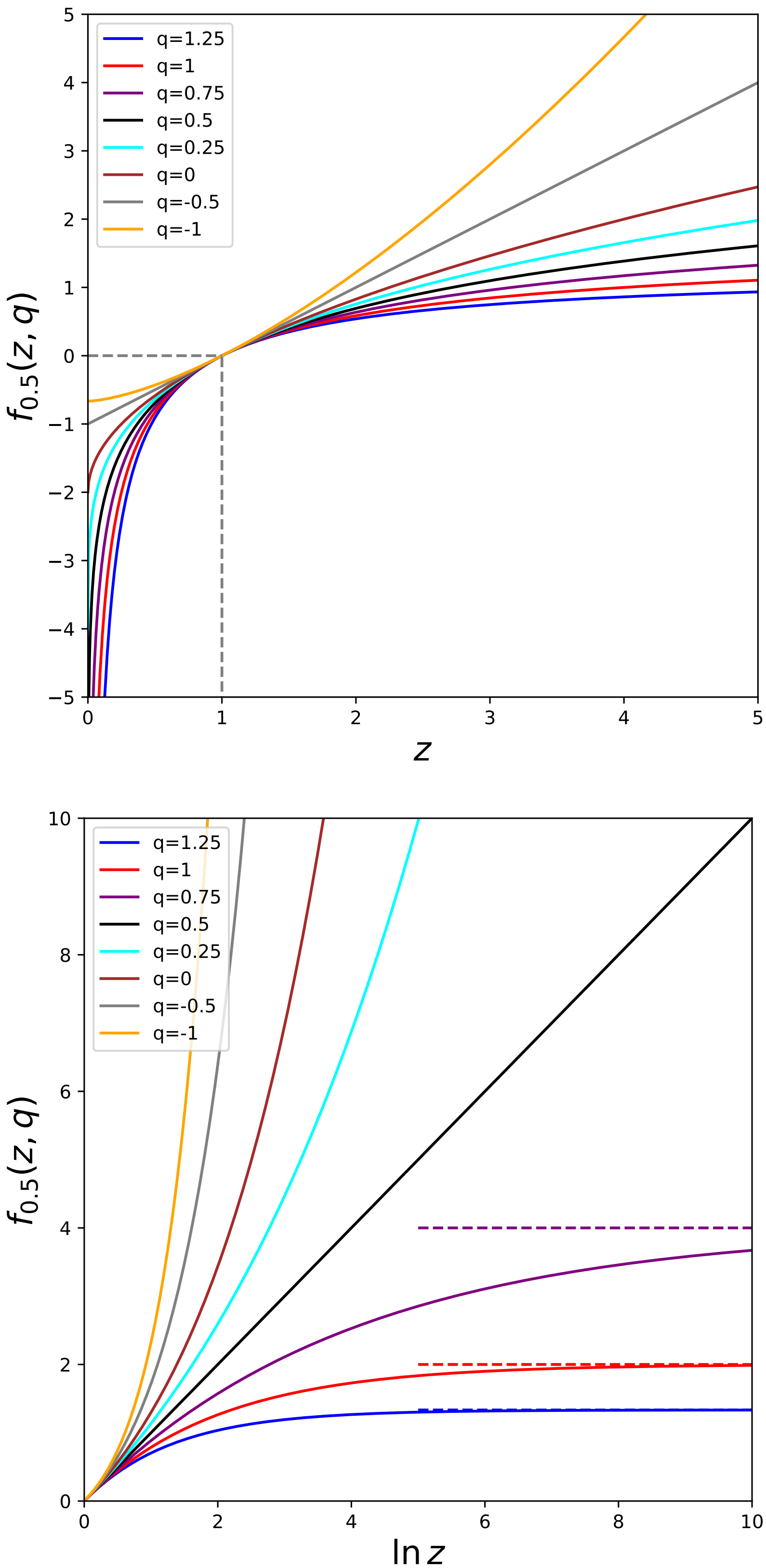

FIGURE 10. The function $f_{q^*}(z,q)$ defined in Eq. (4.1 for $q^* = 0.5$. *Top:* as function of z; for $q = q^* - 1$ we have the straight line $f_{q^*}(z, q^* - 1) = z - 1$; in the $z \to \infty$, the slopes asymptotically become zero, 1, and infinity for $q > q^*$, $q = q^*$, and $q < q^*$ respectively. *Bottom*: as function of $\ln z$; ; for $q = q^*$ we have the straight line line $f_{q^*}(z, q^*) = \ln z$; the dashed lines correspond to the $z \to \infty$ asymptotic behaviors $f_{q^*}(z,q) \sim 1/(q - q^*)$.

$$\frac{df_{q^*}(z,q)}{d(\ln z)} \to \begin{cases} 0 & \text{if} \quad q > q^*\,; \\ 1 & \text{if} \quad q = q^*\,; \\ \infty & \text{if} \quad q < q^*\,. \end{cases} \tag{4.4}$$

The conjecture that is being used in [38] (see also [6, 19, 39, 40, 41]) is that, at the (1+1)-dimensional quantum critical point, the Gruneisen parameter is given by

$$\Big[\frac{dS_q}{d\lambda}\Big]^{-1} \sim A\frac{df_{0.0828}(N,q)}{d(\ln N)} \to A\lim_{N\to\infty}\frac{df_{0.0828}(N,q)}{d(\ln N)} \quad (A>0)\,, \tag{4.5}$$

where the prefactor A depends on nonuniversal information such as the size of the spin and the number of short-range interactions assumed in the model. Consequently

$$\Big[\frac{dS_q}{d\lambda}\Big]^{-1} \to \begin{cases} 0 & \text{if} \quad q > q^*\,; \\ A & \text{if} \quad q = q^*\,; \\ \infty & \text{if} \quad q < q^*\,. \end{cases} \tag{4.6}$$

4.2. **Clarifications.** To avoid the possibility of confusion for the reader, let us make at this point some clarifications:

(i) We refer here as *entropic functional* to a function $S(\{p_i\})$ of the probabilities set $\{p_i\}$ [S must vanish for certainty and be maximal for equal probabilities], and *entropy* to an entropic functional applied to a specific classical or quantum system whose size is finite, typically having N microscopic elements. This distinction is rarely done in the literature. It is, however, important. For example, the entropic functionals (2.7) and (2.10) are definitively different but, when applied to a $d=3$ *black hole* (under the hypothesis of equal probabilities), they both lead to the same entropy

$$S_{bh} = k\Big(\frac{S^{BH}}{k}\Big)^{\delta} = k\Big[\frac{A}{4l_P^2}\Big]^{\delta}\,, \tag{4.7}$$

where A is the area of the event horizon, l_P is the Planck length, and S^{BH} is currently referred to as the *Bekenstein-Hawking entropy* [8, 9, 23, 24] (see, for instance, [53, 50, 54] and also [25, 28]). By conjecturally assuming that the event horizon is a (rough) fractal, Barrow proposed [5]

$$S^B = k\Big(\frac{S^{BH}}{k}\Big)^{1+\Delta/2} \;\; (\Delta>0)\,. \tag{4.8}$$

By identifying Eqs. (4.7) and (4.8) we obtain

$$\delta = 1+\frac{\Delta}{2}\,. \tag{4.9}$$

It is therefore possible to have an entropy (the *Barrow entropy* in this example) even if a Barrow entropic functional does not exist. We verify that $\delta=3/2$, hence $\Delta=1$, makes S_{bh} to be thermodynamically extensive, as argued in [53]. On mathematical grounds, the Barrow entropy might well be grounded on an entropic functional such as S_δ^T or $S_{\alpha,\delta}^{TTJ}$ or a similar one.

(ii) In 2003, Gell-Mann and myself published [18] a book entitled *Nonextensive Entropy - Interdisciplinary Applications*. The correct title ought to be *Nonadditive Entropy - Interdisciplinary Applications* but a clerical inadvertence occurred. In its first chapter, the possibility was for the first time advanced that $W(N)$ could be given by $W(N)\propto N^\rho$, which meant that $S_q^T \propto N$ for $q=1-1/\rho$. In other words, S_q^T is nonadditive for all values of $q\neq 1$, but, for such class of systems, the corresponding entropy is nonextensive for all values of q excepting for that very special one, for which S_q^T is extensive.

(iii) Thanks to the generous and highly honoring collegiality of many scientists, S_q^T is referred, in the literature, to as *Tsallis entropy*. However, since S_δ^T was published in [49], several authors also refer to this one as *Tsallis entropy*. This fact not rare generates confusion in the readers, which could be simply avoided by being slightly more specific, say by perhaps using instead, – if I dare to suggest, with my apologies –, expressions such as *Tsallis q-entropy* and *Tsallis δ-entropy*, or just *q-entropy* and *δ-entropy*, for the benefit of clarity for the readers of those many papers.

5. Some open mathematical problems

A new physical theory which is being developed along more than three decades raises naturally various interesting open mathematical problems. Some of them have recently been focused and listed in [51]. Let us add here some more in the same list. But before extending this list, let us remind a most important open problem which remains elusive until today. A q-generalization of the classical Central Limit Theorem (CLT) has been already studied in [60, 59, 58], namely that q-Gaussian distributions are, for a specific class of strong correlations between the random variables, the $N \to \infty$ attractors in the space of probabilities for the sum of N random variables (Gaussians corresponding to the case of nearly independent variables). However, there is evidence that the existing theorem describes conditions which are sufficient but not necessary. The general necessary and sufficient conditions for q-Gaussians to be the attractors remains today as a most important challenge in the theory of probabilities.

Let us now present a few more problems which would greatly benefit from rigorous proofs.

5.1. **q-Pesin-like behavior for D dynamical variables at the edge of chaos.** Let us focus on a classical D-dimensional nonlinear dynamical system at a situation where the entire Lyapunov spectrum vanishes (edge of chaos) (see details in [1, 52]).

The entropy which linearly increases with time t $(t \to \infty)$ for a vast class of such systems is not $S_{BG}(t)$ but $S_{q^{entropy}}(t)$ instead with $q^{entropy}$ given by

$$\frac{1}{1-q^{entropy}} = \sum_{k=1}^{D} \frac{1}{1-q_k^{sensitivity}} \quad (D \geq 1)\,. \tag{5.1}$$

where $\{q_k^{sensitivity}\}$ characterizes the asymptotic power-law growth of the sensitivity to the initial conditions $\xi_k \sim t^{\frac{1}{1-q_k^{sensitivity}}}$ along the k-th direction. This relation has interesting particular cases, namely

(i) If $D = 1$ we recover $q^{entropy} = q^{sensitivity}$, well verified numerically or analytically for many examples. This is so for the logistic map $x_{t+1} = 1 - ax_t^2$ $(t = 0, 1, 2, ...;\ a \in [0, 2]); x_t \in [-1, 1])$ at the Feigenbaum point (edge of chaos). The corresponding entropic index is $q^{entropy} = q^{sensitivity} = 0.2444877013412820661 98\ldots$ (more than 10^4 exact digits are known today, through the relation $q^{entropy} = 1 - \ln 2/\ln\alpha_F$, where α_F is the Feigenbaum universal constant; see details in [50]).

(ii) It suffices one of the D Lyapunov exponents to be strictly positive, then $q^{entropy} = 1$, i.e., the BG entropy is the correct entropy to be used to exhibit the linear increase with time, consistently with the Pesin identity.

(iii) If all D values of $q_k^{sensitivity}$ coincide, then $(1 - q^{entropy}) \propto \frac{1}{D}$, which leads to $q^{entropy} = 1$ in the $D \to \infty$ limit.

The general conditions for the validity of relation (5.1) have not yet been rigorously established.

5.2. **CLT behavior of the z-logistic map.** At the Feigenbaum point, the logistic map exhibits a neat CLT-like behavior. Indeed, numerical studies [44, 56] suggest that a $q^{attractor}$-Gaussian attractor emerges with $q_{attractor} \simeq 5/3$. The z-logistic map is defined as follows: $x_{t+1} = 1 - a|x_t|^z$ $(t = 0, 1, 2, ...;\ a \in [0, 2]); x_t \in [-1, 1],\ z > 1)$, which recovers the standard logistic map for $z = 2$. Ongoing calculations [35] suggest the analytical expression

$$q^{attractor} = 1 + \frac{2}{z+1} \quad (z > 1)\,, \tag{5.2}$$

which yields $q^{attractor} = 2, 5/3, 1$ for $z = 1, 2, \infty$ respectively. As an analytical extension, we obtain for $z = 0$, $q^{attractor} = 3$, which corresponds to the highest value of q guaranteeing normalizability of q-Gaussians.

The proof of the conjectural Eq. (5.2) would be most welcome.

5.3. **Classical many-body Hamiltonian systems with arbitrarily-ranged two-body interactions.** The classical d-dimensional α-XY, α-Heisenberg and α-Fermi-Pasta-Ulam Hamiltonian models involve two-body interactions decaying with the distance r proportionally to $1/r^\alpha$ with $\alpha \geq 0$ (see [33] and references therein; see also [29, 55, 30]). A variety of first-principle numerical indications (uniquely based on $\vec{F} = m\vec{a}$) are available in the literature suggesting that, in longstanding quasi-stationary regimes for isolated systems, the one-particle energies are q_E-exponentially distributed with

$$q_E = \begin{cases} \frac{4}{3} & \text{if} \quad 0 \leq \alpha/d \leq 1\,; \\ 1 + \frac{1}{3}e^{1-\alpha/d} & \text{if} \quad \alpha/d > 1\,. \end{cases} \tag{5.3}$$

In the same physical regime, similar numerical approaches suggest that the one-particle momenta are distributed through q_p-Gaussians, with

$$\frac{q_p - 1}{q_E - 1} = 2\,. \tag{5.4}$$

These behaviors are referred to as *very-long-range interactions* ($0 \leq \alpha/d \leq 1$), *long-range interactions* ($1 < \alpha/d < \infty$) and *short-range interaction* ($\alpha/d \to \infty$). The proof of the above two relations would be of paramount importance within nonextensive statistical mechanics.

5.4. **Crossover between BG and q-statistical behaviors as a nonuniform convergence in the $(N, T) \to (\infty, \infty)$ limit.** We focus on the same class of systems of Subsection 5.3. During the longstanding quasi-stationary state, two regimes appear to exist, namely the q-statistical regime (characterized by $q_p > q_E > 1$) for $1 << t << N$ and the BG regime (characterized by $q_p = q_E = 1$) for $1 << N << t$. These two regimes are connected [12, 34] through a crossover occurring at

$$N \sim B(d)\, t^{\gamma(d)} \;\; [(N, T) \to (\infty, \infty)]\,, \tag{5.5}$$

where $0 < B(d) << 1$ and $\gamma(d) > 1$ are d- and model-dependent quantities.

The mathematical establishment of the above crossover would definitively illustrate the interplay and the respective domains of validity or failure of BG statistical mechanics ($q = 1$) and the nonextensive one ($q \neq 1$).

6. Final words

Within the present plethoric scenario where close to fifty entropic functionals are available in the literature, we have here selected and reviewed the few forms and paths which have proved to be useful at the level of applications in natural, artificial and social sciences. Figs. 6 to 8 illustrate the structure of this complex network, and relevant properties (trace-form, composability, concavity, nonnegativity, monotonicity) have been focused on. We have concluded by indicating some new open mathematical problems (in addition to those already indicated in [51]) whose mathematical handling would be most welcome in order to fix crucial issues of the present statistical mechanical theory grounded on nonadditive entropic functionals.

Finally, let us emphasize that the choice of the entropic indices is done in order to satisfy, for specific classes of systems, the entropic extensivity mandated by the Legendre structure of classical thermodynamics. We have sacrificed the simplicity of the Boltzmann-Gibbs *additive* entropic functional but have achieved a higher goal, namely the preservation of the thermodynamical structure, similarly to sacrificing the simplicity of the Galilean *additive* composition of velocities but achieving a higher goal, namely the unification of Maxwell electromagnetism with Newtonian mechanics through the Lorentz transform.

Acknowledgments

I am indebted to H.S. Lima for kind and efficient assistance concerning various figures. I also benefited from fruitful conversations with him and with M. de Souza, L.C.G. Squillante and S.M. Soares, as well as from partial financial support by the Brazilian Agencies CNPq and FAPERJ.

References

[1] Ananos G.F.J., Baldovin F., Tsallis C.; Anomalous sensitivity to initial conditions and entropy production in standard maps: Nonextensive approach. Eur. Phys. J. B **46**, 409 (2005).

[2] Anteneodo C., Plastino A.R.; Maximum entropy approach to stretched exponential probability distributions. J. Phys. A **32**, 1089 (1999).

[3] Abe S.; A note on the q-deformation theoretic aspect of the generalized entropies in nonextensive physics. Phys. Lett. A **224**, 326-330 (1997).

[4] Barlow H.; Conditions for versatile learning, Helmholtz's unconscious inference, and the task of perception. Vision. Res. **30**, 1561 (1990).

[5] Barrow J.D.; The area of a rough black hole. Phys. Lett. B 808, 135643 (2020).

[6] Bartosch L., de Souza M., Lang M.: Scaling theory of the Mott transition and breakdown of the Gruneisen scaling near a finite-temperature critical end point. Phys. Rev. Lett. **104**, 245701 (2010).

[7] Beck C., Tsallis C.: Anomalous velocity distributions in slow quantum-tunneling chemical reactions. Phys. Rev. Research / Letter **7**, L012081 (2025).

[8] Bekenstein J.D.; Black holes and entropy. Phys. Rev. D **7**, 2333 (1973).

[9] Bekenstein J.D.;, Generalized second law of thermodynamics in black-hole physics. Phys. Rev. D **9**, 3292 (1974).

[10] Boltzmann L.; Weitere Studien ùber das Wȧrmegleichgewicht unter Gas molekùlen [Further Studies on Thermal Equilibrium Between Gas Molecules], Wien, Ber. **66**, 275 (1872). Boltzmann, L.; Uber die Beziehung eines allgemeine mechanischen Satzes zum zweiten Haupsatze der Warmetheorie. Sitzungsberichte, K. Akademie der Wissenschaften in Wien, Math.-Naturwissenschaften **75**, 67-73 (1877).

[11] Borges E.P., Roditi I.; A family of non-extensive entropies. Phys. Lett. A **246**, 399-402 (1998).

[12] Christodoulidi H., Tsallis C., Bountis T.; Fermi-Pasta-Ulam model with long-range interactions: Dynamics and thermostatistics. EPL **108**, 40006 (2014).

[13] Curado E.M.F., Tsallis C.; Generalized statistical mechanics: connection with thermodynamics. J. Phys. A **24**, L69-72 (1991); Corrigenda: **24**, 3187 (1991) and **25**, 1019 (1992).

[14] Curado E.M.F., Tempesta P., Tsallis C.; A new entropy based on a group-theoretical structure. Annals Phys. **366**, 22-31 (2016).

[15] Combe G., Richefeu V., Stasiak M., Atman A.P.F.; Experimental validation of nonextensive scaling law in confined granular media. Phys. Rev. Lett. **115**, 238301 (2015).

[16] Douglas P., Bergamini S., Renzoni F.; Tunable Tsallis distributions in dissipative optical lattices. Phys. Rev. Lett. **96**, 110601 (2006).

[17] Enciso A., Tempesta P.; Uniqueness and characterization theorems for generalized entropies. J. Stat. Mech. 123101 (2017).

[18] Gell-Mann M., Tsallis C., eds.; Nonextensive Entropy - Interdisciplinary Applications. (Oxford University Press, New York, 2004).

[19] Gomes G.O., Stanley H.S., de Souza M.; Enhanced Gruneisen parameter in supercooled water. Sci. Rep. **9**, 12006 (2019).

[20] Gibbs J.W.; Elementary Principles in Statistical Mechanics – Developed with Especial Reference to the Rational Foundation of Thermodynamics. C. Scribner's Sons, New York, 1902.

[21] Hanel R., Thurner S.; A comprehensive classification of complex statistical systems and an axiomatic derivation of their entropy and distribution functions. Europhys. Lett. **93**, 20006 (2011).

[22] Hanel R., Thurner S.; When do generalised entropies apply? How phase space volume determines entropy. Europhys. Lett. **96**, 50003 (2011).

[23] Hawking S.W., Black hole explosions? Nature **248**, 30 (1974).

[24] Hawking S.W.; Black hole and thermodynamics. Phys. Rev. D **13**, 191 (1976).

[25] 't Hooft G.; The black hole interpretation of string theory. Nucl. Phys. B **355**, 138 (1990), and references therein.

[26] Kaniadakis G.; Non linear kinetics underlying generalized statistics. Physica A **296**, 405 (2001); Statistical mechanics in the context of special relativity. Phys. Rev. E **66**, 056125 (2002); Statistical mechanics in the context of special relativity. II. Phys. Rev. E **72**, 036108 (2005).

[27] Lutz E., Renzoni F.; Beyond Boltzmann-Gibbs statistical mechanics in optical lattices. Nature Physics **9**, 615-619 (2013).

[28] Maldacena J. and Strominger A.; Statistical entropy of de Sitter space. J. High Energy Phys. **2**, 014 (1998).

[29] de Oliveira R.M., Brito S., da Silva L.R., Tsallis C.; Connecting complex networks to nonadditive entropies. Scientific Reports **11**, 1130 (2021).

[30] Oliveira R.M., da Silva L.R., Tsallis C.; Energy distribution in long-range-interacting weighted geographic networks. Eur. Phys. J. Plus **139**, 975 (2024).

[31] Penrose O.; Foundations of statistical mechanics: A deductive treatment. (Pergamon, Oxford, 1970), page 167.

[32] Renyi A.; On measures of information and entropy, in *Proceedings of the Fourth Berkeley Symposium*, **1**, 547 (University of California Press, Berkeley, Los Angeles, 1961). Renyi A.; Probability theory (North-Holland, Amsterdam, 1970).

[33] Rodriguez A., Pluchino A., Tirnakli U., Rapisarda A., Tsallis C.; Nonextensive footprints in dissipative and conservative dynamical systems. Symmetry **15**, 444 (2023).

[34] Rodriguez A., Tsallis C.; Diffusion crossover from/to q-statistics to/from Boltzmann-Gibbs statistics in the classical inertial α-XY ferromagnet. Phys. Rev. E **111**, 024143 (2025).

[35] Saberi A.A., Tirnakli U., Tsallis C.; Central Limit Theorem behavior at the edge of chaos in the z-logistic map. (2025), in preparation.

[36] Schwammle V., Tsallis C.; Two-parameter generalization of the logarithm and exponential functions and Boltzmann-Gibbs-Shannon entropy. J. Math. Phys. **48**, 113301 (2007).

[37] Sharma B.D., Mittal D.P.; New non-additive measures of entropy for discrete probability distributions. J. Math. Sci. **10**, 28 (1975).

[38] Soares S.M., Squillante L., Lima H.S., Tsallis C., de Souza M.; Universally non-diverging Gruneisen parameter at critical points. Phys. Rev. B / Letter **111**, L060409 (2025).

[39] Squillante L., Mello I.F., Seridonio A.C., de Souza M.; Giant caloric effects close to any critical end point. Mat. Res. Bull. **142**, 111413 (2021).

[40] Squillante L., Ricco L.S., Ukpong A.M., Lagos-Monaco R.E., Seridonio A.C., de Souza M.; Gruneisen parameter as an entanglement compass and the breakdown of the Hellmann-Feynman theorem. Phys. Rev. B **108**, L140403 (2023).

[41] Squillante L., Gomes G.O., Mello I.F., Nogueira G., Seridonio A.C., Lagos-Monaco R.E., de Souza M.; Exploring the expansion of the universe using the Gruneisen parameter. Results Phys. **57**, 107344 (2024).

[42] Tempesta P.; Multivariate group entropies, super-exponentially growing complex systems, and functional equations. Chaos **30**, 123119 (2020).

[43] `https://tsallis.cbpf.br/biblio.htm`.

[44] Tirnakli U., Tsallis C., Beck C.; A closer look at time averages of the logistic map at the edge of chaos. Phys. Rev. E **79**, 056209 (2009).

[45] Tsallis C.; Possible generalization of Boltzmann-Gibbs statistics. J. Stat. Phys. **52**, 479-487 (1988) [First appeared as preprint in 1987: CBPF-NF-062/87, ISSN 0029-3865, Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro].

[46] Tsallis C., Mendes R.S., Plastino A.R.; The role of constraints within generalized nonextensive statistics. Physica A **261**, 534-554 (1998).

[47] Tsallis C.; Nonextensive statistics: Theoretical, experimental and computational evidences and connections. In *Nonextensive Statistical Mechanics and Thermodynamics*, eds. Salinas S.R.A. and Tsallis C., Braz. J. Phys. **29** (1), 1-35 (1999).

[48] Tsallis C., Talk at the IMS Winter School on *Nonextensive generalization of Boltzmann-Gibbs Statistical mechanics and its applications* (Institute for Molecular Science, 15-18 February 1999, Okazaki, Japan).

[49] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics - Approaching a Complex World* (Springer, New York, 2009).

[50] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics - Approaching a Complex World*, Second Edition (Springer-Nature, 2023).

[51] Tsallis C.; Open mathematical issues in nonextensive statistical mechanics. Matematica Contemporanea **58**, 316-337 (2023).

[52] Tsallis C., E.P. Borges E.P., Plastino A.R.; Entropy evolution at generic power-law edge of chaos. Chaos, Solitons and Fractals **174**, 113855 (2023).

[53] Tsallis C., Cirto L.J.L.; Black hole thermodynamical entropy. Eur. Phys. J. C **73**, 2487 (2013).

[54] Tsallis C., Jensen H.J.; Extensive composable entropy for the analysis of cosmological data. Phys. Lett. B **861**, 139238 (2025).

[55] Tsallis C., de Oliveira R.M.; Complex network growth model: Possible isomorphism between nonextensive statistical mechanics and random geometry. Chaos **32**, 053126 (2022).

[56] Tsallis C., Tirnakli U.; Nonadditive entropy and nonextensive statistical mechanics - Some central concepts and recent applications. J. Phys. C Series **201**, 012001 (2010).

[57] Ubriaco M.R.; Entropies based on fractional calculus. Phys. Lett. A **373**, 2516 (2009).

[58] Umarov S., Tsallis C.; Mathematical Foundations of Nonextensive Statistical Mechanics. (World Scientific, Singapore, 2022).

[59] Umarov S., Tsallis C., Gell-Mann M., Steinberg S.; Generalization of symmetric α-stable Lévy distributions for $q > 1$. J. Math. Phys. **51**, 033502 (2010).

[60] Umarov S., Tsallis C., Steinberg S.; On a q-central limit theorem consistent with nonextensive statistical mechanics. Milan J. Math. **76**, 307-328 (2008).

[61] Watanabe S.; Knowing and Guessing. (Wiley, New York, 1969).

[62] Wild R., Notzold M., Simpson M., Tran T.D., Wester R.; Tunnelling measured in a very slow ion-molecule reaction. Nature **615**, 425 (2023).

[63] Wong C.Y., Wilk G., Cirto L.J.L., Tsallis C.; From QCD-based hard-scattering to nonextensive statistical mechanical descriptions of transverse momentum spectra in high-energy pp and $p\bar{p}$ collisions. Phys. Rev. D **91**, 114027 (2015).

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.300-306
DOI: 10.29229/uzmj.2025-2-29

The Goursat's problem for generalized (fractional) hyperbolic-type equation

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. We start by defining a novel bivariate Mittag-Leffler function represented by an infinite double series. Several key properties of this function are established, including differentiation formulas and an upper bound estimate. Subsequently, we consider Goursat's problem for a fractional generalization of a hyperbolic-type equation. An explicit solution to this problem is constructed using the newly defined bivariate Mittag-Leffler function. The analytical characteristics derived for this function allow us to prove the unique solvability of the proposed problem.

Keywords: Bivariate Mittag-Leffler type function; Goursat's problem; hyperbolic-type equation; regularized Prabhakar fractional derivative.

MSC (2020): 33E12, 35R11, 45D05.

1. Introduction

The study of special functions is crucial in analyzing fractional differential equations, particularly those involving nonlocal operators and memory effects. The Mittag-Leffler function and its generalizations have received significant attention due to their fundamental role in expressing solutions to fractional evolution equations. While the classical (univariate) Mittag-Leffler function has been widely studied and utilized in the context of Caputo and Riemann-Liouville derivatives, the multivariate extensions, such as the bivariate Mittag-Leffler function $E_2(x, y)$ - have garnered growing interest in recent years for their potential in modeling more complex systems involving multiple fractional parameters or variables.

Our interest in the bivariate Mittag-Leffler function is motivated by its application to fractional partial differential equations (PDEs), in particular, to the generalized (fractional) hyperbolic-type equation involving the regularized Prabhakar fractional derivative.

Prabhakar's fractional operators are widely used in many scientific areas, including physics, engineering, and mathematics. They are particularly relevant in fields such as media studies, the time evolution of polarization processes, fractional Poisson processes, fractional diffusion and telegraph equations, fractional Maxwell models of linear viscoelasticity, birth-death processes, and control theory [1], [4], [5], [6], [7], [8], [13].

The behavior of electromagnetic waves traveling through transmission lines is characterized by wave equations, which were originally referred to as telegraph equations. This terminology stems from their initial application in describing wave propagation in telegraph lines.

Earlier in work [15], the existence and uniqueness theorem for the analogue of the Goursat problem for an equation of the form

$$\partial_{0x}^{\alpha}\partial_{0y}^{\beta}u\left(x,y\right)+\lambda u\left(x,y\right)=f\left(x,y\right), \tag{1.1}$$

with the Riemann-Liouville derivatives was proven. Here $\alpha, \beta \in (0,1)$, $\lambda \in \mathbb{R}$ and ∂_{0s}^{γ} is the Caputo fractional derivative of order γ with respect to the variable s. In work [3], analogues of the Cauchy problem and the Goursat problem were considered for equation (1.1) in the case of the Riemann-Liouville operator and $\lambda = 0$. Work [16] solved the analogue of the Goursat problem with an integral condition for equation (1.1). Note that non-local boundary problems for equation (1.1) have been studied in publications [10], [11]. Finally, in work [17], the Goursat problem for equation (1.1) was resolved.

In this work, we study Goursat's problem for a generalized hyperbolic-type equation with the regularized Prabhakar fractional derivative. A bounded rectangular domain of the plane with two independent variables is used to consider the equation. In solving the problem, we reduce it to a

Volterra integral equation of the second kind and employ the method of successive approximations to find its solution. The solution is obtained in explicit form and expressed in terms of $E_{12}(\cdot)$ is a new type of the bivariate Mittag-Leffler type functions. As a result, a theorem demonstrating both the existence and uniqueness of a solution to the problem under investigation has been established; its representation is determined through the solutions to the resulting integral equation.

2. Bivariate Mittag-Leffler type function

In this section, we introduce a new bivariate Mittag-Leffler function and study its main properties. Namely, we determine regions of convergence of double series, integral representation, and estimate.

$E_{12}(\cdot)$ is a new type of the bivariate Mittag-Leffler type function

$$E_{12}\left(\begin{matrix}\alpha_1,\beta_1,\delta_1;\\ \alpha_2,\beta_2,\delta_2;\alpha_3,\delta_3;\alpha_4,\delta_4;\beta_3,\delta_5\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=$$

$$=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\Gamma\left(\alpha_1 n+\beta_1 m+\delta_1\right)x^n y^m}{\Gamma\left(\alpha_2 n+\beta_2 m+\delta_2\right)\Gamma\left(\alpha_3 n+\delta_3\right)\Gamma\left(\alpha_4 n+\delta_4\right)\Gamma\left(\beta_3 m+\delta_5\right)}, \tag{2.1}$$

$$\left(x,y,\alpha_l,\beta_i,\delta_j\in\mathbb{R};\ \min\left\{\alpha_l,\beta_i\right\}>0;\ \left(l=\overline{1,4},\ i=\overline{1,3},\ j=\overline{1,5}\right)\right)$$

in which the double series converges for $x,y\in\mathbb{R}$, if $\Delta_1>0$, and $\Delta_2>0$. Here

$$\Delta_1=\alpha_3+\alpha_4+\alpha_5-\alpha_1-\alpha_2,\ \ \Delta_2=\beta_2+\beta_3-\beta_1.$$

Previously, in [9], 11 similar functions were introduced and studied.

In the following, we present differentiation formulas for E_{12}, which will be used further.

2.1. **Differentiation and integration formulas.** In the following, we present differentiation and integration formulas for E_{12}, which will be used further.

Lemma 2.1. *For the function (2.1), the following differentiation and integration formulas hold:*

$$\frac{\partial}{\partial x}\left(x^{\alpha_4-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,\alpha_4;1,1\end{matrix}\middle|\begin{matrix}-\lambda x y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right)\right)=$$

$$=x^{\alpha_4-2}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,\alpha_4-1;1,1\end{matrix}\middle|\begin{matrix}-\lambda x y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right),\quad \alpha_4\neq 1, \tag{2.2}$$

$$\frac{\partial}{\partial y}\left(y^{\beta_1-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta_2,\alpha,\beta_1;\gamma,\gamma;\alpha_4,\gamma_2;1,1\end{matrix}\middle|\begin{matrix}-\lambda x y^{\beta_2}\\ \delta y^{\alpha}\end{matrix}\right)\right)=$$

$$=y^{\beta_1-2}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta_2,\alpha,\beta_1-1;\gamma,\gamma;\alpha_4,\gamma_2;1,1\end{matrix}\middle|\begin{matrix}-\lambda x y^{\beta_2}\\ \delta y^{\alpha}\end{matrix}\right),\quad \beta_1\neq 1, \tag{2.3}$$

$$I_{0y}^{\alpha,1-\beta_3,-\gamma,\delta}\left(y^{\beta_1-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta_2,\alpha,\beta_1;\gamma,\gamma;\alpha_4,\gamma_2;1,1\end{matrix}\middle|\begin{matrix}-\lambda x y^{\beta_2}\\ \delta y^{\alpha}\end{matrix}\right)\right)=$$

$$=y^{\beta_1-\beta_3}E_{12}\left(\begin{matrix}\gamma,1,0;\\ \beta_2,\alpha,\beta_1-\beta_3+1;\gamma,0;\alpha_4,\gamma_2;1,1\end{matrix}\middle|\begin{matrix}-\lambda x y^{\beta_2}\\ \delta y^{\alpha}\end{matrix}\right). \tag{2.4}$$

`Proof.` Let us prove (2.2):

$$\frac{\partial}{\partial x}\left(x^{\alpha_4-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,\alpha_4;1,1\end{matrix}\middle|\begin{matrix}-\lambda x y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right)\right)=$$

$$=\frac{\partial}{\partial x}\left(\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\Gamma\left(\gamma n+m+\gamma\right)(-\lambda)^n\delta^m x^{n+\alpha_4-1}y^{\beta n+\alpha m}}{\Gamma\left(\beta n+\alpha m+\beta\right)\Gamma\left(\gamma n+\gamma\right)\Gamma\left(n+\alpha_4\right)\Gamma\left(m+1\right)}\right)=$$

$$=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\Gamma\left(\gamma n+m+\gamma\right)(-\lambda)^n\delta^m(n+\alpha_4-1)x^{n+\alpha_4-2}y^{\beta n+\alpha m}}{\Gamma\left(\beta n+\alpha m+\beta\right)\Gamma\left(\gamma n+\gamma\right)\Gamma\left(n+\alpha_4-1+1\right)\Gamma\left(m+1\right)}=$$

$$= x^{\alpha_4-2}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,\alpha_4-1;1,1\end{matrix}\middle|\begin{matrix}-\lambda x y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right),\quad \alpha_4\neq 1.$$

The remaining formulas can be proved similarly.

Lemma 2.1 is proved. □

2.2. **Upper bound.** We present in the following the statement on the estimation of $E_{12}(x,y)$:

Lemma 2.2. *If $\alpha>0$, $0<\beta<1$, $\gamma>0$, $\delta<0$, then for any $\lambda>0$, the following*

$$\left|E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right)\right|\leq C. \tag{2.5}$$

is true.

Proof. Using (2.1), we express $E_{12}\left(\cdot\right)$ in the form

$$E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right)=$$

$$=\sum_{\substack{n=0\\m=0}}^{+\infty}\frac{\Gamma\left(\gamma n+m+\gamma\right)\left(-\lambda\right)^n\left(x-\xi\right)^n\delta^m y^{\alpha m+\beta n}}{\Gamma\left(\alpha m+\beta n+\beta+1\right)\Gamma\left(\gamma n+\gamma\right)\Gamma\left(n+1\right)\Gamma\left(m+1\right)}=$$

$$=\sum_{\substack{n=0\\m=0}}^{+\infty}\frac{\Gamma\left(\gamma n+m+\gamma\right)B\left(\beta n+\beta,\alpha m+1\right)\left(-\lambda\right)^n\left(x-\xi\right)^n\delta^m y^{\alpha m+\beta n}}{\Gamma\left(\alpha m+1\right)\Gamma\left(\beta n+\beta\right)\Gamma\left(\gamma n+\gamma\right)\Gamma\left(n+1\right)\Gamma\left(m+1\right)}=$$

$$=\sum_{\substack{n=0\\m=0}}^{+\infty}\frac{\left(-\lambda\right)^n\left(x-\xi\right)^n\delta^m y^{\alpha m+\beta n}}{\Gamma\left(\alpha m+1\right)\Gamma\left(\beta n+\beta\right)\Gamma\left(\gamma n+\gamma\right)\Gamma\left(n+1\right)\Gamma\left(m+1\right)}\times$$

$$\times\int\limits_0^{+\infty}t^{\gamma n+m+\gamma-1}e^{-t}dt\int\limits_0^1 z^{\beta n+\beta-1}(1-z)^{\alpha m+1-1}dz=$$

$$=\int\limits_0^{+\infty}\int\limits_0^1 e^{-t}t^{\gamma-1}z^{\beta-1}\sum_{n=0}^{+\infty}\frac{t^{\gamma n}z^{\beta n}(-\lambda)^n(x-\xi)^n y^{\beta n}}{\Gamma\left(\beta n+\beta\right)\Gamma\left(\gamma n+\gamma\right)\Gamma\left(n+1\right)}\sum_{m=0}^{+\infty}\frac{t^m(1-z)^{\alpha m}\delta^m y^{\alpha m}}{\Gamma\left(m+1\right)\Gamma\left(\alpha m+1\right)}dtdz. \tag{2.6}$$

Taking into account the notation (2.7), we write (2.6) in the form

$$\left|E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right)\right|\leq$$

$$\leq\int\limits_0^{+\infty}\int\limits_0^1 e^{-t}t^{\gamma-1}z^{\beta-1}\left|e^{\beta,\gamma}_{\beta,-\gamma}\left[-\lambda\left(x-\xi\right)t^{\gamma}z^{\beta}y^{\beta}\right]\right|\left|e^{1,1}_{1,-\alpha}\left[\delta t(1-z)^{\alpha}y^{\alpha}\right]\right|dtdz.$$

Here

$$e^{\beta_1,\beta_2}_{\alpha_1,-\alpha_2}\left(z\right)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\alpha_1 n+\beta_1\right)\Gamma\left(\alpha_2 n+\beta_2\right)}. \tag{2.7}$$

is the Wright function [18].

Due to the inequalities $\left|e^{\mu_1,\delta_1}_{\alpha_1,\beta_1}(-z)\right|<\dfrac{1}{\Gamma(\mu_1)\Gamma(\delta_1)}$, $\mu_1\geqslant 0$, $\delta_1\geqslant\beta_1$ and $z>0$ [18] it follows from the above that

$$\left|E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right)\right|\leq C.$$

Lemma 2.2 is proved. □

3. Application in the Goursat problem for fractional generalization of hyperbolic-type equation

In this section, we will consider the Goursat problem for a fractional generalization of a hyperbolic-type equation in a rectangular domain.

3.1. **Goursat problem.** We consider the following generalized hyperbolic-type equation

$$\frac{\partial}{\partial x}\, {}^{PC}D_{0y}^{\alpha,\beta,\gamma,\delta} u\left(x,y\right) + \lambda u\left(x,y\right) = f\left(x,y\right), \tag{3.1}$$

in a domain $\Omega = \{(x,y) : 0 < x < a, 0 < y < b\}$ Here $f\left(x,y\right)$ is a given function and

$${}^{PC}D_{0y}^{\alpha,\beta,\gamma,\delta} u\left(x,y\right) = I_{0y}^{\alpha,1-\beta,-\gamma,\delta} \frac{\partial}{\partial y} u\left(x,y\right) \tag{3.2}$$

represents regularized Prabhakar fractional derivative [2] and

$$I_{0y}^{\alpha,\beta,\gamma,\delta} g\left(y\right) = \int\limits_0^y \left(y-z\right)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left[\delta(y-z)^{\alpha}\right] g\left(z\right) dz, \quad y > 0 \tag{3.3}$$

represents Prabhakar fractional integral [19]. We note that above-given definitions are valid for α, β, γ, δ, λ, a, $b \in \mathbb{R}$, such that $\alpha > 0$, $0 < \beta < 1$, $a > 0$ and $b > 0$.

Here $E_{\alpha,\beta}^{\gamma}\left(z\right)$ is the generalized Mittag-Leffler (Prabhakar) function [19]

$$E_{\alpha,\beta}^{\gamma}\left(z\right) = \sum_{m=0}^{\infty} \frac{\left(\gamma\right)_m}{\Gamma\left(\alpha m + \beta\right)} \frac{z^m}{m!}.$$

We note that at $\beta = 1$, $\delta = 0$, Eq. (3.1) becomes classical hyperbolic-type equation:

$$u_{xy}(x,y) + \lambda u(x,y) = f(x,y).$$

Problem G**.** We are interested in finding a regular solution of the equation (3.1) with $0 < \beta < 1$ in a domain Ω, satisfying initial condition

$$u\left(x,0\right) = \varphi\left(x\right), \quad 0 \le x \le a\,, \tag{3.4}$$

and boundary condition

$$u\left(0,y\right) = \psi\left(y\right), \quad 0 \le y \le b\,, \tag{3.5}$$

where $\varphi(x)$, $\psi(y)$ are given functions, such that $\varphi\left(0\right) = \psi\left(0\right)$.

We call a function $u\left(x,y\right)$ as **a regular solution** of problem (3.1), (3.4)-(3.5), if $u(x,y) \in C(\overline{\Omega})$, $(\partial/\partial x)^{PC}D_{0y}^{\alpha,\beta,\gamma,\delta} u(x,y) \in C(\Omega)$.

By integrating equation (3.1) over $[0,x]$, we obtain the equation:

$${}^{PC}D_{0y}^{\alpha,\beta,\gamma,\delta} u\left(x,y\right) + \lambda I_{0x}^{1} u\left(x,y\right) = I_{0x}^{1} f\left(x,y\right) + \Phi\left(y\right), \tag{3.6}$$

where $\Phi\left(y\right)$ is an arbitrary smooth function, and I_{0x}^{1} is the Riemann-Liouville fractional integral:

$$I_{0x}^{\beta^*} f\left(x\right) = \frac{1}{\Gamma\left(\beta^*\right)} \int\limits_0^x \left(x-t\right)^{\beta^*-1} f\left(t\right) dt.$$

Applying the operator $I_{0y}^{\alpha,\beta,\gamma,\delta}$ to equation (3.6), and considering the equality:

$${}^{P}I_{0y}^{\alpha,\beta,\gamma,\delta}\, {}^{PC}D_{0y}^{\alpha,\beta,\gamma,\delta} u\left(x,y\right) = u\left(x,y\right) - u\left(x,0\right),$$

and the conditions (3.4), we obtain a Volterra integral equation of the second kind:

$$u\left(x,y\right)+\lambda I_{0y}^{\alpha,\beta,\gamma,\delta}I_{0x}^{1}u\left(x,y\right)=g\left(x,y\right), \tag{3.7}$$

where $g\left(x,y\right)=\varphi\left(x\right)+\Phi_1\left(y\right)+I_{0y}^{\alpha,\beta,\gamma,\delta}I_{0x}^{1}f\left(x,y\right)$, and $\Phi_1\left(y\right)=I_{0y}^{\alpha,\beta,\gamma,\delta}\Phi\left(y\right)$.

To solve equation (3.7), we apply the method of successive approximations:

$$u_0\left(x,y\right)=g\left(x,y\right)=\varphi\left(x\right)+\Phi_1\left(y\right)+I_{0y}^{\alpha,\beta,\gamma,\delta}I_{0x}^{1}f\left(x,y\right), \tag{3.8}$$

$$u_n\left(x,y\right)=u_0\left(x,y\right)-\lambda I_{0y}^{\alpha,\beta,\gamma,\delta}I_{0x}^{1}u_{n-1}\left(x,y\right). \tag{3.9}$$

Using the formulas $I_{0y}^{\alpha,\beta_1,\gamma_1,\delta}I_{0y}^{\alpha,\beta_2,\gamma_2,\delta}=I_{0y}^{\alpha,\beta_2,\gamma_2,\delta}I_{0y}^{\alpha,\beta_1,\gamma_1,\delta}=I_{0y}^{\alpha,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta}$ and $I_{0x}^{\alpha}I_{0x}^{\beta}=I_{0x}^{\beta}I_{0x}^{\alpha}=I_{0x}^{\alpha+\beta}$, we compute $u_n\left(x,y\right)$:

$$u_n\left(x,y\right)=u_0\left(x,y\right)-\lambda I_{0y}^{\alpha,\beta,\gamma,\delta}I_{0x}^{1}u_0\left(x,y\right)+\lambda^2 I_{0y}^{\alpha,2\beta,2\gamma,\delta}I_{0x}^{2}u_0\left(x,y\right)-$$

$$-\lambda^3 I_{0y}^{\alpha,3\beta,3\gamma,\delta}I_{0x}^{3}u_0\left(x,y\right)+\cdots+\left(-\lambda\right)^n I_{0y}^{\alpha,n\beta,n\gamma,\delta}I_{0x}^{n}u_0\left(x,y\right). \tag{3.10}$$

According to the theory of integral equations [14], if the limit $\lim\limits_{n\to\infty}u_n\left(x,y\right)$ exists uniformly in (x,y), then the limiting function is a solution to the integral equation (3.7).

Taking the limit as $n\to\infty$ in (3.10) and substituting the expression for $u_0\left(x,y\right)$, we obtain the solution to equation (3.7) in the form:

$$u\left(x,y\right)=u_0\left(x,y\right)+\sum_{n=1}^{+\infty}\left(-\lambda\right)^n I_{0y}^{\alpha,n\beta,n\gamma,\delta}I_{0x}^{n}u_0\left(x,y\right). \tag{3.11}$$

Using the function (3.8) given above, we can write the function (3.11) in the following form:

$$u(x,y)=\vartheta_1(x,y)+\vartheta_2(x,y)+\vartheta_3(x,y), \tag{3.12}$$

where

$$\vartheta_1(x,y)=\varphi\left(x\right)+\sum_{n=1}^{+\infty}\left(-\lambda\right)^n I_{0y}^{\alpha,n\beta,n\gamma,\delta}I_{0x}^{n}\varphi\left(x\right)$$

$$\vartheta_2(x,y)=\Phi_1\left(y\right)+\sum_{n=1}^{+\infty}\left(-\lambda\right)^n I_{0y}^{\alpha,n\beta,n\gamma,\delta}I_{0x}^{n}\Phi_1\left(y\right)$$

$$\vartheta_3(x,y)=\sum_{n=0}^{+\infty}\left(-\lambda\right)^n I_{0y}^{\alpha,n\beta+\beta,n\gamma+\gamma,\delta}I_{0x}^{n+1}f\left(x,y\right)$$

Now performing some calculations, we can write the functions $\vartheta_1(x,y)$, $\vartheta_2(x,y)$ and $\vartheta_3(x,y)$ as follows:

$$\vartheta_1(x,y)=\varphi\left(x\right)+\sum_{n=1}^{+\infty}\left(-\lambda\right)^n I_{0y}^{\alpha,n\beta,n\gamma,\delta}I_{0x}^{n}\varphi\left(x\right)=\varphi\left(x\right)-$$

$$-\lambda y^{\beta}\int_0^x\varphi\left(\xi\right)E_{12}\left(\begin{array}{l}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1\end{array}\middle|\begin{array}{c}-\lambda\left(x-\xi\right)y^{\beta}\\ \delta y^{\alpha}\end{array}\right)d\xi, \tag{3.13}$$

$$\vartheta_2(x,y)=\Phi_1\left(y\right)+\sum_{n=1}^{+\infty}\left(-\lambda\right)^n I_{0y}^{\alpha,n\beta,n\gamma,\delta}I_{0x}^{n}\Phi_1\left(y\right)=\Phi_1\left(y\right)-\lambda x\int_0^y\left(y-\eta\right)^{\beta-1}\times$$

$$\times E_{12}\left(\begin{array}{l}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1\end{array}\middle|\begin{array}{c}-\lambda x\left(y-\eta\right)^{\beta}\\ \delta\left(y-\eta\right)^{\alpha}\end{array}\right)\Phi_1\left(\eta\right)d\eta, \tag{3.14}$$

$$\vartheta_3(x,y)=\sum_{n=0}^{+\infty}(-\lambda)^n\, I_{0y}^{\alpha,n\beta+\beta,n\gamma+\gamma,\delta} I_{0x}^{n+1} f\left(x,y\right)=\int\limits_0^x\int\limits_0^y (y-\eta)^{\beta-1}\times$$

$$\times f\left(\xi,\eta\right)E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)\left(y-\eta\right)^{\beta}\\ \delta\left(y-\eta\right)^{\alpha}\end{matrix}\right)d\xi d\eta. \tag{3.15}$$

Using the functions (3.13), (3.14) and (3.15), we write (3.12) in the form:

$$u\left(x,y\right)=\varphi\left(x\right)+\,\Phi_1\left(y\right)-\lambda y^{\beta}\int\limits_0^x\varphi\left(\xi\right)E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right)d\xi-$$

$$-\lambda x\int\limits_0^y\left(y-\eta\right)^{\beta-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1\end{matrix}\middle|\begin{matrix}-\lambda x(y-\eta)^{\beta}\\ \delta(y-\eta)^{\alpha}\end{matrix}\right)\Phi_1\left(\eta\right)d\eta+$$

$$+\int\limits_0^x\int\limits_0^y\left(y-\eta\right)^{\beta-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)\left(y-\eta\right)^{\beta}\\ \delta(y-\eta)^{\alpha}\end{matrix}\right)f\left(\xi,\eta\right)d\xi d\eta. \tag{3.16}$$

For the function (3.16), the following equalities hold:

$$u\left(0,y\right)=\varphi\left(0\right)+\,\Phi_1\left(y\right)=\psi\left(y\right),\;\Phi_1\left(y\right)=\psi\left(y\right)-\varphi\left(0\right)$$

or since $\varphi\left(0\right)=\psi\left(0\right)$, we have

$$\Phi_1\left(y\right)=\psi\left(y\right)-\psi\left(0\right). \tag{3.17}$$

Using (3.17), we find the functions $\Phi_1\left(y\right)$, substitute them into (3.16), perform some calculations, and determine function $u(x,y)$ in the form

$$u\left(x,y\right)=\varphi\left(x\right)+\,\psi\left(y\right)-\psi\left(0\right)-$$

$$-\lambda y^{\beta}\int\limits_0^x\varphi\left(\xi\right)E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)y^{\beta}\\ \delta y^{\alpha}\end{matrix}\right)d\xi-$$

$$-\lambda x\int\limits_0^y\left(\psi\left(\eta\right)-\psi\left(0\right)\right)\left(y-\eta\right)^{\beta-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1\end{matrix}\middle|\begin{matrix}-\lambda x(y-\eta)^{\beta}\\ \delta(y-\eta)^{\alpha}\end{matrix}\right)d\eta+$$

$$+\int\limits_0^x\int\limits_0^y\left(y-\eta\right)^{\beta-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)\left(y-\eta\right)^{\beta}\\ \delta(y-\eta)^{\alpha}\end{matrix}\right)f\left(\xi,\eta\right)d\xi d\eta. \tag{3.18}$$

Theorem 3.1. *If $\varphi\left(0\right)=\psi\left(0\right)$, $\varphi\left(x\right)\in C\left[0,a\right]\cap C^1(0,a)$, $\psi\left(y\right)\in C\left[0,b\right]\cap C^1(0,b)$, and $f\left(x,y\right)=x^{-\varepsilon_1}y^{-\varepsilon_2}\widetilde{f}_1\left(x,y\right)$, where $\widetilde{f}_1\left(x,y\right)\in C\left(\overline{\Omega}\right)$ and $0\le\varepsilon_1<1$, $0\le\varepsilon_2<\beta$, then the solution to problem G exists and is unique, determined by formula (3.18).*

Observe that this result is presented for the first time in the international conference [12].

Theorem 3.2. *If all condition of the Theorem 3.1 and $\lambda>0$, $\delta<0$ are valid, then for the solution of equation* (3.1)*, the following inequality holds:*

$$\|u(x,y)\|_{C(\overline{\Omega})}\le C_1\|\varphi(x)\|_{C[0,a]}+C_2\|\psi(y)\|_{C[0,b]}+$$

$$+C_3|\psi(0)|+C_4\|x^{\varepsilon_1}y^{\varepsilon_2}f(x,y)\|_{C(\overline{\Omega})},$$

where $C_j,\ j=\overline{0,4}$ are some positive constants, $0\le\varepsilon_1<1$, $0\le\varepsilon_2<\beta$.

References

[1] Chu Y.M., Inc M., Hashemi M.S., Eshaghi S.; Analytical treatment of regularized Prabhakar fractional differential equations by invariant subspaces. Comput.Appl.Math.,–2022.–41.–6.P.1–17.

[2] DOvidio M, Polito F. Fractional diffusion-telegraph equations and their associated stochastic solutions. Theory Probab.Appl.–2013.–62.–4.P.552–574.

[3] Eremin A.S.; Three Problems for One Equation in Partial Fractional Derivatives. Proceedings of the All-Russian Scientific Conference "Differential Equations and Boundary Value Problems." Part 3. Mathematical Modeling and Boundary Value Problems. Samara State Technical University (SamSTU). Samara.–2004.P.94–98. [in Russian]

[4] Eshaghi S., Khoshsiar Ghaziani R., Ansari A.; Stability and chaos control of regularized Prabhakar fractional dynamical systems without and with delay. Math. Methods Appl.Sci.–2019.–42.–7.P.2302–2323.

[5] Eshaghi S., Ansari A., Khoshsiar Ghaziani R.; Generalized Mittag-Leffler stability of nonlinear fractional regularized Prabhakar differential systems. Int.J.Non.Anal.Appl.–2021.–12.–2.P.665–678.

[6] Garrappa R.; Grünwald-Letnikov operators for fractional relaxation in Havriliak-Negami models. Commun.Non.Sci.Numer.Simul.–2016.–38.P.178–191.

[7] Garrappa R., Kaslik E.; Stability of fractional-order systems with Prabhakar derivatives. Nonlinear Dyn.–2020.–102.P.567–578.

[8] Giusti A., Colombaro I.; Prabhakar-like fractional viscoelasticity. Commun.Non.Sci.Numer.Simulat.–2018.–56.P.138–143.

[9] Hasanov A., Karimov E.T.; On generalized Mittag-Leffler type functions of two variables. https://arxiv.org/pdf/2501.03918

[10] Kilbas A.A., Repin O.A.; An Analog of the Bitsadze-Samarskii Problem for a Mixed Type Equation with a Fractional Derivative. Differential Equations.–2003.–39.–5.P.638–644. [in Russian]

[11] Kilbas A.A., Repin O.A.; A Nonlocal Problem for an Equation of Mixed Type with a Riemann-Liouville Partial Derivative and Operators of Generalized Fractional Integration in the Boundary Condition. Proceedings of the Institute of Mathematics. Minsk.–2004.–12.–2.P.75–81. [in Russian]

[12] Karimov E.T., Turdiev K.N., Usmonov D.A.; Boundary-value problem for a telegraph equation of fractional order with the Prabhakar operator. Proceedings of the International Conference Differential Equations and their Applications, Kazan Federal University, Kazan, 6-7 June 2024.

[13] Mainardi F., Garrappa R.; On complete monotonicity of the Prabhakar function and non-Debye relaxation in dielectrics. J.Comput.Phys.–2015.–293.P.70–80.

[14] Mikhlin S.G. (ed.); Linear Integral Equations. Dover Publications, New York.–2020.

[15] Pshibikhova R.A.; The Goursat Problem for the Fractional Telegraph Equation with Caputo Derivatives. Matematicheskie Zametki.–2016.–99.–4.P.559–563. [in Russian]

[16] Pshibikhova R.A.; Goursat's task for the fractional telegraph equation with Caputo's derivatives and with the integrated. News of the Kabardin-Balkar scientific center of RAS.–2016.–2.P.25–29. [in Russian]

[17] Pshibikhova R.A.; An Analog of the Goursat Problem for the Generalized Fractional-Order Telegraph Equation. Differential Equations.–2014.–50.–6.P.839–843. [in Russian]

[18] Pskhu A.V. Partial Differential Equations of Fractional Order. Nauka, Moscow. – 2005. [in Russian]

[19] Prabhakar T.R.; A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math.J.–1971.–19.P.:7–15.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.307-326
DOI: 10.29229/uzmj.2025-2-30

CTRW-limits in the the Skorokhod topology and FPK equations associated with SDEs driven by CTRW-limit processes

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this paper we study conditions under which a wide class of continuous time random walk stochastic processes have limits in the J_1- and M_1-topologies of the Skorokhod space of càdlàg processes. The approach used for the proof is based on the theory of convergence of composite stochastic processes developed by Silvestrov in the 1970th. Stochastic differential equations driven by such limits are introduced. Fokker-Planck-Kolmogorov equations associated with the corresponding SDEs driven by CTRW-limits are established.

Keywords: continuous time random walk, stochastic process, time-changed process, Lévy process, stable subordinator, mixed Lévy processes, stochastic differential equation, Fokker-Planck-Kolmogorov equation

MSC (2020): 35Q84, 85C31

1. Introduction

A continuous time random walk (CTRW), by definition, is a random walk subordinated to a renewal process. More precisely CTRW is a stochastic process constructed as sums of an i.i.d. sequence of random variables (vectors) $\{X_n, n \geq 0\}$, expressing jumps with the given initial state X_0, and with random waiting times t_n between two consecutive jumps X_{n-1} and X_n. The sequence $\{t_n, n \geq 0\}$, itself is an i.i.d. sequence of non-negative random variables with the convention $t_0 = 0$. I.i.d. assumptions imply that the pairs $\{(X_n, t_n), n \geq 0\}$, also form an i.i.d. sequence. Since each jump X_n occurs at a random time $T_n = t_0 + \cdots + t_n$ the CTRW is expressed as a stochastic process

$$X_t = \sum_{n=0}^{N_t} X_n,$$

where $N_t = \max\{n \geq 0 : T_n \leq t\}$. The latter can also be expressed as the composition

$$X_t = (S \circ N)_t = S_{N_t} = S(N_t),$$

of two stochastic processes S_t and N_t. The first of these processes,

$$S_t = \sum_{n=0}^{t} X_n,$$

is a discrete stochastic process defined at discrete times $t = 0, 1, \ldots,$ and the second one N_t is defined for all $t \geq 0$. Both processes can be interpreted as càdlàg processes. A stochastic process is càdlàg, if its sample paths are right-continuous and have left-limits. Therefore, the process X_t is a càdlàg stochastic process as well, as a composition of two càdlàg processes. If the sequences X_n and T_n are independent, then CTRW X_t is called decoupled, otherwise coupled. Historically, CTRW concept was first introduced by Montrol and Weiss [25] in 1965. Nowadays CTRW approach to mathematical modeling of random processes is widely used in various applications in science and engineering (see, e.g. [2, 8, 21]). CTRW have been extensively investigated recent years by many authors; see [6, 7, 8, 19, 20, 22, 23, 21, 41] and references therein.

We note that if the random variable t_n has a finite mean, then T_n is asymptotically equivalent to ct with some constant $c > 0$ [17]. In this case CTRW behaves (asymptotically) like an ordinary random walk. In addition, if X_n has a finite variance, then due to Donsker's theorem the scaled limit of CTRW in the uniform topology represents Brownian motion. In this paper we are interested in scaled limits of CTRW in the Skorokhod topologies of càdlàg processes in the case when t_n has infinite mean and X_n has infinite variance.

The limiting stochastic processes of various CTRWs in the sense of convergence in law are well studied; see, for example, works [7, 8, 20, 39, 40, 37], and references therein. There are different approaches in this direction among which are constructive and abstract methods. Constructive methods are especially useful in creating simulation models for various applied processes.

Convergence of sequences of càdlàg stochastic processes in J- and M-topologies was studied by Skorokhod [35]. The definitions of J_1- and M_1-topologies see in Section 2. The main conditions ensuring such convergences were the identification of limiting process in a weaker topology (say convergence in law) and stochastic compactness of the sequence of stochastic processes under consideration. Compositions of càdlàg stochastic processes by nondecreasing càdlàg stochastic processes, interpreted as stopping-time stochastic processes, were studied by Silvestrov [31, 33]. The survey of works in this direction including various applications one can find in [32]. Convergence of CTRWs formed with Lévy stable jump processes and one dimensional Lévy stable subordinated waiting times in the M_1-Skorokhod topology was studied by Meerschaert and Scheffler in papers [22, 23] using the coninuity mapping theorem in Skorokhod spaces. Note that M_1-topology is weaker than the J_1-topology. The continuity mapping theorem does not work for the stronger J_1-convergence in the case when a subordinator of limiting process is not strictly increasing. In this paper we will use a different approach which does not use the continuity mapping theorem. Namely, our method is based on the theorem of convergence of compositions of càdlàg stochastic processes in the J_1 and M_1-topologies established in [31, 33]. We will also consider CTRWs with waiting times represented by mixtures of stable subordinaters. In particular, in the case of M_1-convergence the corresponding results generalize theorems obtained in [22, 23].

The scaled limit processes of CTRWs play an important role in modeling of various complex random processes. For example, they serve as a driving process in the stochastic differential equation models of the form

$$dY_t = b(Y_t)dt + \sigma(Y_t)dW_t, \quad \lim_{t\to 0+} Y_t = Y_0,$$

where $b(\cdot)$ and $\sigma(\cdot)$ are given functions and W_t is a driving process, which is a scaled limit of CTRW. The deterministic description of such processes, depending on some classes of CTRW limits, can be given by fractional order pseudo-differential equations. The latter is refereed as Kolmogorov-Fokker-Planck (FPK) equations [10, 9, 39, 38].

The paper is organized as follows. In Section 2 the space of càdlàg stochastic processes and Skorokhod's J_1 and M_1 topologies in it are introduced. The characterization of these spaces given in this section is based on the spirit of projective limits of decreasing topological spaces. The necessary known facts are also presented there. In Section 3 relevant facts on Lévy stochastic processes and subordinators are presented. In Section 4 we prove theorems on convergence of continuous time random walk stochastic processes in Skorokhod topologies. We note that CTRW-limits considered in this paper generalizes results obtained in [22, 23]. In Section 5 we discuss stochastic integrals driven by cádlág processes. Here new SDEs driven by CTRW-limit processes are introduced as well. As a time-changed subordinated process here we consider the inverse of a positive stable Lévy process or mixtures of such processes. Finally, in Section 6 we establish Fokker-Planck-Kolmogorov (FPK) equations associated with SDEs driven by CTRW-limit processes.

2. Skorokhod space and Skorokhod topologies

The Skorokhod spaces were introduced in [35]. In this section we briefly describe these spaces for completeness. A function $x : [0,\infty) \to \mathbb{R}^d$ is said to be *càdlàg* if x is right-continuous at any $t \geq 0$, that is $\lim_{s\to t+} x(s) = x(t)$, and has left limits at any $t > 0$, that is $\lim_{s\to t-} x(s) = x(t-)$ exists. A function $x : [0,\infty) \to \mathbb{R}^d$ is said to be *càglàd* if x is left-continuous at any $t > 0$, that is $\lim_{s\to t-} x(s) = x(t)$, and has right limits at any $t \geq 0$, that is $\lim_{s\to t+} x(s) = x(t+)$ exists. In this paper we consider only càdlàg functions and accept $x(0-) = 0$. The space of all càdlàg functions defined on $[0,\infty)$ and denoted by $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is called *Skorokhod space.* $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is complete with respect to the locally uniform topology generated by the metric

$$d(x,y)=\sum_{n=1}^{\infty}\frac{\min(1,\|x-y\|_n)}{2^n},$$

where

$$\|x\|_n=\sup_{0\le t\le n}|x(t)|=\sup_{0\le t\le n}(|x_1(t)|^2+\dots+|x_d(t)|^2)^{1/2}.$$

However, $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is not separable with respect to this topology. Recall that a metric space is separable if it contains a countable dense subset with respect to the corresponding metric. To see that $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is not separable, for simplicity assume $d=1$. Assume, in contrary, that $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is separable with respect to the uniform metric $d(x,y)$. It is known that any subset of a separable set is also separable. Consider, the set $\mathbb{D}_0:=\{x_a(t)=I_{[a,2]}(t), 1\le a<2\}$, where $I_A(t)$ is the indicator function of A. Obviously, $\mathbb{D}_0$ is a subset of $\mathbb{D}([0,\infty),\mathbb{R})$ and must be separable. However, it is not, since $d(x_a,x_b)=1$ for all $a,b\in[1,2),\ a\neq b$.

The uniform topology $d(x,y)$ works well in the subspace $C([0,\infty),\mathbb{R}^d)$ of continuous functions of $\mathbb{D}([0,\infty),\mathbb{R}^d)$. However, due to above fact on non-separability, it fails to be good in approximations of functions in $\mathbb{D}([0,\infty),\mathbb{R}^d)$. Therefore, one needs better topologies to meet approximation issues. Below we provide some facts related to the Skorokhod space with nonuniform topologies.

There are two frequently used Skorokhod topologies (introduced by A.V. Skorokhod [35] in 1956) called J_1-topology and M_1-topology, which are useful in approximation of càdlàg functions. To define J_1-topology we introduce the set Λ of continuous strictly increasing functions λ defined on $[0,\infty)$ and satisfying the conditions $\lambda(0)=0$ and $\lim_{t\to\infty}\lambda(t)=\infty$. The J_1-*topology* is defined by the metric

$$\delta(x,y)=\sum_{n=1}^{\infty}\frac{\min(1,\omega_n(x,y))}{2^n}, \tag{2.1}$$

where

$$\omega_n(x,y)=\inf_{\lambda\in\Lambda}\left(\|\lambda-t\|_n+\|x-y\circ\lambda\|_n\right).$$

We denote the Skorokhod space endowed with the J_1-topology by $\mathbb{D}([0,\infty),\mathbb{R}^d,J_1)$.

The letter can be defined as a projective limit of a sequence of Skorokhod spaces defined on finite intervals. This approach works for other topologies as well. Let $\{t_n\}$ be an increasing sequence of positive (non-random) numbers:

$$0<t_1<t_2<\dots,\quad\text{and}\quad t_n\to\infty,\quad n\to\infty. \tag{2.2}$$

Let $\mathbb{D}_n=\mathbb{D}([0,t_n),\mathbb{R}^d,\tau_n)$ be the space of càdlàg functions defined on $[0,t_n)$ and endowed with a topology τ_n, such that τ_{n+1} is weaker than τ_n (denoted by $\tau_{n+1}\prec\tau_n$) for all $n\ge 1$. As the sequence of sets, $\mathbb{D}_n$ satisfies the condition

$$\mathbb{D}_1\supset\mathbb{D}_2\supset\dots\supset\mathbb{D}_n\supset\mathbb{D}_{n+1}\supset\dots, \tag{2.3}$$

and due to the condition on the sequence of topologies

$$\tau_1\succ\tau_2\succ\dots\succ\tau_n\succ\tau_{n+1}\succ\dots, \tag{2.4}$$

we have that the space $\mathbb{D}_{n+1}$ is continuously embedded to the space $\mathbb{D}_n$ for each $n\ge 1$. Hence, we can define the projective limit of $\mathbb{D}_n$:

$$\mathbb{D}_\infty(\tau)=\mathrm{pr}\lim_{n\to\infty}\mathbb{D}_n:=\cap_{n=1}^{\infty}\mathbb{D}_n,$$

endowed with the coarsest topology $\tau=\lim_{n\to\infty}\tau_n$. If the topology τ_n is induced by the J_1-metric

$$\omega_n^{'}(x,y)=\inf_{\lambda\in\Lambda_n}\left(\|\lambda-t\|_{t_n}+\|x-y\circ\lambda\|_{t_n}\right),$$

where Λ_n is the set of continuous functions mapping $[0,t_n]$ onto $[0,t_n]$ such that $\lambda(0)=0, \lambda(t_n)=t_n$, and $\|a\|_{t_n}=\sup_{0\le t\le t_n}|a(t)|$, then the sequence τ_n satisfies (2.4). The coarsest topology in this case is the topology induced by the metric

$$\delta^{'}(x,y)=\sum_{n=1}^{\infty}\frac{\min(1,\omega_n^{'}(x,y))}{2^n},$$

and is equivalent to the J_1-topology induced by the metric $\delta(x,y)$ defined in (2.1). Hence, in this case we have

$$\mathbb{D}_\infty(J_1)=\mathbb{D}([0,\infty),\mathbb{R}^d,J_1).$$

To define the M_1-topology, we first introduce the notion of *completed graph* of a function $x\in\mathbb{D}([0,t_n),\mathbb{R}^d)$, which takes into account straight lines connecting $(t,x(t-))$ with $(t,x(t))$ in the cross sections $t\times\mathbb{R}^d$ of the space $[0,t_n)\times\mathbb{R}^d$ with the discontinuity points $t\in[0,t_n)$. Namely, the completed graph Γ_x of $x\in\mathbb{D}([0,t_n),\mathbb{R}^d)$ is defined as

$$\Gamma_x=\{(t,z)\in\mathbb{R}_+\times\mathbb{R}^d : z=\alpha x(t-)+\beta x(t), \quad \alpha+\beta=1\}.$$

One can define an order relation in Γ_x. We say that $(t_1,z_1)<(t_2,z_2)$ if either $t_1<t_2$, or if $t_1=t_2$, then $|x(t-)-z_1|<|x(t-)-z_2|$. By the *parametric representation* of the function x we understand the nondecreasing function (r,u) mapping $[0,t_n)$ onto Γ_x, where r is the time component and u is the spatial component of the completed graph Γ_x. Denote by Π_x the set of parametric representations of x. The M_1-*topology* in $\mathbb{D}([0,t_n),\mathbb{R}^d)$ is induced by the metric

$$\rho_n(x,y)=\inf_{\substack{(r,u)\in\Pi_x\\(s,v)\in\Pi_y}}\left(\|r-s\|_{t_n}+\|u-v\|_{t_n}\right),$$

with an increasing sequence $\{t_n\},\ t_n\to\infty$, being continuity points of $x,\ y$. For the fact that $\rho_n(x,y)$ indeed is a metric, see [42]. Moreover, one can easily verify that topologies induced by $\rho_n(x,y)$ satisfy the condition (2.4). Hence, we can define

$$\mathbb{D}([0,\infty),\mathbb{R}^d,M_1):=\mathrm{pr}\lim_{n\to\infty}\mathbb{D}([0,t_n),\mathbb{R}^d,M_1). \tag{2.5}$$

with the coarsest topology of the projective limit, which is equivalent to the topology induced by the metric

$$m(x,y)=\sum_{n=1}^{\infty}\frac{\min(1,\rho_n(x,y))}{2^n}.$$

Further, we introduce the sequence of continuous functions $\kappa_n,\ n=1,2,\cdots:$

$$\kappa_n(t)=\begin{cases}1, & \text{if } \ t<t_n,\\ \frac{t_{n+1}-t}{t_{n+1}-t_n}, & \text{if } \ t_n\le t<t_{n+1},\\ 0, & \text{if } \ t\ge t_{n+1},\end{cases}$$

and define the following metric in $\mathbb{D}([0,t_{n+1}),\mathbb{R}^d)$

$$\pi_n(x,y)=\inf_{\lambda\in\Lambda_{n+1}}\left(|||\lambda|||_n+\|\kappa_n x-(\kappa_n y)\circ\lambda\|\right),$$

where

$$|||\lambda|||_n=\sup_{0\le s<t\le t_{n+1}}\left|\ln\frac{\lambda(t)-\lambda(s)}{t-s}\right|.$$

For the fact that $\pi_n(x,y)$ is a metric, see [14]. One can easily verify that topologies induced by $\rho_n(x,y)$ satisfy the condition (2.4). Hence, we can define

$$\mathbb{D}([0,\infty),\mathbb{R}^d,\pi):=\mathrm{pr}\lim_{n\to\infty}\mathbb{D}([0,t_n),\mathbb{R}^d,\pi_n). \tag{2.6}$$

with the coarsest topology of the projective limit, which is equivalent to the topology induced by the metric

$$\pi(x,y)=\sum_{n=1}^{\infty}\frac{\min(1,\pi_n(x,y))}{2^n}.$$

This metric was introduced by Prokhorov [27] in 1956. We note that even though the Skorokhod space is separable with respect to J_1- and M_1-topologies, it is not complete under these topologies. However, the space $\mathbb{D}([0,\infty),\mathbb{R}^d,\pi)$ is complete under the topology π.

We say that a sequence $x_k\in\mathbb{D}([0,\infty),\mathbb{R}^d,J_1)$, $k=1,2,\dots$, converges to $x\in\mathbb{D}([0,\infty),\mathbb{R}^d,J_1)$ in the J_1-topology if $\delta(x_k,x)\to 0$ as $k\to\infty$. Similarly, we say that a sequence $x_k\in\mathbb{D}([0,\infty),\mathbb{R}^d,M_1)$ ($\mathbb{D}([0,\infty),\mathbb{R}^d,\pi)$), $k=1,2,\dots$, converges to $x\in\mathbb{D}([0,\infty),\mathbb{R}^d,M_1)$ ($x\in\mathbb{D}([0,\infty),\mathbb{R}^d,\pi)$) in the M_1-topology (π-topology) if $m(x_k,x)\to 0$ ($\pi(x_k,x)\to 0$) as $k\to\infty$. We also use the notations $\xrightarrow{J_1}$ and $\xrightarrow{M_1}$ for convergences in the J_1-topology and in the M_1-topology, respectively. The projective limit structure used above for the definition of the Skorokhod space of càdlàg functions defined on the semi-infinite interval $[0,\infty)$ allows a characterization of the convergence via the corresponding convergence on finite intervals. For instance, a sequence $x_k\in\mathbb{D}([0,\infty),\mathbb{R}^d,J_1)$ converges to $x_0\in\mathbb{D}([0,\infty),\mathbb{R}^d,J_1)$ in the J_1-topology, if restrictions of x_k to $\mathbb{D}([0,t_n),\mathbb{R}^d,J_1)$ converge to the restriction of x_0 to $\mathbb{D}([0,t_n),\mathbb{R}^d,J_1)$ in the J_1-topology for any sequence t_n satisfying the condition (2.2). The same characterization is true for convergences in the M_1- and π-topologies.

Let $Disc(x)$ be the set of discontinuity points of $x\in\mathbb{D}([0,\infty),\mathbb{R}^d)$. Namely, $Disc(x)=\{t\in\mathbb{R}_+ : x(t-)\neq x(t)\}$.

Proposition 2.1. (1) *The set $Disc(x)$ for $x\in\mathbb{D}([0,\infty),\mathbb{R}^d)$ is at most countable;*

(2) *For any $x,y\in\mathbb{D}([0,\infty),\mathbb{R}^d)$ the relation $m(x,y)\le\delta(x,y)$ holds;*

(3) *Let τ be either J_1- or M_1-topology and $x_k\xrightarrow{\tau}x$ in $\mathbb{D}([0,\infty),\mathbb{R}^{d_1},\tau)$ and $y_k\xrightarrow{\tau}y$ in $\mathbb{D}([0,\infty),\mathbb{R}^{d_2},\tau)$. If $Disc(x)\cap Disc(y)=\emptyset$, then $(x_k,y_k)\xrightarrow{\tau}(x,y)$ in $\mathbb{D}([0,\infty),\mathbb{R}^{d_1+d_2},\tau)$;*

(4) *Let $x_k\xrightarrow{M_1}x$ in $\mathbb{D}([0,\infty),\mathbb{R}^d,M_1)$ and $y_k\xrightarrow{M_1}y$ in $\mathbb{D}([0,\infty),\mathbb{R}^d,M_1)$. If $Disc(x)\cap Disc(y)=\emptyset$, then $x_k-y_k\xrightarrow{M_1}x-y$ in $\mathbb{D}([0,\infty),\mathbb{R}^d,M_1)$.*

(5) *If $x_k\xrightarrow{J_1}x$ in $\mathbb{D}([0,\infty),\mathbb{R}^d,J_1)$, then $x_k\xrightarrow{M_1}x$ in $\mathbb{D}([0,\infty),\mathbb{R}^d,M_1)$;*

(6) *For any nondecreasing $\lambda\in\mathbb{D}([0,\infty),\mathbb{R}_+)$ and any $x\in\mathbb{D}([0,\infty),\mathbb{R}^d)$ the function $x\circ\lambda\in\mathbb{D}([0,\infty),\mathbb{R}^d)$.*

For proofs and further details we refer the reader to books [4, 14, 42, 33].

A stochastic process Z is said to be *càdlàg* (resp. *càglàd*) if Z has (resp. *càglàd*) sample paths. It follows from Proposition 2.1 that the assumption that Z is càdlàg or càglàd requires its sample paths to have at most countably many finite jumps. Associated to a càdlàg process Z is its jump process $(\Delta Z_t)_{t\ge 0}$ where $\Delta Z_t:=Z_t-Z_{t-}$ with Z_{t-} denoting the left limit at t and $Z_{0-}=0$ by convention.

The convergence of càdlàg stochastic processes in the Skorokhod topologies is characterized by recognizing the limiting process by the weak convergence (or in law) and compactness of the sequence of stochastic processes. These characterizations were given by Skorokhod [35]. Below we follow criteria for J_1- and M_1-convergences of càdlàg stochastic processes given in [33].

Let a family of càdlàg stochastic processes $X_\varepsilon(t)$, $t\ge 0$, $\varepsilon>0$, and a càdlàg stochastic process $X_0(t)$, $t\ge 0$, satisfy the following two conditions:

(C) weak convergence: $X_\varepsilon(t)$ weakly converges to the process $X_0(t)$ as $\varepsilon\to 0$ for all $t\in S$, where S is a subset of $[0,\infty)$ that is everywhere dense in this interval and contains the point 0;

(K_J) J_1-compactness:

$$\lim_{c\to 0}\limsup_{\varepsilon\to 0}\ \mathbb{P}\{\omega_J(X_\varepsilon(\cdot),c,T)>\delta\}=0,\quad\text{for all}\quad \delta,T>0, \tag{2.7}$$

where

$$\omega_J(x(\cdot),c,T)=\sup_{t-c\le t_1<t<t_2\le t+c}\inf(|x(t_1)-x(t)|,|x(t_2)-x(t)|). \tag{2.8}$$

(K_M) M_1-compactness:

$$\lim_{c\to 0}\limsup_{\varepsilon\to 0}\ \mathbb{P}\{\omega_M(X_\varepsilon(\cdot),c,T)>\delta\}=0, \quad \text{for all} \quad \delta,T>0,$$

where

$$\omega_M(x(\cdot),c,T)=\sup_{t-c\le t_1<t_2<t_3\le t+c}|x(t_2)-[x(t_1),x(t_3)]|,$$

with $[x(t_1),x(t_3)]=\{ax(t_1)+(1-a)x(t_3):\ 0\le a\le 1\}$.

Proposition 2.2. (1) *Conditions (C) and (K_J) are necessary and sufficient for the J_1-convergence $X_\varepsilon \xrightarrow{J_1} X_0$ as $\varepsilon\to 0$.*

(2) *Conditions (C) and (K_M) are necessary and sufficient for the M_1-convergence $X_\varepsilon \xrightarrow{M_1} X_0$ as $\varepsilon\to 0$.*

3. Lévy's stable processes and their mixtures

A càdlàg process $L_t,\ t\ge 0$, in $\mathbb{R}^d$ is called a *Lévy process* if L has independent and stationary increments and the mapping $t\to L_t$ is continuous in probability; i.e. $\lim_{s\to t}\mathbb{P}(|L_t-L_s|>\varepsilon)=0$ for all $\varepsilon>0$ and $t\ge 0$. Note that the continuity in probability does not imply that the sample paths are continuous; in fact, sample paths of many important Lévy processes have jumps. The simplest examples of Lévy processes include Brownian motion and Poisson processes. The sum of independent identically distributed Lévy processes is again a Lévy process, as well as their scalar multiples. Lévy processes are characterized by three parameters: a vector $b_0\in\mathbb{R}^d$, a nonnegative definite $d\times d$ matrix Σ, and a measure ν defined on $\mathbb{R}^d\setminus\{0\}$ such that

$$\int(1\wedge|w|^2)\nu(dw)<\infty,$$

called the Lévy measure, where $a\wedge b=\min\{a,b\}$. Examples of Lévy measures include any Lebesgue measure with compact support and measures integrable at infinity.

The Lévy–Itô decomposition states that a given Lévy process L_t is represented as

$$L_t=\tilde b_0 t+\sigma_0 B_t+\int_{|w|<1}w\tilde N(t,dw)+\int_{|w|\ge 1}wN(t,dw), \tag{3.1}$$

where $\tilde b_0\in\mathbb{R}^n$, σ_0 is an $n\times m$-matrix, B is an m-dimensional Brownian motion, and N and $\tilde N$ are a Poisson random measure and a compensated Poisson martingale-valued measure on $[0,\infty)\times(\mathbb{R}^n\setminus\{0\})$, respectively (for details see [1, 3, 29]). Namely, $N(t,A)$ represents the number of jumps of size A up to time t, and $\left(\int_{|w|\ge 1}wN(t,dw)\right)_{t\ge 0}$ is a compound Poisson process describing large jumps, whereas $\left(\int_{|w|<1}w\tilde N(t,dw)\right)_{t\ge 0}$ is the compensated sum of small jumps. The matrices σ_0 and Σ in equations (3.1) and (3.2), respectively, are related as $\Sigma=\sigma_0\times\sigma_0^T$, where σ_0^T is the transpose of σ_0. Vectors b_0 and $\tilde b_0$ responsible for the drift are not necessarily the same.

The *Lévy–Khintchine formula* characterizes the Lévy process L as an infinitely divisible process. Namely, the characteristic function for L_t is given by

$$\varphi_{L_t}(\xi)=\mathbb{E}[e^{i(\xi,L_t)}]=e^{t\Psi(\xi)},$$

where

$$\Psi(\xi)=i(b_0,\xi)-\frac{1}{2}(\Sigma\xi,\xi)+\int_{\mathbb{R}^d\setminus\{0\}}(e^{i(w,\xi)}-1-i(w,\xi)I_{(-1,1)}(w))\nu(dw). \tag{3.2}$$

The function Ψ is called a *Lévy symbol* of L. The Lévy symbol Ψ is continuous, hermitian, conditionally positive definite and $\Psi(0)=0$.

Lévy processes are infinite divisible [1, 29]. For a Lévy process L_t for a fixed $t\ge 0$ one has

$$L_t=Y_1(t)+\cdots+Y_n(t),$$

in distribution, where $Y_k(t) = L_{(k/n)t} - L_{((k-1)/n)t},\ k = 1,\dots,n$. Due to independence of increments of L_t, the variables $Y_k(t)$ are independent, and due to stationarity of L_t, each of $Y_k(t)$ is distributed as $L_{t/n}$. Hence, $Y_k(t)$ are i.i.d. Thus we have

$$\{\tilde{S}_n(t)\}_{t\geq 0} \overset{FD}{\Rightarrow} \{L_t\}_{t\geq 0} \quad \text{as } n \to \infty,$$

where

$$\tilde{S}_n(t) = Y_1(t) + \cdots + Y_n(t).$$

An important subclass of Lévy processes are *stable* Lévy processes. In 1-D case, by definition, a Lévy process L_t is *stable* [1, 24, 26] if

$$L_{ct}{=}c^{1/\alpha}L_t, \tag{3.3}$$

in distribution for all $t > 0$ with some $0 < \alpha \leq 2$. The number α is called a *stability index* of L_t. The property (3.3) is called a self-similarity of the process L_t. Due to the self-similarity property of stable Lévy processes it suffices to study L_t for $t = 1 : X = L_1$. This random variable is a Lévy stable distribution. For complete description of Lévy stable distributions four parameters are used: a stability index $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\sigma > 0$, and a location parameter $\mu \in \mathbb{R}$. Among these parameters, α is the most important one, determining a peculiar class of stability. To emphasize this, the term α-stable distribution is used for distributions with the stability index α. We denote the class of α-stable distributions by $S_\alpha(\beta, \sigma, \mu)$. In literature there are different parameterizations used for the description of stable distributions; see e.g. [24, 26, 28]. For us it is convenient to adopt the parameterization used in [28]. Namely, the Lévy symbol (3.2) of α-stable distribution X has the form

$$\Psi(\xi) = \begin{cases} i\xi\mu - \sigma^\alpha|\xi|^\alpha(1 - i\beta\text{sign}(\xi)\tan\frac{\alpha\pi}{2}), & \text{if } \alpha \neq 1, \\ i\xi\mu - \sigma|\xi|(1 + i\beta\text{sign}(\xi)\frac{2}{\pi}\ln|\xi|), & \text{if } \alpha = 1, \end{cases} \tag{3.4}$$

where

$$\text{sign}(\xi) = \begin{cases} 1, & \text{if } \xi \geq 0, \\ -1, & \text{if } \xi < 0. \end{cases}$$

It follows from (3.4) that if $\alpha = 2$, then $\varphi_X(\xi) = \exp(i\xi\mu - \sigma^2\xi^2)$, which is the characteristic function of the normal distribution with mean μ and variance $2\sigma^2$. As it is well known, the density function of the latter is an exponential function

$$f(x) = \frac{1}{2\sigma\sqrt{\pi}}e^{-\frac{(x-\mu)^2}{4\sigma^2}}, \quad x \in \mathbb{R}. \tag{3.5}$$

If $\alpha < 2$, then the density $f_X(x)$ of α-stable distribution X has power law decay at infinity, and [28]

$$f_X(x) = O(|x|^{-1-\alpha}), \quad |x| \to \infty. \tag{3.6}$$

In the case $1 < \alpha < 2$ the variance does not exist, but the mean exists and equals μ. If $\beta = 0$ then the Lévy distribution X is called *symmetric* . The support of X in the symmetric case is the whole real axis $\mathbb{R}$. In the case $0 < \alpha \leq 1$ both the variance and mean do not exist. In this case if $\beta = 1$, then the support is the interval $[\mu, \infty)$; if $\beta = -1$, then the support is $(-\infty, -\mu]$. The case $0 < \alpha < 1,\ \beta = 1, \mu = 0$, provides important stable subordinators widely used in the study of anomalous diffusion processes. Note that the probability density function of α-stable distribution has a closed form only in three particular cases:

(i) $\alpha = 2, \beta = 0$ (normal distribution), given by (3.5);

(ii) $\alpha = 1, \beta = 0$ (Cauchy distribution), given by

$$f(x) = \frac{1}{\pi}\frac{\sigma}{(x-\mu)^2 + \sigma^2};$$

and

(iii) $\alpha = 1/2, \beta = 1$ (Lévy distribution), given by

$$f(x) = \sqrt{\frac{\sigma}{2\pi}} \frac{e^{-\frac{\sigma}{2(x-\mu)}}}{(x-\mu)^{3/2}}, \quad , x > \mu.$$

The central limit theorem for α-stable random variables is formulated as follows; see [26].

Theorem 3.1. *([26], Theorem 4.1) A nondegenerate random variable Z is α-stable for some $0 < \alpha \le 2$ if and only if there is an independent, identically distributed sequence of random variables $X_1, X_2, \dots,$ and constants $a_n > 0,\ b_n \in \mathbb{R}$, such that*

$$a_n(X_1 + \cdots + X_n) - b_n \overset{d}{\Rightarrow} Z, \quad n \to \infty, \tag{3.7}$$

where $\overset{d}{\Rightarrow}$ means the convergence in distribution.

Remark 3.2. Theorem 3.1 is about description of α-stable distributions. Namely, if Z is an α-stable distribution, then one can construct a sequence of independent random variables X_n, and sequences $a_n > 0, b_n$, such that the limit (3.7) holds. On the other hand if there is a sequence of random variables X_n and numbers $a_n > 0, b_n$, such that the limit (3.7) holds, then Z is an α-stable random variable.

The multi-variate Lévy processes are defined similarly. For completeness, we give a brief description of Lévy processes in the multidimensional case following [22]. Assume that the members of an i.i.d. X_n belong to the strict generalized domain of attraction of a full operator stable law ν. This means that ν is not supported on any proper hyperplane of $\mathbb{R}^d$ and there exists a $d \times d$-matrix A such that $\nu^t = t^A \nu$ for all $t > 0$. Here ν^t means the t-fold convolution power of ν and

$$t^A \nu(dx) = \nu(t^{-A} dx)$$

is the image measure of ν. The expression t^{-A} is the matrix power, i.e.

$$t^{-A} = e^{-A \ln t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} A^k \ln^k t.$$

It is known [24] that there exists an invertible matrix $B(n)$ such that $B(\lambda n) B(n)^{-1} \to \lambda^{-A}$, as $n \to \infty$, for all $\lambda > 0$, such that

$$B(n) S_n \overset{d}{\Rightarrow} L, \tag{3.8}$$

where $S_n = X_1 + \cdots + X_n$ and L has the distribution ν. Moreover, as is shown in [22], if one defines the stochastic process

$$S_t = \sum_{k=1}^{\lfloor t \rfloor} X_k, \tag{3.9}$$

then

$$\{B(c) S_{ct}\}_{t \ge 0} \overset{FD}{\Longrightarrow} \{L_t\}_{t \ge 0}, \text{ as } c \to \infty, \tag{3.10}$$

where L_t is a stochastic process continuous in measure, with stationary and independent increments, $L_0 = 0$ a.s., and for any fixed t L_t has the distribution $\nu^t = t^A \nu$. Here $\overset{FD}{\Longrightarrow}$ stands for the convergence in the finite dimensional sense, meaning that for any partition $0 < t_1 < \cdots < t_k, k = 1, 2, \dots,$ the random vector $(B(c) S_{ct_1}, \dots, B(c) S_{ct_k})$, converges to $(L_{t_1}, \dots, L_{t_k})$ in distribution. Due to Theorem 5 in [11] the stochastic process L_t is strict operator self-similar, that is $L_{ct} \overset{FD}{=} c^A L_t$ for $c > 0$. The latter can be expressed as follows: the characteristic function $\varphi_{L_1}(\xi) = \mathbb{E}(\exp(i \xi L_1))$ of L_1 satisfies

$$[\varphi_{L_1}(\xi)]^t = \varphi_{L_1}(t^A \xi), \quad t > 0,\ \xi \in \mathbb{R}^d. \tag{3.11}$$

This class of operator stable and operator self-similar processes defined with the self-similarity operator exponent A we denote by $\mathbb{OSS}(A)$. The stochastic process $L_t \in \mathbb{OSS}(A)$ is called operator Lévy motion.

In the particular case of $A = \frac{1}{\alpha} I$, where I s the identity matrix, the process L_t is called symmetric α-stable, and its characteristic function has the form

$$\varphi_{L_t}(\xi) = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d,$$

where $|\xi|^2 = \xi_1^2 + \cdots + \xi_d^2$. Thus, for $\alpha = 1$ and $\alpha = 2$ one obtains the multi-dimensional versions of symmetric Cauchy and Wiener processes.

Let $A_j, j \in J$, be invertible operators in $\mathbb{R}^d$ and $L_t^{(j)} \in \mathbb{OSS}(A_j),\ j \in J$, be independent (but not identically distributed) stochastic processes (Lévy motions) with self-similarity exponents A_j. Let $\mathbb{S}$ be the class of stochastic processes defined as a linear combination of stochastic processes $L_t^{(j)} \in \mathbb{OSS}$:

$$Z_t = \sum_{j \in J} C_j L_t^{(j)}, \quad t > 0, \quad Z_0 = 0. \tag{3.12}$$

The class of stochastic processes $\mathbb{S}$ can be characterized as mixtures of operator stable processes with some mixing measure. It is not hard to verify that the characteristic function of the process $Z_t \in \mathbb{S}$ defined in (3.12), due to independence of $L_t^{(j)},\ j \in J$, satisfies the condition

$$\varphi_{Z_t}(\xi) = \prod_{j \in J} \varphi_{L_t^{(j)}}(C_j \xi), \quad \xi \in \mathbb{R}^d. \tag{3.13}$$

Lemma 3.3. *The stochastic process $Z_t = \sum_{j \in J} C_j L_t^{(j)}$ is a Lévy process with the symbol*

$$\begin{aligned} \Psi_Z(\xi) &= \sum_{j \in J} \Psi_j(C_j \xi) \\ &= \sum_{j \in J} \Big[i C_j (b_{0,j}, \xi) - \frac{C_j^2}{2} (\Sigma_j \xi, \xi) + \int_{\mathbb{R}^d \setminus \{0\}} (e^{i(w, C_j \xi)} - 1 - i(w, C_j \xi) I_{(-1,1)}(w)) \nu_j(dw) \Big], \end{aligned}$$

where $\Psi_j(\xi)$ is the symbol of $L_t^{(j)}$ with parameters $(b_{0,j}, \Sigma_j, \nu_j)$.

Proof. This immediately follows from (3.13).

Corollary 3.4. *The Lévy process $\mathcal{Z}_t$ is uniquely defined by parameters*

$$\Big(\sum_{j \in J} C_j b_{0,j}, \quad \sum_{j \in J} C_j^2 \Sigma_j, \quad \sum_{j \in J} C_j \nu_j \Big).$$

In particular, if $A_j = \frac{1}{\alpha_j} I,\ j \in J$, then

$$\varphi_{Z_t}(\xi) = e^{-t \sum_{j \in J} C_j^{\alpha_j} |\xi|^{\alpha_j}}, \quad \xi \in \mathbb{R}^d.$$

In terms of symbols the latter is written as

$$\Psi_Z(\xi) = \sum_{j \in J} C_j^{\alpha_j} |\xi|^{\alpha_j}.$$

In [9] the class $\mathbb{SS}$ of mixed symmetric stochastic processes was introduced. By definition, this class contains d-dimensional stable processes Z_t, $Z_0 = 0$, whose characteristic functions are given by

$$\varphi_{Z_t}(\xi) = e^{-t \Psi(\xi)},\ \xi \in \mathbb{R}^d, \tag{3.14}$$

where

$$\Psi(\xi) = \int_0^2 |\xi|^\alpha d\rho(\alpha),$$

and ρ is a finite Borel measure with the support $\operatorname{supp} \rho \subseteq [0, 2]$. It is obvious that $\mathbb{SS} \subset \mathbb{S}$. The class $\mathbb{SS}$ obviously contains symmetric α-stable Lévy processes and all mixtures of their finitely many independent representatives. For a process $Z_t \in \mathbb{SS}$ corresponding to a measure ρ, we use the notation $Z_t = L_t^{\rho}$ to indicate this correspondence.

Similarly one can introduce a mixture of Lévy stable subordinators. Let

$$\mu(\beta) = \sum_{k\in K} a_k \delta(\beta - \beta_k), \tag{3.15}$$

where K is a finite subset of the set of natural numbers, a_k are positive constants, $\delta(\beta-\beta_k)$ is the Dirac delta function concentrated at $\beta_k \in (0,1),\ k \in K$, and $D_{t,\beta}$ a stable subordinator with a stability index $\beta \in (0,1)$. Assume the processes $D_{t,\beta},\ 0<\beta<1$, are independent for any finite collection of $\beta_1,\dots,\beta_k,\ k\geq 2$. Then, one can define a stochastic process $D(\mu;t)$ as a process with the Laplace transform

$$\phi_{D(\mu;t)}(s) = \mathrm{e}[e^{-sD(\mu;t)}] = e^{-t\Phi(s)}, \quad s>0.$$

where

$$\Phi(s) = \sum_{k\in K} a_k^{\beta_k} s^{\beta_k} = -t^{-1}\ln \mathrm{e}[e^{-s\sum_{k=1}^{K} a_k D_{t,k}}], \quad s>0, (\forall t>0),.$$

where $D_{t,k}$ are Lévy stable subordinators with stability index $\beta_k,\ k\in K$.

The set of all mixed subordinators $D(\mu;t)$ we denote by $\mathbb{SO}$. The inverse process to $D(\mu;t)\in\mathbb{SO}$ is denoted by E_t^{μ}. Namely,

$$E_t^{\mu} = \inf\,\{\tau\geq 0 : D(\mu;\tau) > t\}.$$

Lemma 3.5. (1) *A stochastic process $Z_t \in \mathbb{S}$ is a Lévy process;*

(2) *A stochastic process $Z_t\in\mathbb{S}$ is strict operator stable and operator self-similar with stability index A if and only if $A_j = A$ for all $j\in J$;*

(3) *A stochastic process $F_t \in \mathbb{SO}$ is a Lévy subordinator;*

(4) *A stochastic process $F_t\in\mathbb{SO}$ is stable and self-similar if and only if $\mu = C\delta(\beta-\beta_0)$ for some $\beta_0\in(0,1)$ and constant $C>0$, where $\delta(\beta-\beta_0)$ is the Dirac delta-function concentrated at β_0;*

(5) *The inverse to any $F_t\in\mathbb{SO}$ is positive, continuous and nondecreasing;*

Proof.

(1) By definition, each of stochastic processes $L_t^j,\ j\in J$, has independent increments, i.e. for any partition $0\leq t_1 < s_1 \leq t_2 < s_2 \leq \cdots \leq t_k < s_k < \infty$, one has $L^{(j)}_{s_1-t_1}, L^{(j)}_{s_2-t_2},\dots,L^{(j)}_{s_k-t_k}$, are independent. Therefore, it follows from (3.13) that for any $m\neq n$

$$\begin{aligned}
\varphi_{Z_{s_m-t_m}+Z_{s_n-t_n}}(\xi) &= \mathrm{e}[e^{i\xi(Z_{s_m-t_m}+Z_{s_n-t_n})}]\\
&= \mathrm{e}\left[e^{i\xi\left(\sum_{j\in J} C_j L^{(j)}_{s_m-t_m} + \sum_{j\in J} C_j L^{(j)}_{s_n-t_n}\right)}\right]\\
&= \mathrm{e}\left[e^{i\xi\sum_{j\in J} C_j\left(L^{(j)}_{s_m-t_m}+L^{(j)}_{s_n-t_n}\right)}\right]\\
&= \left(\prod_{j\in J}\varphi_{L^{(j)}_{s_m-t_m}}(C_j\xi)\right) * \left(\prod_{j\in J}\varphi_{L^{(j)}_{s_n-t_n}}(C_j\xi)\right)\\
&= \varphi_{Z_{s_m-t_m}}(\xi) * \varphi_{Z_{s_n-t_n}}(\xi),
\end{aligned}$$

showing that $Z^{(j)}_{s_1-t_1}, Z^{(j)}_{s_2-t_2},\dots,Z^{(j)}_{s_k-t_k}$ are independent. Here "$*$" is the convolution operation.

Similarly, by definition, each of stochastic processes $L_t^j,\ j \in J$, has stationary increments, i.e. for any $h > 0$ one has $L_{t+h}^{(j)} - L_t^{(j)} \stackrel{FD}{=} L_h^{(j)},\ j \in J$. This implies

$$\begin{aligned} Z_{t+h} - Z_t &= \sum_{j \in J} C_j L_{t+h}^{(j)} - \sum_{j \in J} C_j L_t^{(j)} \\ &= \sum_{j \in J} C_j \left(L_{t+h}^{(j)} - L_t^{(j)} \right) \\ &\stackrel{FD}{=} \sum_{j \in J} C_j L_h^{(j)} = Z_h, \quad h > 0. \end{aligned} \tag{3.16}$$

showing that Z_t has stationary increments. Since each of $L_t^{(j)} \to 0$ as $t \to 0$ in probability, one has $\mathcal{Z}_t \to 0$ as $t \to 0$ in probability. This and (3.16) imply the desired result.

(2) *Sufficiency:* Let $L_t^{(j)} \in \mathbb{OSS}$ with self-similarity index $A_j = A$ for all $j \in J$, i.e. by definition, $L_{ct}^{(j)} \stackrel{FD}{=} c^A L_t^{(j)},\ j \in J$. This implies

$$Z_{ct} = \sum_{j \in J} C_j L_{ct}^{(j)} \stackrel{FD}{=} \sum_{j \in J} C_j c^A L_t^{(j)} = c^A Z_t, \tag{3.17}$$

meaning that Z_t is self-similar with self-similarity index A. Due to Theorem 5 in [11] relationship (3.17) implies that the stochastic process Z_t is strict operator self-similar.

Necessity: Suppose $L_t^{(j)} \in \mathbb{OSS}(A_j)$ and there exists an invertible matrix A, such that $\mathcal{Z}_{ct} \stackrel{FD}{=} c^A \mathcal{Z}_t$ for $c > 0$. Then, we have

$$\mathcal{Z}_{ct} \stackrel{FD}{=} \sum_{j \in J} C_j c^{A_j} L_t^{(j)} = c^A \sum_{j \in J} C_j L_t^{(j)}.$$

It follows that

$$\sum_{j \in J} C_j (c^{A_j} - c^A) L_t^{(j)} = 0_t, \quad t \geq 0,$$

implying $c^{A_j} = c^A, j \in J$. Hence, $A_j = A$ for all $j \in J$.

(3) Since, by definition, each $D_{t,\beta}$ is a strictly increasing and positive cádlág process for any $\beta \in (0,1)$, the mixture process $F_t = D(\mu; t)$ also inherits these properties. The fact that F_t has independent and stationary increments can be proved as in Part 1.

(4) Let $d\mu(\beta) = C\delta(\beta - \beta_0)d\beta$ with som $C > 0$ and $\beta_0 \in (0,1)$. Then, $F_t = D_{t,\beta_0}$, which is a Lévy stable subordinator with the stability index β_0. On the other hand if F_t is stable with a stability index β_0, then $d\mu(\beta) = C\delta(\beta - \beta_0)$ with some constant $C > 0$.

(5) It follows from Part 3 that the process E_t^μ (inverse to $F_t = D(\mu; t)$) exists, positive, continuous (with possible flat pieces), and non-decreasing.

4. CTRW limits in Skorokhod topologies

Since a CTRW is a stochastic process being a composition of two càdlàg stochastic processes, the following two general theorems on convergence of sequences of composite stochastic processes in the Skorokhod space supply *sufficient* conditions for the convergence of CTRW in the J_1- and M_1-topologies of the Skorokhod space. We recall that the set $Disc(X)$ for a càdlàg process $X(t)$ means the set of its discontinuity points: $X(t-) - X(t) \neq 0$.

Before formulating our theorem on J_1-convergence of CTRW process we would like to display the following to known results on the J_1 and M_1-convergence of composite stochastic cádlág processes obtained by Silvestrov [33] and Whitt [42]. These theorems will be used in the proof of our theorem.

Theorem 4.1. *([33, Thm. 3.4.2.]) Let $X_\varepsilon(t)$ and $X_0(t)$ be càdlàg d-dimensional stochastic processes, and $U_\varepsilon(t)$ and $U_0(t)$ be non-negative and non-decreasing one-dimensional càdlàg stochastic processes. Assume that the following conditions hold:*

(1) *$(X_\varepsilon(t), U_\varepsilon(s))$ converges weakly to $(X_0(t), U_0(s))$ as $\varepsilon \to 0$ for all $(t,s) \in T \times S$, where T and S are some subsets of $[0,\infty)$ that are dense in this interval and contain point 0,*

(2) *$\lim\limits_{c\to 0}\limsup\limits_{\varepsilon\to 0}\mathbb{P}\left[\omega_J(X_\varepsilon(\cdot), c, T) > \delta\right] = 0$ for all $\delta > 0$ and $T > 0$,*

(3) *$U_0(t)$ is a.s. continuous process,*

(4) *$\mathbb{P}\left[U_0(t^{'}) = U_0(t^{''}) \in Disc(X_0)\right] = 0$ for $0 \le t^{'} < t^{''} < \infty$,*

(5) *$\mathbb{P}\left[U_0(0) \in Disc(X_0)\right] = 0$.*

Then $\Big(X_\varepsilon \circ U_\varepsilon\Big)(t) \xrightarrow{J_1} \Big(X_0 \circ U_0\Big)(t)$ as $\varepsilon \to 0$.

Now we proceed to the convergence of CTRW processes in the J_1-topology. Since the method used in [23] is not applicable in this case, we will use Theorem 4.1 to establish the convergence of CTRW triangular arrays in the Skorokhod's J_1-topology.

Consider the following sequence of random variables T_n^μ and $\mathcal{S}_n$ defined as follows. Let $J_{i,\beta}$ be a sequence of independent and identically distributed random variables in the domain of attraction of β-stable Lévy subordinators and $T_{n,\beta} = J_{1,\beta} + \dots + J_{n,\beta}$. Then, there exists $b(n,\beta)$ such that

$$b(n,\beta)T_{nt,\beta} \overset{FD}{\Rightarrow} D_{t,\beta}, \tag{4.1}$$

as $n \to \infty$, where $D_{t,\beta}$ is the β-stable subordinator (see [22]). The stochastic process $T_n(\mu;t)$ is the μ-mixture of $b(n,\beta)T_{nt,\beta}$ with a mixing measure μ defined in (3.15). Namely, the process $T_n(\mu;t)$ is defined as

$$T_n(\mu;t) = \sum_{k\in K} a_k b(n,\beta_k)T_{nt,\beta_k}. \tag{4.2}$$

Further, let $X_n^{(j)}, j \in J$, be sequences of i.i.d. random variables belonging to the strict generalized domains of attraction of a full operator stable laws $\nu_j,\ j \in J$, with corresponding indices $A_j,\ j \in J$. Then, in accordance with (3.8), there exists $B_j(n)$ such that $B_j(n)S_n^{(j)} \overset{d}{\Rightarrow} L^j$, where $S_n^{(j)} = X_1^{(j)} + \dots + X_n^{(j)}$, and L^j has the distribution ν_j. Define the sequence

$$\mathcal{S}_n := \sum_{j\in J} C_j B_j(n) S_n^{(j)}, \quad n \ge 1, \tag{4.3}$$

where $C_j, j \in J$, are real constants, and corresponding sequence of stochastic process

$$\mathcal{S}_n(t) = \sum_{j\in J} C_j B_j(n) S_{nt}^{(j)}, \quad n \ge 1, t > 0, \quad S_n(0) = 0. \tag{4.4}$$

Lemma 4.2. *The following convergences are valid:*

(1) *$T_n(\mu;t) \overset{FD}{\Rightarrow} D(\mu;t) \in \mathbb{SO}$, as $n \to \infty$;*

(2) *$\mathcal{S}_n(t) \overset{FD}{\Rightarrow} \mathcal{S}(t) \in \mathbb{S}$, as $n \to \infty$.*

Proof.

(1) Using the convergence (4.1) we have

$$T_n(\mu;t) = \sum_{k\in K} a_k b(n,\beta_k)T_{nt,\beta_k} \overset{FD}{\Rightarrow} \sum_{k\in K} a_k D_{t,\beta_k} = D(\mu;t).$$

as $n \to \infty$. The fact that $D(\mu;t) \in \mathbb{SO}$ follows from the definition of $\mathbb{SO}$.

(2) Since due to (3.10) we have the convergence $B_j(c)S^{(j)}_{ct} \overset{FD}{\Rightarrow} L^{(j)}_t$, where $L^{(j)}_t \in \mathbb{OSS}$ with self-similarity index A_j, one has

$$\mathcal{S}_n(t) = \sum_{j\in J} C_j B_j(n) S^{(j)}_{nt} \overset{FD}{\Rightarrow} \sum_{j\in J} C_j L^{(j)}_t = \mathcal{S}_t, \quad \mathcal{S}(0) = 0,$$

as $n \to \infty$. Obviously $\mathcal{S} \in \mathbb{S}$; see Definition (3.12).

Theorem 4.3. *Let $X^j_i, i = 1, 2, \ldots,$ for each $j \in J$ be an i.i.d. random vectors belonging to the strict generalized domains of attraction of a full operator stable laws with stability indices A_j, $j \in J$, and $\mathcal{J}_{i,\beta}$ be a positive i.i.d. random variables in the domain of attraction of β-stable Lévy subordinators, satisfy the following conditions:*

(1) *X^j_i and $\mathcal{J}_{i,\beta}$ are independent i.i.d.s;*

(2) *the sequence $\{(S_n(t), T_n(\mu; t) : t \geq 0\}$ weakly converges to $\{(\mathcal{S}_t, D^\mu_t) : t \geq 0\}$ as $n \to \infty$;*

(3) $\mathbb{P}\Big[Disc\big(\mathcal{S}(t)\big) \cap Disc\big(D(\mu; t)\big)\Big] = 0.$

Then the CTRW process

$$\{W^{(n)}_t \equiv \Big(S_n \circ \big(T_n(\mu; \cdot)\big)^{-1}\Big)(t) : t \geq 0\} \tag{4.5}$$

converges as $n \to \infty$ to the time-changed stochastic process $\{V(t) = \mathcal{S}(E^\mu_t) : t \geq 0\}$, in the J_1-topology on $\mathcal{D}\big([0, \infty), \mathbb{R}^d\big)$, where $E^\mu_t = \inf\{u \geq 0 : D(\mu; u) > t\}$.

Proof. We set $X_\varepsilon(t) = S_{1/\varepsilon}(t)$ and $U_\varepsilon(t) = N^{(1/\varepsilon)}_t = \big(T_{1/\varepsilon}(\mu; t)\big)^{-1}$ in which $\varepsilon = 1/n$. Then, the composition $\{(X_\varepsilon \circ U_\varepsilon)(t) : t \geq 0\}$ represents the CTRW process $\{W^{(n)}_t : t \geq 0\}$ with $n = 1/\varepsilon$. Therefore, it suffices to check out each conditions of Theorem 4.1. Condition (1) of this theorem is fulfilled, since by construction of processes $S_n(t)$ and $N^{(n)}_t$, we have that X_ε converges weakly to $\mathcal{S}(t)$ and U_ε converges weakly to E^μ_t as $\varepsilon \to 0$. Now we show that $\{X_\varepsilon(t) = S_{1/\varepsilon}(t) : t \geq 0\}$ satisfies condition (2) of Theorem 4.1. Following [22], it suffices to show that

$$\lim_{h\to 0} \limsup_{n\to\infty} \sup_{|t-s|\leq h} \mathbb{P}(|S_n(t) - S_n(s)| > \delta) = 0 \tag{4.6}$$

for any δ and $T > 0$, where $0 \leq s < t \leq T$. For each $Y^{(j)}_n(t) = B_j(n)S^{(j)}_{tn}$, $j \in J$, where $S^{(j)}_n = X^j_1 + \cdots + X^j_n$, this fact can be shown exactly as in the proof of Theorem 4.1 in [22]. Obviously, this condition is also valid for $C_jY^{(j)}_n$ with any constant C_j. Thus, for all $j \in J$ we have

$$\lim_{h\to 0} \limsup_{n\to\infty} \sup_{|t-s|\leq h} \mathbb{P}(|C_jY^{(j)}_n(t) - C_jY^{(j)}_n(s)| > \delta) = 0 \tag{4.7}$$

for any δ and $T > 0$, where $0 \leq s < t \leq T$. It follows from the independence of $Y^{(j)}_n, j \in J$, that the events $\Omega_j = \{|C_jY^{(j)}_n(t) - C_jY^{(j)}_n(s)| > \delta\}$, $j \in J$, are also independent. Hence,

$$\begin{aligned}
\mathbb{P}(|S_n(t) - S_n(s)| > \delta) &= \mathbb{P}\left(|\sum_{j\in J} C_jY^{(j)}_n(t) - \sum_{j\in J} C_jY^{(j)}_n(s)| > \delta\right) \\
&= \mathbb{P}\left(|\sum_{j\in J}(C_jY^{(j)}_n(t) - C_jY^{(j)}_n(s))| > \delta\right) \\
&\leq \mathbb{P}\left(\sum_{j\in J} |C_jY^{(j)}_n(t) - C_jY^{(j)}_n(s)| > \delta\right) && (4.8)\\
&\leq \mathbb{P}(\cup_{j\in J} \Omega_j) && (4.9)\\
&= \sum_{j\in J} \mathbb{P}\Big(|C_jY^{(j)}_n(t) - C_jY^{(j)}_n(s)| > \delta\Big) && (4.10)
\end{aligned}$$

In relations (4.8), (4.9) and (4.10) we used the inequality

$$\sum_{j\in J}|C_jY_n^{(j)}(t)-C_jY_n^{(j)}(s)|\leq \sum_{j\in J}|C_jY_n^{(j)}(t)-C_jY_n^{(j)}(s)|,$$

the fact that if at least one of the events Ω_j is fulfilled, then the event $\Omega=\{|S_n(t)-S_n(s)|>\delta\}$ is also fulfilled, and the independence of events Ω_j, respectively. Now, taking into account finiteness of the set J, and using (4.7)-(4.10), we obtain (4.6).

Further, the process $\{E_t^{\mu}:t\geq 0\}$ as the inverse of strictly increasing subordinator is a nondecreasing continuous process (see Lemma 3.5). Hence, condition (3) of Theorem 4.1 is fulfilled by construction. The process $\{E_t^{\mu}:t\geq 0\}$ may have flat pieces. It is not hard to verify that the assumption $\mathbb{P}\Big[Disc(\mathcal{S})\cap Disc(D(\mu;\cdot))\Big]=0$ implies Condition (4) of Theorem 4.1. Finally, since $E_0^{\mu}=0$, condition (5) is reduced to $\mathbb{P}[0\in Disc(X_0)]=0$. Due to construction of our CTRW we have $X_0(0)=(\mathcal{S}\circ E^{\mu})(0)=\mathcal{S}(0)=0$. Further, since the process $\mathcal{S}$ as a càdlàg process is right continuous, we have $X_0(0+)-X_0(0)=0$. This implies condition (5) of Theorem 4.1. Hence, in accordance with this theorem the composite process $\{(X_{\varepsilon}\circ U_{\varepsilon})(t):t\geq 0\}$ converges to the composite process $\{(X_0\circ U_0)(t):t\geq 0\}$ as $\varepsilon\to 0$ in the J_1 topology, that is, CTRW process $\{W_t^{(n)}:t\geq 0\}$ converges to $\{\mathcal{S}(E_t^{\mu}):t\geq 0\}$ as $n\to\infty$ in the J_1 topology on $\mathcal{D}\big([0,\infty),\mathbb{R}^d\big)$.

Since the convergence in J_1 topology implies the convergence in M_1 topology (see Proposition (2.1)). Theorem 4.3 implies the following statement.

Corollary 4.4. *Under the conditions of Theorem 4.3 the CTRW process* $\{W_t^{(n)}\equiv(S^{(n)}\circ(T^{(n)})^{-1})_{nt}:t\geq 0\}$ *converges as* $n\to\infty$ *to the time-changed stochastic process* $\{V(t)=L_{E_t}:t\geq 0\}$ *in the* M_1 *topology on* $\mathcal{D}\big([0,\infty),\mathbb{R}^d\big)$.

The M_1-convergence of the CTRW process $\{W_t^{(n)}:t\geq 0\}$ in (4.5) can also be proved based on the following theorem on the M_1-convergence of composite cádlág processes.

Theorem 4.5. *(*[42, Thm. 13.2.4.]*) Let $X_n(t)$ and $X_0(t)$ be càdlàg d-dimensional stochastic processes, and $U_n(t)$ and $U_0(t)$ be non-negative and non-decreasing one-dimensional càdlàg stochastic processes satisfying the following conditions:*

(1) *The sequence* $(X_n(t),U_n(t))$ *weakly converges to* $(X_0(t),U_0(t))$ *as* $n\to\infty$;

(2) $U_0(t)$ *is continuous and strictly increasing at* t *whenever* $U_0(t)\in Disc(X_0)$;

(3) $X_0(t)$ *is monotone on* $[U_0(t-),U_0(t)]$ *and* $U_0(t-),\ U_0(t)\notin Disc(X_0)$, *whenever* $t\in Disc(U_0)$.

Then $X_n\circ U_n\xrightarrow{M_1}X_0\circ U_0$ *as* $n\to\infty$.

Indeed, if one takes $\{X_0(t):t\geq 0\}\equiv\{\mathcal{S}_t:t\geq 0\}$ and $\{U_0(t):t\geq 0\}\equiv\{E_t^{\mu}:t\geq 0\}$ in Theorem 4.5, then condition (3) is unnecessary, since the process $\{E_t^{\mu}:t\geq 0\}$ is continuous for all $t\geq 0$. The inverse process $\{D(\mu;t):t\geq 0\}$ to $\{E_t^{\mu}:t\geq 0\}$ is strictly increasing as a mixture of Lévy subordinators. Therefore condition (2) of Theorem 4.5 in our case takes the form $Disc(\mathcal{S}\cap Disc(D(\mu;\cdot))=\emptyset$. Condition (1) is shown in the proof of Theorem 4.1. Thus the following theorem holds:

Theorem 4.6. *Let* $X_i^j, i=1,2,\ldots,$ *for each* $j\in J$ *be an i.i.d. random vectors belonging to the strict generalized domains of attraction of a full operator stable laws with stability indices* $A_j,\ j\in J$, *and* $\mathcal{J}_{i,\beta}$ *be a positive i.i.d. random variables in the domain of attraction of* β*-stable Lévy subordinators, satisfy the following conditions:*

(1) Y_i *and* J_i *are independent i.i.d.s;*

(2) *the sequence* $\{(S_n(t),T_n(\mu;t):t\geq 0\}$ *weakly converges to* $\{(\mathcal{S}_t,D_t^{\mu}):t\geq 0\}$ *as* $n\to\infty$;

(3) $Disc\big(\mathcal{S}(t)\big)\cap Disc\big(D(\mu;t)\big)=\emptyset$.

Then the CTRW process $\{W_t^{(n)} \equiv \Big(S_n \circ (T_n(\mu;\cdot))^{-1}\Big)(t) : t \geq 0\}$ *converges as* $n \to \infty$ *to the time-changed stochastic process* $\{V(t) = \mathcal{S}(E_t^{\mu}) : t \geq 0\}$, *in the* M_1*-topology on* $\mathcal{D}\big([0,\infty), \mathbb{R}^d\big)$, *where* $E_t^{\mu} = \inf\{u \geq 0 : D(\mu; u) > t\}$.

Properties of the process W_t were studied in [22] in the particular case of $\mathcal{S}_t = L_t \in \mathbb{OSS}(A)$ with some A and $E_t^{\mu} = E_T^{\beta}$ with some $\beta \in (0,1)$. Below we list some properties of $W_t = \mathcal{S}(E_t^{\mu})$, where $\mathcal{S} \in \mathbb{S}$ and μ is a bounded measure on $[0,1]$.

Theorem 4.7. *The limiting process* $\{W_t = \mathcal{S}(E_t^{\mu}) : t \geq 0\}$ *obtained in Theorems 4.1 and 4.6 possesses the following properties:*

(1) $\{W_t : t \geq 0\}$ *is operator self-similar if all* $L_t^{(j)}$, $j \in J$, *have the same self-similarity index* $A_j = A$, $j \in J$, *and* $\mu(d\beta) = \delta(\beta - \beta_0)d\beta$ *for some* $\beta_0 \in (0,1)$. *If this condition is fulfilled, then* $\{W_{ct} : t \geq 0\} = \{c^{\beta_0 A} W_t : t \geq 0\}$.

(2) $\{W_t : t \geq 0\}$ *has no independent increments and has no stationary increments;*

(3) $\{W_t : t \geq 0\}$ *is not operator stable for any* $t > 0$.

Proof. The proof immediately follows from Corollaries 4.1, 4.3, and Theorem 4.3 of [22].

5. SDEs driven by CTRW-limit processes

In this section we briefly review stochastic differential equations (SDEs) driven by time-changed driving processes. An important class of driving processes is the CTRW limit processes and the corresponding SDEs are used in modeling of various engineering and scientific problems. Let the driving process (integrator) $W_t = \mathcal{S}(E_t^{\mu})$ be the CTRW limit obtained in Theorems 4.1 and 4.6. Consider a class of SDEs of the form

$$dX_t = \sigma(t, E_t^{\mu}, X_t)dW_t, \quad X_0 = x_0, \tag{5.1}$$

where x_0 is a random variable with a density function $f_{x_0}(x), x \in \mathbb{R}^d$, and $\sigma(t,s,x) : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a measurable function for which there is a positive constant K such that

$$\begin{aligned}
&|\sigma(t,s,x) - \sigma(t,s,y)| \leq K|x-y|,\\
&|\sigma(t,s,x)| \leq K(1+|x|),\\
&|\sigma(t,s,x) - \sigma(u,v,x)| \leq K(1+|x|)(|t-u| + |s-v|),
\end{aligned}$$

for all $x, y \in \mathbb{R}^d$ and $s,t,u,v \geq 0$.

Even though SDE (5.1) is not yet studied in such a generalization, but in various particular cases it is studied. There are different approaches for the study of SDEs. One of these approaches assumes approximation of stochastic integrals of the form

$$\int_0^t H_{s-} dZ_s, \tag{5.2}$$

where H_t and Z_t are cádlág stochastic processes in the Skorokhod space $\mathbb{D}([0,\infty), \mathbb{R}^d)$. This is reasonable, because the meaning of SDE (5.1) is given by the following expression

$$X_t = x_0 + \int_0^t \sigma(s-, E_s^{\mu}, X_{s-})dW_s,$$

where the integral is of the form (5.2). The stochastic integral (5.2) driven by Brownian motion $Z_t \equiv B_t$ was first introduced by Itô [13, 12]. The idea used by Itô for definition of the stochastic integral, in fact, works for any semimartingale integrator Z_t. For cádlág stochastic processes Z_t stochastic integrals were considered by Kurtz and Protter in [18]. The authors applied this approach for approximation of SDEs of the form

$$X_t = U_t + \int_0^t \sigma(s-, X_s)dZ_s,$$

Weak and J_1-convergences of stochastic integrals driven by scaled CTRWs to a stochastic integral driven by time-changed Brownian motion is studied in [5]. Another particular case of approximation of stochastic integrals driven by a time-changed symmetric α-stable Lévy processes is studied by Scalas and Viles in [30]. In the general case, however, the approximation of stochastic integrals driven by CTRW-limits still remains open.

The SDE driven by the time-changed Brownian motion

$$dX_t = b(E_t, X_t)dE_t + \sigma(E_t, X_t)dB_{E_t} \quad \text{with} \quad X_0 = x_0, \tag{5.3}$$

where $x_0 \in \mathbb{R}^d$ is a non-random constant, and $b(t,x) : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(t,x) : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable functions satisfying growth and Lipschitz conditions, was solved in papers [15] (see also [16]) using the numerical Euler-Maruyama method.

Due to (3.1), SDEs driven by a Lévy process L_t, has the form

$$\begin{aligned} Y_\tau = y + \int_0^\tau b(Y_{s-})ds + \int_0^\tau \sigma(Y_{s-})dB_s \\ + \int_0^\tau \int_{|w|<1} H(Y_{s-}, w)\tilde{N}(ds, dw) + \int_0^\tau \int_{|w|\geq 1} K(Y_{s-}, w)N(ds, dw), \end{aligned} \tag{5.4}$$

where the continuous mappings $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $H : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, and $K : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ are bounded and satisfy growth and Lipschitz conditions. Namely, there exist positive constants C_1 and C_2 such that

$$\begin{aligned} |b(y_1) - b(y_2)|^2 + \|\sigma(y_1) - \sigma(y_2)\|^2 + \int_{|w|<1} |H(y_1, w) - H(y_2, w)|^2 \nu(dw) \\ \leq C_1|y_1 - y_2|^2 \quad \text{for all } y_1, y_2 \in \mathbb{R}^n; \end{aligned} \tag{5.5}$$

$$\int_{|w|<1} |H(y, w)|^2 \nu(dw) \leq C_2(1 + |y|^2) \quad \text{for all } y \in \mathbb{R}^n. \tag{5.6}$$

As it is known [1, 34], under these conditions SDE (5.4) has a unique solution.

It follows from Lévy-Itô representation (3.1) of Lévy processes and SDE (5.4) that if SDE is driven by the process $\mathcal{S} = \sum_{j\in J} C_j L_t^{(j)}$, then one obtains SDE

$$\begin{aligned} V_t = v + \sum_{j\in J} C_j \bigg[\int_0^t b_j(V_{s-})ds + \int_0^t \sigma_j(V_{s-})dB_s^{(j)} \\ + \int_0^t \int_{|w|<1} H_j(V_{s-}, w)\tilde{N}_j(s, dw) + \int_0^t \int_{|w|\geq 1} K_j(V_{s-}, w)N_j(s, dw) \bigg]. \end{aligned} \tag{5.7}$$

Finally, SDE driven by CTRW-limit process $W_t = (\mathcal{S} \circ E^\mu)_t = \Big((\sum_{j\in J} C_j L^{(j)}) \circ E^\mu \Big)_t$, takes the form

$$\begin{aligned} X_t = x + \sum_{j\in J} C_j \bigg[\int_0^t b_j(X_{s-})dE_s^\mu + \int_0^t \sigma_j(X_{s-})dB_{E_s^\mu}^{(j)} \\ + \int_0^t \int_{|w|<1} H_j(X_{s-}, w)\tilde{N}_j(dE_s^\mu, dw) + \int_0^t \int_{|w|\geq 1} K_j(X_{s-}, w)N_j(dE_s^\mu, dw) \bigg]. \end{aligned} \tag{5.8}$$

6. Fokker-Planck-Kolmogorov equations associated with SDEs driven by CTRW-limit processes

Fokker-Planck-Kolmogorov (FPK) equations are deterministic form of description of stochastic processes modeled through SDEs. In this section we derive the FPK equation associated with SDE (5.8) driven by CTRW-limit process W_t.

It is known that if SDE is driven by a Lévy process with parameters $(b(y), \Sigma(y), \nu(y))$, $y \in \mathbb{R}^d$, then $u(t,y) = \mathrm{e}[f(Y_t)|Y_0 = y]$, where Y_t is a solution of this SDE and $f(y)$ is the density function of Y_0, satisfies the following Cauchy problem for a pseudo-differential equation (see [1])

$$\frac{\partial v}{\partial t} = \mathcal{A}(y, D_x)v(t,y),\ t > 0,\ y \in \mathbb{R}^n, \tag{6.1}$$

$$v(0,y) = \varphi(y),\ y \in \mathbb{R}^n. \tag{6.2}$$

Here $\mathcal{A}(y, D_y)$ is the pseudo-differential operator

$$\begin{aligned}\mathcal{A}(y, D_y)\varphi(x) &= i(b(y), D_y)\varphi(y) - \frac{1}{2}(\Sigma(x)D_y, D_y)\varphi(y) \\ &+ \int_{\mathbb{R}^n\setminus\{0\}} \big[\varphi(y + G(y,w)) - \varphi(y) - i\, I_{(-1,1)}(w)(G(y,w), D_y)\varphi(y)\big]\nu(dw)\end{aligned} \tag{6.3}$$

with symbol

$$\begin{aligned}\Psi(y,\xi) &= i(b(y),\xi) - \frac{1}{2}(\Sigma(y)\xi,\xi) \\ &+ \int_{\mathbb{R}^n\setminus\{0\}} (e^{i(G(y,w),\xi)} - 1 - i(G(y,w),\xi)I_{(-1,1)}(w))\,\nu(dw),\end{aligned} \tag{6.4}$$

where $\Sigma(y) = \sigma(y) \times \sigma(y)^T$ and $G(y,w) = H(y,w)$ if $|w| < 1$ and $G(y,w) = K(y,w)$ if $|w| \geq 1$.

We will also use the following fact (see [39]) in the proof of the main theorem of this section.

Theorem 6.1. *Let $U_t = \sum_{k=1}^N c_k U_{k,t}$, where $U_{k,t}$, $k = 1, \dots, N$, are independent stable subordinators with respective indices $\beta_k \in (0,1)$ and constants $c_k > 0$. Let E_t be the inverse process to U_t. Suppose $\{T_t, t \geq 0\}$ is a strongly continuous semigroup in a Banach space $\mathcal{X}$ and has infinitesimal generator $\mathcal{A}$ with $Dom(\mathcal{A}) \subset \mathcal{X}$. Then for each fixed $t \geq 0$ and $\varphi \in Dom(\mathcal{A})$, the integral $\int_0^\infty f_{E_t}(\tau)T_\tau\varphi\, d\tau$, where $f_{E_t}(\tau)$ is the density function of E_t, exists and the vector-function $v(t) = \int_0^\infty f_{E_t}(\tau)T_\tau\varphi\, d\tau$ satisfies the abstract Cauchy problem*

$$\sum_{k=1}^N C_k D_*^{\beta_k} v(t) = \mathcal{A}v(t),\ t > 0, \tag{6.5}$$

$$v(0) = \varphi, \tag{6.6}$$

where D_*^β is the fractional derivative in the sense of Caputo[a] *and $C_k = c_k^{\beta_k}$, $k = 1, \dots, N$.*

Now we are ready to formulate the theorem on FPK equation associated with SDE (5.8).

Theorem 6.2. *Let $D(\mu;t) \in \mathbb{SO}$ with a mixing measure $\mu = \sum_{k=1}^N a_k\delta(\beta - \beta_k)$, $\beta_k \in (0,1)$, where a_k are positive constants, and let E_t^μ is the inverse of $D(\mu;t)$. Suppose that a stochastic process $\mathcal{S}(t) = \sum_{j\in J} L_t^{(j)} \in \mathbb{S}$, where $L_t^{(j)} \in \mathbb{OSS}(A_j), j \in J$, is independent of the stochastic process E_t^μ.*

Then if X_t is a solution of SDE (5.8), driven by the time-changed process $W_t = (\mathcal{S} \circ E^\mu)(t)$, then the function $u(t,x) = \mathbb{E}[\varphi(X_t)|X_0 = x]$, where $\varphi(x) = f_{X_0}(x) \in C_0^2(\mathbb{R}^n)$, satisfies the following Cauchy problem

$$\sum_{k=1}^N \alpha_k D_*^{\beta_k} u(t,x) = \sum_{j\in J} \mathcal{A}_j(x, D_x)u(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{6.7}$$

$$u(0,x) = \varphi(x),\ x \in \mathbb{R}^n. \tag{6.8}$$

[a] see e.g. [36, 39] for the fractional derivative in the sence of Caputo

where $\alpha_k = a_k^{\beta_k}$, $k = 1, \dots, N$, and the pseudo-differential operator $\mathcal{A}_j(x, D_x), j \in J$, has the symbol

$$\Psi_j(x,\xi) = iC_j(b_j(x),\xi) - \frac{C_j^2}{2}(\Sigma(y)\xi,\xi) \qquad (6.9)$$
$$+ \int_{\mathbb{R}^n\setminus\{0\}} (e^{iC_j(G(y,w),\xi)} - 1 - iC_j(G(y,w),\xi)I_{(-1,1)}(w))\,\nu(dw), \quad j \in J.$$

Proof. In accordance with Corollary 3.4 the driving stochastic process $\mathcal{S}_t$ in SDE (5.7) is a Lévy process with parameters $(b(x), \Sigma(x), \nu(x))$, where

$$b(x) = \sum_{j\in J} C_j b_j(x), \quad \Sigma(x) = \sum_{j\in J} C_j^2 \Sigma_j(x), \quad \nu(x) = \sum_{j\in J} C_j \nu_j(x).$$

Therefore, if Y_t is a solution to SDE (5.7), then $\mathrm{e}[\varphi(Y_t)|Y_0 = x]$ satisfies Cauchy problem (6.1), (6.2), in which the operator $\mathcal{A}(x, D)$ has the symbol

$$\Psi(x,\xi) = \sum_{j\in J} C_j i(b_j(x),\xi) - \sum_{j\in J}\frac{C_j^2}{2}(\Sigma_j(x)\xi,\xi) \qquad (6.10)$$
$$+ \sum_{j\in J}\int_{\mathbb{R}^n\setminus\{0\}} (e^{iC_j(G_j(y,w),\xi)} - 1 - iC_j(G_j(y,w),\xi)I_{(-1,1)}(w))\,\nu_j(dw),$$

Now, consider the expression $T_\tau\varphi(x) = \mathbb{E}[\varphi(Y_\tau)|Y_0 = x]$, where Y_τ is a solution of SDE (5.7). Then, it is known [1, 29, 39], that T_τ is a strongly continuous contraction semigroup in the Banach space $C_0(\mathbb{R}^n)$. Moreover, since the function $v(\tau, x) = T_\tau\varphi(x)$ with $\varphi \in C_0^2(\mathbb{R}^n)$ satisfies equation (6.1), the semigroup T_τ has infinitesimal generator given by the pseudo-differential operator $\mathcal{A}(x, D_x)$ with the symbol $\Psi(x, \xi)$ defined in (6.10). Obviously, $C_0^2(\mathbb{R}^n) \subset \mathrm{Dom}(\mathcal{A}(x, D_x))$.

Finally, consider the stochastic process $X_t = Y_{E_t^\mu}$. This process is a unique solution to SDE (5.8) (see [9, 39]). Further, let

$$u(t,x) = \mathbb{E}[\varphi(X_t)|X_0 = x] = \mathbb{E}[\varphi(Y_{E_t^\mu})|Y_0 = x], \quad t > 0,\ x \in \mathbb{R}^d. \qquad (6.11)$$

Recall that $E_0^\mu = 0$. In (6.11), conditioning on the events $\{E_t^\mu = \tau\}$ and using independence of the processes Y_τ and E_t^μ, we have

$$u(t,x) = \int_0^\infty \mathbb{E}[\varphi(Y_\tau)|E_t^\mu = \tau, Y_0 = x] f_{E_t^\mu}(\tau)d\tau = \int_0^\infty f_{E_t^\mu}(\tau) T_\tau\varphi(x)d\tau. \qquad (6.12)$$

Now, in accordance with Theorem 6.1, $u(t, x)$ satisfies the Cauchy problem (6.7),(6.8).

Remark 6.3. Theorem 6.2 holds true for arbitrary measure μ bounded on $[0, 1]$, as well. Namely, in this case the FPK equation (6.7) takes the form

$$D_\mu u(t,x) = \sum_{j\in J}\mathcal{A}_j(x,D_x)u(t,x), \quad t>0,\ x\in\mathbb{R}^n,$$

where D_μ is the distributed fractional order differential operator [39], defined by

$$D_\mu f(t) = \int_0^1 D_*^\beta f(t)\mu(d\beta), \quad t > 0,$$

where $f(t)$ is a differentiable function on $[0, \infty)$.

Example. Let $L_t^{(j)}, j \in J$, be symmetric α_j-stable (with $\beta = \mu = 0$ in equation (3.4)) Lévy processes. Then the symbol of the process $L_t = \sum_{j\in J} c_j L_t^{(j)}$ has the form $\Psi(\xi) = \sum_{j\in J} c_j^{\alpha_j}|\xi|^{\alpha_j}$. The corresponding to the process $(L \circ E^\mu)_t$ FPK equation then takes the form

$$D_\mu u(t,x) = \sum_{j\in J} c_j^{\alpha_j} D_{|x|}^{\alpha_j} u(t,x), \quad t>0,\ x\in\mathbb{R},$$

where $D_{|x|}^\alpha$ is the Riesz-Feller derivative of order $0 < \alpha < 2$ defined by

$$D_{|x|}^\alpha f(x) = \frac{\alpha\Gamma(\frac{\alpha}{2})\Gamma(\frac{1+\alpha}{2})\sin\frac{\alpha\pi}{2}}{\pi^2 2^{2-\alpha}} \int\limits_{\mathbb{R}} \frac{f(x-y) - 2f(x) + f(x+y)}{|y|^{1+\alpha}}dy, \quad x \in \mathbb{R}.$$

References

[1] Applebaum D.; Lévy processes and stochastic calculus. Cambridge University Press, –2004.

[2] Andries E., Umarov S., Steinberg St.; Monte-Carlo random walk simulations based on distributed order differential equations with applications to cell biology. Frac. Calc. Appl. Anal. –2006, –9, –(4), –351-369.

[3] Barndorff-Nielsen O.E., Mikosch T., Resnick S.I.; Lévy processes: Theory and Applications. Birkháuser, –2001.

[4] Billingsley, P.; Convergence of Probability Measures. Wiley Series in Prob. and Stat., –1999.

[5] Burr M. N.; Weak convergence of stochastic integrals driven by continuous-time random walks. https://doi.org/10.48550/arXiv.1110.0216, –2011.

[6] Fa K. S.; Uncoupled continuous-time random walk: Finite jump length probability density function. J. Phys. A: Math. Theor. –2012, –45, –195002.

[7] Gorenflo R., Mainardi F., Moretti D., Pagnini G., Paradisi P.; Discrete random walk models for space-time fractional diffusion. Chemical Physics, –2000, –284, –521-541.

[8] Scalas E., Gorenflo R., Mainardi F.; Fractional calculus and continuous-time finance. Physica A, –2000, –284, –(1), –376-384.

[9] Hahn M., Kobayashi K., Umarov S.; SDEs driven by a time-changed Lévy process and their associated time-fractional order pseudo-differential equations. J. Theoret. Probab. –2012, –25, –(1), –262-279.

[10] Hahn M., Umarov S.; Fractional Fokker-Planck-Kolmogorov type equations and their associated stochastic differential equations. Frac. Calc. Appl. Anal. –2011, –14, –(1), –56-79.

[11] Hudson W. N., Mason J.D.; Operator-self-similar processes in a finite-dimensional space. Trans. AMS, –1982, –273, –(1), 281-297.

[12] Ikeda N., Watanabe S.; Stochastic differential equations and diffusion processes. North-Holland/Kodansha, –1989.

[13] Itô K.; Stochastic integral. Proc. of the Imperial Academy, –1944, –20, –(8), –519-524.

[14] Jacod J., Shiryaev A.N.; Limit theorems for stochastic processes. Springer-Verlag, –1987.

[15] A strong and weak approximation scheme for stochastic differential equations driven by a time-changed Brownian motion. Prob. Math. Stat. –2016, –36, –(2), –201-220.

[16] Jin S., Kobayashi K.; Strong approximation of stochastic differential equations driven by a time-changed Brownian motion with time-space-dependent coefficients. J. Math. Anal. Appl. –2019, –476, –(2), –619-636.

[17] Kotulski, M.; Asymptotic distributions of the continuous-time random walks: a probabilistic approach. J. Stat. Phys. –1995, –81, –777-792.

[18] Kurtz T.G., Protter P.; Weak limit theorems for stochastic integrals and stochastic differential equations. Ann. Probab., –1991, –19, –1035-1070.

[19] Liu F., Shen S., Anh V., Turner I.; Analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation. ANZIAM J. –2004, –46, –488-504.

[20] Mainardi F., Raberto M., Gorenflo R., Scalas E.; Fractional calculus and continuous-time finance II: the waiting-time distribution. Physica A, –2000, –287, –(3), –468-481.

[21] Metzler, R., Klafter, J.; The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. –2000, –339, –(1), –1-77.

[22] Meerschaert M. M., Scheffler H.-P.; Limit theorems for continuous-time random walks with infinite mean waiting times. J. Appl. Probab. –2004, –41, –623-638.

[23] Meerschaert M. M., Scheffler H.-P.; Triangular array limits for continuous time random walks. Stoch. Proc. Appl. –2008, –118, –1606-1633.

[24] Meerschaert M.M., Scheffler H.-P; Limit distributions for sums of independent random vectors. Wiley, –2001.

[25] Montroll E. W., Weiss G. H.; Random walks on lattices. II. J. Mathematical Phys. –1965, –6, –167-181.

[26] Nolan L.P.; Univariate Stable Distributions: Models for Heavy Tailed Data. Springer, –2020.

[27] Prokhorov Yu.V.; Convergence of Random Processes and Limit Theorems in Probability Theory. Theory Probab. Appl. –1956, –1, –(2), –157-214.

[28] Samorodnitsky G., Taqqu M.S.; Stable non-Gaussian random processes. Chapman & Hall, –1994.

[29] Sato K.-i.; Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, –1999.

[30] Scalas E., Viles N.; A functional limit theorem for stochastic integrals driven by a time-changed symmetric α-stable Lévy process. Stoch. Proc. Appl. –2014, –124, –385-410.

[31] Silvestrov D.S.; Conditions for convergence of the superposition of random processes in the J-topology. Theory Probab. Appl. –1974, –18, –(3), –579-582.

[32] Silvestrov D.S.; Convergence in Skorokhod J-topology for compositions of stochastic processes. Theory of Stochastic Processes, –2008, –14–30, –(1), –126-143.

[33] Silvestrov D.S.; Limit Theorems for Randomly Stopped Stochastic Processes. Springer, –2004.

[34] Situ R.; Theory of stochastic differential equations with jumps and applications. Springer, –2005.

[35] Skorokhod A.V.; Limit theorems for stochastic processes. Theor. Prob. Appl., –????, –1, –261-290.

[36] Samko S. G., Kilbas A. A., Marichev O. I.; Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, –1993.

[37] Umarov S., Gorenflo R.; On multi-dimensional symmetric random walk models approximating fractional diffusion processes. Frac. Calc. Appl. Anal., –2005, –8, –(1), –73-88.

[38] Umarov S., Hahn M., Kobayashi K.; Beyond the triangle: Brownian motion, Itô calculus, and Fokker-Planck equation - Fractional generalizations. World Scientific, –2018.

[39] Umarov S.; Introduction to fractional and pseudo-differential equations with singular symbols. Springer, –2015.

[40] Umarov S., Steinberg St.; Random walk models associated with distributed fractional order differential equations. IMS Lect. Notes Monograph Series, –2006, –51, –117-127.

[41] Weron K., Jurlewich A., Magdziarz M.; HavriliakNegami response in the framework of the continuous-time random walk. Acta Phys. Polonica B, –2005, –36, –1855-1868.

[42] Whitt W.; Stochastic-process Limits: an Introduction to Stochastic-Process Limits and their Application to Queues. Springer, –2002.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.327-336
DOI: 10.29229/uzmj.2025-2-31

Initial-boundary-value problem for inhomogeneous mixed type fourth-order differential equation with power degeneration

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this paper an initial boundary value problem is studied for a fourth-order nonhomogeneous equation of mixed type with a degenerate parabolic part. Using the spectral analysis method, a criterion for the uniqueness of the solution to the problem was established and the solution is constructed as the sum of a Fourier series. The work also studied the spectral properties of the Samarskii - Ionkin type problem for an ordinary differential equation of the fourth order, found the eigenvalues and corresponding eigenfunctions, proved their completeness and basis property, and investigated the adjoint problem.

Keywords: degenerate equation; mixed type equation; Samarskii-Ionkin type problem; spectral method; adjoint problem; completeness; Riss basis; existence; uniqueness.
MSC (2020): 34B10, 34L10, 35K35, 35K65, 35M13.

1. Introduction and problem statement

Among the works devoted to the study of boundary value problems for differential equations, initial-boundary-value problems for a degeneration mixed type differential equations play a special role. Taking into account the vastness of the bases of such works, we cite only those that are close to our work on the subject. In the work [7], for a mixed parabolic-hyperbolic equation with power degeneration of the form

$$F(x,t) = Lu \equiv \begin{cases} t^n u_{xx} - u_t - b^2 t^n u, t > 0 \\ (-t)^m u_{xx} - u_{tt} - b^2 (-t)^m u,\ t < 0 \end{cases}, \tag{*}$$

in rectangular domain $\Omega = \{(x,t)\,|0 < x < 1, -\alpha < t < \beta\}$ in case $F(x,t) = 0$ the first mixed problem has been studied $(n, m, \alpha, \beta \in R^+)$. Using spectral method, the criteria for the uniqueness of the solutions of the given problems are established, which are constructed in the form of a Fourier series. The stability of the solution for the nonlocal condition is also established. In [8], for the equation (*) in the same domain Ω in case $n = 0$ and $F(x,t) = f(x)$, the inverse problem for finding the unknown right-hand side was considered. The solution is constructed as the sum of the series of eigenfunctions corresponding to a one-dimensional spectral problem, and the uniqueness of the solution of the problem is established by the method of spectral analysis. A similar nonlocal inverse problem for finding unknown right-hand sides for equation (*) was studied in [10]. Note also the work [11], where for the equation (*) in the case of $F(x,t) = 0$ in the domain Ω, nonlocal boundary value problems were studied. In [9] the initial-boundary value problem for three classes of non-homogeneous degenerate mixed parabolic-hyperbolic type equations is considered. In each case, the criterion of uniqueness of the solution is established, and the solution is constructed in the form of series according to the system of eigenfunctions, corresponding to a one-dimensional spectral problem. In [12] inverse problems were posed and investigated for determining the factors of the right-hand sides for the equation (*) in case $m = 0$, when $F(x,t) = f_1(x)g_1(t)$ for $t > 0$ and $F(x,t) = f_2(x)g_2(t)$ for $t < 0$. Based on the theory of integral equations, corresponding theorems of uniqueness and existence of solutions to inverse problems have been proven, and explicit formulas for the solution have been obtained. We also note the works [5],[6],[15], where boundary value problems in various formulas for degenerate elliptic-hyperbolic equations are investigated.

As for high-order partial differential equations, we note the works [13], [14], where initial-boundary value problems were formulated for a degenerate high-paired partial differential equations with a Bessel operator in a rectangle and the existence, uniqueness, and stability of the solution of the investigated problems were proven.

Let us move on to the formulation of the boundary value problem we are studying. Let

$$\Omega = \{(x,t)\,|0 < x < 1, -\alpha < t < \beta\}, \Omega_1 = \Omega \cap \{t > 0\}, \Omega_2 = \Omega \cap \{t < 0\},$$

$$Lu = \begin{cases} u_t + t^n u_{xxxx} + bt^n u, t > 0, \\ u_{tt} + u_{xxxx} + bu, t < 0. \end{cases}, \quad f(x,t) = \begin{cases} f_1(x,t), t > 0 \\ f_2(x,t), t < 0 \end{cases},$$

where n, α, β, b - are given real positive numbers.

Problem 1. To find a function $u\,(x,t)$ from the class

$$u(x,t) \in C^{3,1}_{x,t}(\bar{\Omega}) \cap C^2_t(\Omega_2) \cap C^4_x(\Omega_1 \cup \Omega_2), \tag{1.1}$$

which satises the equation

$$Lu = f(x,t) \tag{1.2}$$

with boundary value conditions

$$u(1,t) = 0, u_x(0,t) + u_x(1,t) = 0, u_{xx}(0,t) = 0, u_{xxx}(0,t) + u_{xxx}(1,t) = 0, -\alpha \leq t \leq \beta, \tag{1.3}$$

$$u(x,-\alpha) = 0, 0 \leq x \leq 1. \tag{1.4}$$

2. Spectral properties of Problem 1

To solve Problem 1, we apply the spectral method, according to which we seek the non-trivial solutions of Equation (1.2) in the form $u(x,t) = X(x) \cdot T(t)$. By substituting this function into Equation (1.2) and by using conditions (1.3), for the unknown function $X(x)$, we obtain the spectral problem

$$X^{IV}(x) - \mu X(x) = 0,\ 0 < x < 1, \tag{2.1}$$

$$X(1) = 0, X''(0) = 0, X'(0) + X'(1) = 0, X'''(0) + X'''(1) = 0. \tag{2.2}$$

Let us find the eigenvalues and eigenfunctions of the problem (2.1), (2.2). For $\mu \leq 0$ the problem (2.1), (2.2) has a trivial solution, and for $\mu > 0$ this problem has eigenvalues

$$\mu_n = (\pi(2n-1))^4, n = 1, 2, ..., \tag{2.3}$$

and the corresponding eigenfunctions have the form

$$X_{1n}(x) = 2\sin(2n-1)\pi x,\ \ X_{2n}(x) = \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi} + 1} + \cos(2n-1)\pi x. \tag{2.4}$$

Problem (2.1), (2.2) is non-self-adjoint and it is not difficult to see that the following problem will be adjoint to it

$$Y^{IV}(x) - \mu Y(x) = 0,\ 0 < x < 1, \tag{2.5}$$

$$Y(0) + Y(1) = 0, Y'(1) = 0, Y''(0) + Y''(1) = 0, Y'''(0) = 0. \tag{2.6}$$

It is not difficult to show that the problem (2.5), (2.6) has eigenvalues (2.3), and the corresponding eigenfunctions have the form

$$Y_{1n}(x) = \frac{e^{(2n-1)\pi x} - e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi} + 1} + \sin(2n-1)\pi x,\ Y_{2n}(x) = 2\cos(2n-1)\pi x. \tag{2.7}$$

Let us proceed to the study of the basicity of systems (2.4) and (2.7) in $L_2(0,1)$. Using the definition of biorthonormality of two systems of functions, it is easy to show that the following lemma is true.

Lemma 2.1. *The system of functions (2.4) and (2.7) are biorthonormal systems in $L_2(0,1)$.*

Theorem 2.2. *The system of functions (2.4) and (2.7) are complete in the space* $L_2(0,1)$.

Proof. We consider the completeness of system (2.4). Assume, on the contrary, that the system of functions (2.4) is not complete in $L_2(0,1)$. Then there exists a nontrivial function $\varphi(x)$ in $L_2(0,1)$, that is orthogonal to all functions of system (2.4). Let us expand the function $\varphi(x)$ into a Fourier series

$$\varphi(x)=\sum_{n=1}^{\infty}(a_n\cos(2n-1)\pi x+b_n\sin(2n-1)\pi x),$$

according to the complete orthogonal system $\{\cos(2n-1)\pi x,\sin(2n-1)\pi x\}_{n=1}^{\infty}$, which converges in $L_2(0,1)$. Since $\varphi(x)$ is orthogonal to the system $\{\sin(2n-1)\pi x\}_{n=1}^{\infty}$, then the last expansion can be written as

$$\varphi(x)=\sum_{n=1}^{\infty}a_n\cos(2n-1)\pi x. \tag{2.8}$$

Further, multiplying last series by the function $X_{2k}(x)$, and integrating along $[0,1]$, by using the orthogonality of this last function and $\varphi(x)$, we obtain the following equality:

$$0=\int_0^1\varphi(x)\cdot\left(\frac{e^{(2k-1)\pi x}+e^{\pi(2k-1)(1-x)}}{e^{(2k-1)\pi}+1}+\cos(2k-1)\pi x\right)dx=$$

$$=\sum_{n=1}^{\infty}a_n\int_0^1\left(\frac{e^{(2k-1)\pi x}+e^{\pi(2k-1)(1-x)}}{e^{(2k-1)\pi}+1}+\cos(2k-1)\pi x\right)\cos(2n-1)\pi x dx=\frac{1}{2}a_k,\ k=1,2,3,....$$

From here, it follows that $a_k=0$, $k=1,2,....$ Therefore, from (2.8) we conclude that $\varphi(x)=0$ in $[0,1]$, which opposing conditions $\varphi(x)\neq 0$. Thus, the system (2.4) is complete in the space $L_2(0,1)$. The completeness of the system (2.7) is proved similarly. □

Lemma 2.3. *The system of functions (2.4) and (2.7) minimal in* $L_2(0,1)$.

The proof of Lemma 2.3 follows from Lemma 2.1 and Theorem 2.2. [1].

Theorem 2.4. *The system of functions (2.4) and (2.7) are two bases of Riesz in* $L_2(0,1)$.

Proof. In order to prove this statement, it is sufficient to prove the completeness of systems (2.4) and (2.7), and the convergence of the following series for $\varphi(x)\in L_2(0,1)$ according to Theorem 2.1 from [[2], p.375].

$$\sum_{n=1}^{\infty}(\varphi(x),2\sin(2n-1)\pi x)_0^2+\sum_{n=1}^{\infty}\left(\varphi(x),\frac{e^{(2n-1)\pi x}+e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}+\cos(2n-1)\pi x\right)_0^2, \tag{2.9}$$

$$\sum_{n=1}^{\infty}(\varphi(x),2\cos(2n-1)\pi x)_0^2+\sum_{n=1}^{\infty}\left(\frac{e^{(2n-1)\pi x}-e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}+\sin(2n-1)\pi x,\varphi(x)\right)_0^2, \tag{2.10}$$

where $(\varphi,\psi)_0=(\varphi,\psi)_{L_2(a,b)}=\int\limits_a^b\varphi(x)\psi(x)dx$-inner product in $L_2(a,b)$.

Since the completeness of systems (2.4) and (2.7) has been proven in Theorem 2.2 we only must verify the convergence of the previous series. To this end, we consider (2.9) and introduce the following notations

$$I_1=4\sum_{n=1}^{\infty}(\varphi(x),\sin(2n-1)\pi x)_0^2,\ I_2=\sum_{n=1}^{\infty}\left(\varphi(x),\frac{e^{(2n-1)\pi x}+e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}+\cos(2n-1)\pi x\right)_0^2.$$

I_1 represent in the form

$$I_1=4\sum_{n=1}^{\infty}(\varphi(x),\sin(2n-1)\pi x)_0^2=2\sum_{n=1}^{\infty}\left(\varphi(x),\sqrt{2}\sin(2n-1)\pi x\right)_0^2=2\sum_{n=1}^{\infty}c_n^2,$$

where $c_n = (\varphi(x), \sqrt{2}\sin(2n-1)\pi x)_0$ are the Fourier coefficients of the function $\varphi(x)$ on the orthonormal system $\{\sqrt{2}\sin(2n-1)\pi x\}$. From here, applying Bessels inequality, we obtain that $I_1 = 2\sum\limits_{n=1}^{\infty} c_n^2 \leq 2\|\varphi(x)\|^2_{L_2(0,1)}$, i.e. I_1 is finite. Consider I_2. Let

$$A = \left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \cos(2n-1)\pi x\right)_0^2.$$

From here, applying inequality $(a+b)^2 \leq 2(a^2+b^2)$ we obtain that

$$A \leq 2\left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\left(\varphi(x), \cos(2n-1)\pi x\right)_0^2 =$$

$$= 2\left(\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0 + \left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{(2n-1)\pi}+1}\right)_0\right)^2 + 2\left(\varphi(x), \cos(2n-1)\pi x\right)_0^2.$$

Applying the previous inequality again, we get that

$$A \leq 4\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0^2 + 4\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\left(\varphi(x), \cos(2n-1)\pi x\right)_0^2.$$

Thus

$$I_2 \leq 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0^2 + 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\sum_{n=1}^{\infty}\left(\varphi(x), \cos(2n-1)\pi x\right)_0^2 = J_1 + J_2 + J_3.$$

Consider $J_3 = 2\sum\limits_{n=1}^{\infty}(\varphi(x), \cos(2n-1)\pi x)_0^2 = \sum\limits_{n=1}^{\infty} a_n^2$, where $a_n = (\varphi(x), \sqrt{2}\cos(2n-1)\pi x)_0$ are the Fourier coefficients of the function $\varphi(x)$ on the orthonormal system $\{\sqrt{2}\cos(2n-1)\pi x\}$. Then applying Bessel inequality, we get $J_3 = \sum\limits_{n=1}^{\infty} a_n^2 \leq \|\varphi(x)\|^2_{L_2(0,1)}$.

Consider J_2. Since $\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{\pi(2n-1)}+1}\right)_0 = \frac{e^{\pi(2n-1)}}{e^{\pi(2n-1)}+1}\left(\varphi(x)e^{\pi x}, e^{-2\pi n x}\right)_0$, hence

$$J_2 = 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 \leq 4\sum_{n=1}^{\infty}\left(\varphi(x)e^{\pi x}, e^{-2\pi n x}\right)_0^2 = 4\sum_{n=1}^{\infty} c_n^2,$$

where $c_n = (\varphi(x)e^{\pi x}, e^{-2n\pi x})_0$. Then from Lemma 3 from [4], it follows that J_2 is finite. Similarly, we can prove that J_1 is finite too. Thus, the series I_1 and are converge, and therefore series (2.9) also converges. The convergence of series (2.10) is proved similarly. □

3. The uniqueness of the solution of Problem 1

Let $u(x,t)$ is the solution of Problem 1. Consider the functions

$$u_{ik}(t) = \int_0^1 u(x,t)Y_{ik}(x)dx,\ i = 1,2, k = 1,2,... \tag{3.1}$$

and based on this, we introduce functions $v_{i\varepsilon}(t) = \int\limits_{\varepsilon}^{1-\varepsilon} u(x,t)Y_{ik}(x)dx, i = 1,2, k = 1,2,...$, where ε- sufficiently small number. Differentiating this equality by t at $t \in (0,\beta)$ once and at $t \in (-\alpha, 0)$ twice and considering Equation (1.2), we obtain

$$\begin{aligned} v'_{i\varepsilon}(t) &= \int_{\varepsilon}^{1-\varepsilon} u_t(x,t)Y_{ik}(x)dx = -t^n\int_{\varepsilon}^{1-\varepsilon} u_{xxxx}(x,t)Y_{ik}(x)dx - bt^n\int_{\varepsilon}^{1-\varepsilon} u(x,t)Y_{ik}(x)dx, i = 1,2. \\ v''_{i\varepsilon}(t) &= \int_{\varepsilon}^{1-\varepsilon} u_{tt}(x,t)Y_{ik}(x)dx = -\int_{\varepsilon}^{1-\varepsilon} u_{xxxx}(x,t)Y_{ik}(x)dx - b\int_{\varepsilon}^{1-\varepsilon} u(x,t)Y_{ik}(x)dx, i = 1,2. \end{aligned} \tag{3.2}$$

Further, in the integrals from the right-hand sides in (3.2) integrating by parts four times and moving to the limit at $\varepsilon \to 0$ considering the boundary conditions (1.3), we obtain ordinary differential equations to find the unknown functions $u_{in}(t), i = 1, 2$

$$u'_{ik}(t) + \lambda_k t^n u_{ik}(t) = f^i_{1k}(t),\ i = 1, 2, t > 0,\ \ u''_{ik}(t) + \lambda_k u_{ik}(t) = f^i_{2k}(t), i = 1, 2, t < 0,$$

which general solutions have the form

$$u_{ik}(t) = \begin{cases} a_{ik} e^{-\lambda_k t^{n+1}/(n+1)} + \int\limits_0^t f^i_{1k}(s) e^{-\lambda_k \left(t^{n+1}/(n+1) - s^{n+1}/(n+1)\right)} ds,\ t > 0 \\ a_{ik} \cos\sqrt{\lambda_k} t + d_{ik} \sin\sqrt{\lambda_k} t - \frac{1}{\sqrt{\lambda_k}} \int\limits_t^0 f^i_{2k}(s) \sin\left[\sqrt{\lambda_k}(t-s)\right] ds,\ t < 0 \end{cases}, i = 1, 2, \tag{3.3}$$

where $a_{ik}, c_{ik}, d_{ik}, i = 1, 2$ are arbitrary real numbers, $\lambda_k = b + ((2k-1)\pi)^4$, $f^i_{1k} = \int\limits_0^1 f_1(x,t) Y^i_k(x) dx, i = 1, 2$, $f^i_{2k} = \int\limits_0^1 f_2(x,t) Y^i_k(x) dx, i = 1, 2.$

By assumption $u \in C^1(\bar{\Omega})$, then the function (3.3) satisfies the relations

$$u_{ik}(0+0) = u_{ik}(0-0),\ u'_{ik}(0+0) = u'_{ik}(0-0), i = 1, 2, k = 1, 2, ...$$

Satisfying (3.3) these conditions, we find the relationship between the coefficients $a_{ik}, c_{ik}, d_{ik}, i = 1, 2$ in the form $c_{ik} = a_{ik}, d_{ik} = \frac{1}{\sqrt{\lambda_k}} f^i_{1k}(0+0)$. Taking into account the last equalities, the functions (3.3) take the form

$$u_{ik}(t) = \begin{cases} a_{ik} e^{-\lambda_k t^{n+1}/(n+1)} + \int\limits_0^t f^i_{1k}(s) e^{-\lambda_k \left(t^{n+1}/(n+1) - s^{n+1}/(n+1)\right)} ds,\ t > 0 \\ a_{ik} \cos\sqrt{\lambda_k} t + \frac{f^i_{1k}(0+0)}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k} t - \frac{1}{\sqrt{\lambda_k}} \int\limits_t^0 f^i_{2k}(s) \sin\left[\sqrt{\lambda_k}(t-s)\right] ds,\ t < 0 \end{cases}, i = 1, 2. \tag{3.4}$$

To find the constants a_{ik}, we use the boundary condition (1.3) and formula (3.3)

$$u_{ik}(-\alpha) = \int\limits_0^1 u(x, -\alpha) Y_{ik}(x) dx = 0. \tag{3.5}$$

Then from the relations (3.4) and (3.5) we find $a_{ik} = \omega_{ik}(\alpha)/\Delta_k(\alpha)$ under the condition that for all $k = 1, 2, ...$

$$\Delta_k(\alpha) = \cos\sqrt{\lambda_k}\alpha \neq 0, \tag{3.6}$$

where $\omega_{ik}(\alpha) = \frac{f^i_{1k}(0+0)}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\alpha - \frac{1}{\sqrt{\lambda_k}} \int\limits_{-\alpha}^0 f^i_{2k}(s) \sin\left[\sqrt{\lambda_k}(\alpha+s)\right] ds.$

Further, by substituting a_{ik} into (3.4), we obtain

$$u_{ik}(t) = \begin{cases} \frac{\omega_{ik}(\alpha)}{\Delta_k(\alpha)} e^{-\lambda_k t^{n+1}/(n+1)} + \int\limits_0^t f^i_{1k}(s) e^{-\lambda_k \left(t^{n+1}/(n+1) - s^{n+1}/(n+1)\right)} ds,\ t > 0 \\ \frac{\omega_{ik}(\alpha)}{\Delta_k(\alpha)} \cos\sqrt{\lambda_k} t + \frac{f^i_{1k}(0+0)}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k} t - \frac{1}{\sqrt{\lambda_k}} \int\limits_t^0 f^i_{2k}(s) \sin\left[\sqrt{\lambda_k}(t-s)\right] ds,\ t < 0 \end{cases}, i = 1, 2. \tag{3.7}$$

Now, based on formula (3.7), we will prove the uniqueness of the solution to Problem 1. Let $f(x,t) \equiv 0$ on $[0,1]$ and for all $k \in N$, condition (3.6) is satisfied. Then $f_{1k}^i(t) = f_{2k}^i(t) \equiv 0$ for all $k = 1, 2, ...$ and from formulas (3.3) and (3.7) follows $\int\limits_0^1 u(x,t)Y_{ik}(x)dx = 0, i = 1, 2$. From this, due to the completeness of the system (2.7) in the space $L_2[0,1]$, it follows that $u(x,t) = 0$ is almost everywhere on $[0,1]$ for any $t \in [-\alpha, \beta]$. Since $u \in C(\bar{\Omega})$, it follows that $u(x,t) \equiv 0$ in $\bar{\Omega}$. If for some α, β, n, b and $k = s$ the condition (3.6) fails, i.e. $\Delta_k(\alpha) = 0$, then the homogeneous Problem 1 (where $f_i(x,t) \equiv 0, i = 1, 2$) has the nontrivial solution

$$u_{is}(x,t) = \begin{cases} c_s e^{-\lambda_s t^{n+1}/(n+1)}\, X_{is}(x), t > 0 \\ c_s \cos\sqrt{\lambda_s} t X_{is}(x),\ t < 0 \end{cases}, i = 1, 2, \tag{3.8}$$

where $c_s \neq 0$ is an arbitrary constant.

We write the expression $\Delta_k(\alpha)$ in the form

$$\Delta_k(\alpha) = \cos{(2k-1)}^2 \pi \widetilde{\lambda}_k \widetilde{\alpha}, \tag{3.9}$$

where $\widetilde{\alpha} = \pi\alpha$, and $\widetilde{\lambda}_k = \sqrt{1 + \frac{b}{((2k-1)\pi)^4}} = 1 + \theta_k$. Equation $\Delta_k(\alpha) = 0$ has a countable set of zeros with respect to $\widetilde{\alpha}$, which are determined by the formula

$$\widetilde{\alpha} = \frac{1}{2{(2k-1)}^2 \widetilde{\lambda}_k} + \frac{m}{{(2k-1)}^2 \widetilde{\lambda}_k}, k, m \in N. \tag{3.10}$$

Thus, we have established the following uniqueness criterion.

Theorem 3.1. *If there exists a solution of Problem 1, then it is unique only if the condition (3.6) is fulfilled for all $k = 1, 2, ...$*

4. The existence of the solution of Problem 1

Since the expression $\Delta_k(\alpha)$ for the value (3.10) of the parameter $\widetilde{\alpha}$ may vanish, it is necessary to state an estimate for $\Delta_k(\alpha)$ that show that it is separated from zero and obtain the corresponding asymptotics.

Lemma 4.1. *If $b = 0$ and $\widetilde{\alpha}$ is an arbitrary natural number or $b \geq 0$ and $\widetilde{\alpha} = p/t$ is an arbitrary rational number, where $p/t \notin N, (p,t) = 1$, and $t \neq 2r+2$ or $t \neq 2r$ for certain $r = \overline{1, t-1}$, then there exist positive constants C_0 and $k_0, k_0 \in N$, such that for all $k > k_0$ the following estimate holds:*

$$|\Delta_k(\alpha)| \geq C_0 > 0. \tag{4.1}$$

Lemma 4.2. *If $b \geq 0$ and $\widetilde{\alpha} = p/t$ is an arbitrary rational number, where $p/t \notin N$, $(p,t) = 1$ and $(r+1)/t = 1/2$ or $r/t = 1/2$ for certain $1 \leq r \leq t-1$, then there exist positive C_0 and k_0, $k_0 \in N$, such that for all $k > k_0$ the following estimate is valid:*

$$|\Delta_k(\alpha)| \geq \frac{C_0}{{(2k-1)}^2} > 0. \tag{4.2}$$

Lemma 4.1 and Lemma 4.2 are proven similarly to Lemma 3.2 and Lemma 3.3 in [5].

If for the numbers $\tilde{\alpha}$ from Lemma 4.1 the conditions (4.1) for $k = 1, 2, ..., k_0$ are satisfied, then we will seek the solution of Problem 1 in the form the series

$$u(x,t) = \sum_{k=1}^{+\infty} (u_{1k}(t)X_{1k}(x) + u_{2k}(t)X_{2k}(x)), \tag{4.3}$$

where $u_{ik}(t)$ are defined by formulas (3.7).

Lemma 4.3. *If the conditions of Lemma 4.1 are fulfilled, then the following estimates hold for all* $k = 1, 2, ...$:

$$|u_{ik}(t)| \le M_1 \left[(2k-1)^{-2}\|f_{1k}^i\|_C + (2k-1)^{-2}\|f_{2k}^i\|_C\right], t \in [0, \beta], \tag{4.4}$$

$$|u'_{ik}(t)| \le M_2 \left[(2k-1)^{2}\|f_{1k}^i\|_C + (2k-1)^{2}\|f_{2k}^i\|_C\right], t \in [0, \beta], \tag{4.5}$$

$$|u_{ik}(t)| \le M_3 \left[(2k-1)^{-2}\|f_{1k}^i\|_C + (2k-1)^{-2}\|f_{2k}^i\|_C\right], t \in [-\alpha, 0], \tag{4.6}$$

$$|u'_{ik}(t)| \le M_4 \left[\|f_{1k}^i\|_C + \|f_{2k}^i\|_C\right], t \in [-\alpha, 0], \tag{4.7}$$

$$|u''_{ik}(t)| \le M_5 \left[(2k-1)^{2}\|f_{1k}^i\|_C + (2k-1)^{2}\|f_{2k}^i\|_C\right], t \in [-\alpha, 0], \tag{4.8}$$

where

$$\|f_{1k}^i\|_C = \max_{0\le t\le\beta} |f_{1k}^i(t)|, \|f_{2k}^i\|_C = \max_{-\alpha\le t\le 0} |f_{2k}^i(t)|$$

and M_i *are positive constants, depending on* α, β *and* b.

These estimates can be proved by using the inequality (4.1), which is valid for all $k > k_0$.

Lemma 4.4. *If the conditions of Lemma 4.2 are fulfilled, then the following estimates hold for all* $k > k_0$:

$$|u_{ik}(t)| \le M_6 \left[\|f_{1k}^i\|_C + \|f_{2k}^i\|_C\right], t \in [0, \beta], \tag{4.9}$$

$$|u'_{ik}(t)| \le M_7 \left[(2k-1)^{4}\|f_{1k}^i\|_C + (2k-1)^{4}\|f_{2k}^i\|_C\right], t \in [0, \beta], \tag{4.10}$$

$$|u_{ik}(t)| \le M_8 \left[\|f_{1k}^i\|_C + \|f_{2k}^i\|_C\right], t \in [-\alpha, 0], \tag{4.11}$$

$$|u'_{ik}(t)| \le M_9 \left[(2k-1)^{2}\|f_{1k}^i\|_C + (2k-1)^{2}\|f_{2k}^i\|_C\right], t \in [-\alpha, 0], \tag{4.12}$$

$$|u''_{ik}(t)| \le M_{10} \left[(2k-1)^{4}\|f_{1k}^i\|_C + (2k-1)^{4}\|f_{2k}^i\|_C\right], t \in [-\alpha, 0]. \tag{4.13}$$

By Lemma 4.3, the series (4.3) and its first-order derivative with respect to t and its third-order derivative with respect to x in the closed domain $\bar{\Omega}$, the second-order derivative with respect to t in $\bar{\Omega}_2$, as well as the fourth-order derivatives with respect to x in $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, are majorized by the series

$$M_{11} \sum_{k=1}^{\infty} \sum_{i=1}^{2} (2k-1)^2 \left(\|f_{1k}^i\|_C + \|f_{2k}^i\|_C\right). \tag{4.14}$$

Lemma 4.5. *Let* $f_i(x,t) \in^{3,0}_{x,t} (\bar{\Omega}_i)$ $i = 1, 2$ *and satisfy the following conditions*

$$f_i(1,t) = \left.\frac{\partial^2 f_i}{\partial x^2}\right|_{x=0} = 0, \left.\frac{\partial f_i}{\partial x}\right|_{x=0} = \left.\frac{\partial f_i}{\partial x}\right|_{x=1}, \left.\frac{\partial^3 f_i}{\partial x^3}\right|_{x=0} = \left.\frac{\partial^3 f_i}{\partial x^3}\right|_{x=1}, t \in \begin{cases} [0,\beta], i = 1 \\ [-\alpha, 0], i = 2 \end{cases}.$$

Then the following estimates hold

$$\sum_{k=1}^{\infty} \left|f_{1k}^{i(3)}(t)\right|^2 \le \left\|\frac{\partial^3 f_1(x,t)}{\partial x^3}\right\|^2_{L_2(0,1)}, \sum_{k=1}^{\infty} \left|f_{2k}^{i(3)}(t)\right|^2 \le \left\|\frac{\partial^3 f_2(x,t)}{\partial x^3}\right\|^2_{L_2(0,1)}, t \in \begin{cases} [0,\beta], i = 1 \\ [-\alpha, 0], i = 2 \end{cases}. \tag{4.15}$$

Here

$$f_{ik}^1(t) = \frac{f_{ik}^{1(3)}(t)}{((2k-1)\pi)^3}, f_{ik}^{1(3)}(t) = \int_0^1 f'''_{ixxx}(x,t)\sin(2k-1)\pi x dx,$$

$$f_{ik}^2(t) = \frac{f_{ik}^{2(3)}(t)}{((2k-1)\pi)^3}, f_{ik}^{2(3)}(t) = -\int_0^1 f'''_{ixxx}(x,t)\left(\frac{e^{(2k-1)\pi x} + e^{(2k-1)\pi(1-x)}}{e^{(2k-1)\pi x}+1} + \cos(2k-1)\pi x\right)dx.$$

Then by Lemma 4.3 the series (4.14) are majorized by the series

$$M_{12}\sum_{k=k_0+1}^{\infty}\sum_{i=1}^{2}\frac{1}{(2k-1)}\left[\left\|f_{1k}^{i(3)}\right\|_C+\left\|f_{2k}^{i(3)}\right\|_C\right]. \tag{4.16}$$

The convergence of the series (4.16), due to the Weierstrass M-test [3], we obtain the uniform convergence of the series (4.3), as well as series from the first-order derivatives t and the third-order derivatives x of the terms of this series in $\overline{\Omega}$, and the possibility of its term differentiation by t twice and by t four times in $t \le 0$ and by x four times in $t \ge 0$. Thus the following assertion is proved.

Theorem 4.6. *If the functions $f_i(x,t), i=1,2$, satisfy the conditions of Lemma 4.5 and the conditions of Lemma 4.1 are fulfilled, then there exists a unique solution to the Problem 1 and it is determined by the series (4.3).*

Assume that the equality $\Delta_s(\alpha)=0$ holds for certain numbers $\widetilde{\alpha}$ from Lemma 4.1 for some $k=s=k_1,k_2,...,k_m$, where $1\le k_1<k_2<...<k_m\le k_0, k_i, i=\overline{1,m}$, and m are given natural numbers. Then the Problem 1 is solvable if and only if

$$\frac{f_{1k}^i(0+0)}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}\alpha-\frac{1}{\sqrt{\lambda_k}}\int\limits_{-\alpha}^{0}f_{2k}^i(s)\sin\left[\sqrt{\lambda_k}(\alpha+s)\right]ds=0, s=k_1,k_2,...k_m. \tag{4.17}$$

In this case, a solution to the Problem 1 is determined by the series

$$u(x,t)=\left(\sum_{k=1}^{k_1-1}+...+\sum_{k=k_{m-1}+1}^{k_m-1}+\sum_{k=k_{m+1}}^{\infty}\right)\sum_{i=1}^{2}u_{ik}(t)X_{ik}+\sum_{s}\sum_{i=1}^{2}A_s u_{is}(x,t); \tag{4.18}$$

in the last sum s takes $k_1,k_2,...k_m$; $u_{is}(t)$ is defined by the formula (3.8), where k must be replaced by s, and c_s is an arbitrary nonzero constant. As usual, we assume that if in some finite sum in the right-hand side of (4.18) the upper limit is greater than lower, then this sum is equal to zero.

Thus, the following assertion is proved.

Theorem 4.7. *Let the condition of Lemma 4.1 be fulfilled and let functions $f_i(x,t), i=1,2$ satisfy the conditions of Lemma 4.5.*

1) If $\Delta_k(\alpha)\ne 0$ for all $k=\overline{1,k_0}$, then Problem 1 has a unique solution and this solution is defined by the series (4.3) whose coefficients are given by the formulas (3.7);

2) if $\Delta_k(\alpha)=0$ at some $k=k_1,...,k_m\le k_0$, then Problem 1 is solvable only when conditions (4.17) are satisfied and in this case the solution is determined by the series (4.18).

Assume that the conditions of Lemma 4.2 are fulfilled. Due to the Lemma 4.5 the series (4.3) and its first-order derivative with respect to t and its third-order derivative with respect to x in the closed domain $\bar{\Omega}$, the second-order derivative with respect to t in $\bar{\Omega}_2$, as well as the fourth-order derivatives with respect to x in $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, are majorized by the series

$$M_{13}\sum_{k=1}^{\infty}\sum_{i=1}^{2}(2k-1)^4\left\|f_{1k}^i\right\|_C+(2k-1)^4\left\|f_{2k}^i\right\|_C. \tag{4.19}$$

Lemma 4.8. *Let functions $f_i(x,t), i=1,2$ satisfy following conditions*

$$\left.\frac{\partial^k f_i}{\partial x^k}\right|_{x=0}=\left.\frac{\partial^k f_i}{\partial x^k}\right|_{x=1}, f_i(1,t)=0, \left.\frac{\partial^2 f_i}{\partial x^2}\right|_{x=0}=\left.\frac{\partial^4 f_i}{\partial x^4}\right|_{x=1}=0, t\in\begin{cases}[0,\beta], i=1\\ [-\alpha,0], i=2\end{cases}, k=1,3.$$

Then the following estimates hold

$$\sum_{k=1}^{\infty}\left|f_{1k}^{i(5)}(t)\right|^2\le\left\|\frac{\partial^5 f_1(x,t)}{\partial x^5}\right\|_{L_2(0,1)}^2, \sum_{k=1}^{\infty}\left|f_{2k}^{i(5)}(t)\right|^2\le\left\|\frac{\partial^5 f_2(x,t)}{\partial x^5}\right\|_{L_2(0,1)}^2, t\in\begin{cases}[0,\beta], i=1\\ [-\alpha,0], i=2\end{cases}. \tag{4.20}$$

Here

$$f_{ik}^{1} = \frac{1}{((2k-1)\pi)^5} f_{ik}^{1(5)}(t), f_{ik}^{1(5)}(t) = -\int\limits_0^1 f_{1xxxxx}(x,t)\sin(2k-1)\pi x dx,$$

$$f_{ik}^{2} = \frac{f_{ik}^{2(5)}(t)}{((2k-1)\pi)^5}, f_{ik}^{2(5)}(t) = -\int\limits_0^1 f_{ixxxxx}\left(\frac{e^{(2k-1)\pi x}+e^{(2k-1)\pi(1-x)}}{e^{(2k-1)\pi x}+1} - \cos(2k-1)\pi x\right)dx,$$

Then by Lemma 4.8 the series (4.18) are majorized by the series

$$M_{14}\sum_{k=k_0+1}^{\infty}\sum_{i=1}^{2}\frac{1}{(2k-1)}\left(\left\|f_{1k}^{i(5)}\right\|_C + \left\|f_{2k}^{i(5)}\right\|_C\right). \tag{4.21}$$

The convergence of the series (4.21), due to the Weierstrass M-test [3], we obtain the uniform convergence of the series (4.3), as well as series from the first-order derivatives t and the third-order derivatives x of the terms of this series in $\overline{\Omega}$, and the possibility of its term differentiation by t twice and by x four times in $t \le 0$ and by x four times in $t \ge 0$.

Thus, we have proved the following assertions.

Theorem 4.9. *Let the condition of Lemma 4.2 be fulfilled and let functions $f_i(x,t), i = 1,2$ satisfy the conditions of Lemma 4.8.*

1) If $\Delta_k(\alpha) \ne 0$ for all $k = \overline{1,k_0}$, then Problem 1 has a unique solution and this solution is defined by the series (4.3) whose coefficients are given by the formulas (3.7);

2) If $\Delta_k(\alpha) = 0$ at some $k = k_1, ..., k_m \le k_0$, then Problem 1 is solvable only when conditions (4.17) are satisfied and in this case the solution is determined by the series (4.18).

References

[1] Bary N. K . Biorthogonal systems and bases in Hilbert space. Math. Uch. Zap. Mosk. Gos. Univ. 1951. Vol. 148, Issue 4. p. 69-107 (in Russian).

[2] Gohberg I.C., Krein M.G. Introduction to the theory of linear non-selfadjoint operators. Nauka, Moscow 1965, 448 p. (In Russian).

[3] Ilyin V. A., Poznyak E. G. Fundamentals of Mathematical Analysis, Part 2, 7th ed. (Fizmatlit, M., 2005).

[4] Kesel'man G. M. On the unconditional convergence of eigenfunction expansions of certain differential operators, Izv. Vyssh. Uchebn. Zaved. Mat., 1964, . 2, p. 8293. (In Russian).

[5] Ruziev M. Kh. Boundary Value Problem for a Mixed-Type Equation with Singular Coefficients. Russian Mathematics, 2022, Vol. 66, . 7, p. 1424.

[6] Ruziev M. Kh. and Yuldasheva N. T. On a Boundary Value Problem for a Class of Equations of Mixed Type. Lobachevskii Journal of Mathematics, 2023, Vol. 44, . 7, p. 29162929.

[7] Sabitov K. B. Initial-boundary value problem for a parabolic-hyperbolic equation with power-law degeneration on the type change line, Differential Equations. 2011. Vol. 47, p. 1490-1497.

[8] Sabitov K. B., Sidorov S. N. Inverse problem for degenerate parabolic-hyperbolic equation with nonlocal boundary condition, (Iz. VUZ), Russian Math. 2015, 1, p.4659; 59:1. (in Russian).

[9] Sabitov K. B., Sidorov S. N. Initial-boundary-value problem for inhomogeneous degenerate equations of mixed parabolic-hyperbolic type. Differential equations. Journal of Mathematical Sciences, 2019 Vol. 236,-6.

[10] Sidorov S. N. Nonlocal inverse problem of determining the right-hand sides of a degenerate equation of mixed parabolic-hyperbolic type. Nauchnyye vedomosti. Seriya: Matematika. Fizika, 2014, 19 (190), Issue 36.

[11] Sidorov S. N. Nonlocal problems for an equation of mixed parabolic-hyperbolic type with power degeneration, Russian Math. (Iz. VUZ), 59:12 2015, p. 4655. (in Russian).

[12] Sidorov S. N. Inverse problems for a mixed parabolic-hyperbolic equation with a degenerate parabolic part. Sib. Elektron. Mat. Izv., 2019, Vol. 16, p. 144157. (in Russian).

[13] Urinov A. K. Oripov D. D., On the solvability of an initial boundary problem for a high even order degenerate equation, J. Samara State Tech. Univ., Ser. Phys. Math. Sci., 2023, Vol. 27, . 4, p. 621644 (In Russian).

[14] Urinov A. K. Azizov M. S., About an initial boundary problem for a degenerate higher even order partial differential equation. Siberian journal of industrial mathematics. 2023. Vol. 26, . 2, p. 155170.

[15] Vagapov V. Z. Dirichlet problem for a mixed-type equation with two degenerate lines in the half-band . Vest. Sam. state technical university. Series Physics-mathematics, 2019. Vol. 23, . 1, p. 719.

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Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.337-343
DOI: 10.29229/uzmj.2025-2-32

Peridynamic model of membrane heating

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. This paper describes a peridynamic model for membrane heating, formulated as a hyper-singular integro-differential equation in a two-dimensional periodic setting. It states that the existence and uniqueness of the solution have been proven.

Keywords: peridynamics, hyper-singular integro-differential equation, the Cauchy problem, Fourier method

MSC (2020): 74H20, 74J30, 74B20

1. Introduction

We analyze the linearized peridynamic model of membrane heating, which leads to an integro-differential equation of the following form:

$$\frac{\partial\theta}{\partial t}+\int_{T^2} L\left(x,y\right)\left[\theta\left(x,t\right)-\theta\left(y,t\right)\right]dy=f\left(x,t\right), \quad x\in T^2, t>0, \tag{1.1}$$

with given initial conditions

$$\theta\left(x,0\right)=\varphi\left(x\right), \quad x\in T^2. \tag{1.2}$$

Here $x=(x_1,x_2)\in T^2=[-\pi,\pi]\times[-\pi,\pi]$. All functions are 2π-periodic with respect to each x_k for $k=1,2$. We analyze a peridynamic continuum model that incorporates integration over the differences in the displacement field ([6], [7], [10], [11]). We suppose, the kernel L is 2×2 matrix-valued function defined on the domain $T^2\times T^2$ initial function $\varphi: T^2\to R^2$ and $f: T^2\times[0,T]\to R^2$ represents the external heat source ([11]).

We analyze the kernel

$$L\left(x,y\right)=L\left(x-y\right), \quad x\in T^2, y\in T^2,$$

where the periodic function $L\left(x\right)$ has the form

$$L\left(x\right)=\frac{(x\otimes x)}{|x|^{4+\tau}}\chi\left(|x|\right), \quad \tau>0, x\in T^2. \tag{1.3}$$

The function $\chi\in C_0^\infty\left(R\right)$ satisfies the following conditions for some $\delta>0$ and a fixed $0<\alpha<1$:

$$\chi(r) \ = \ \begin{cases} 1 & \text{for} \quad r\leq(1-\alpha)\delta, \\ 0 & \text{for} \quad r\geq\delta, \end{cases}$$

and $0\leq\chi\left(r\right)\leq 1$ for all $r\in R$.

The parameter δ known as horizon, is typically chosen to be sufficiently small. The value of the parameter α is determined by the thickness of the boundary layer, within which the influence of surrounding particles is reduced to zero.

Problems involving similar types of kernels have been studied by various authors [2] -[6].

By utilizing the evenness of the kernel $L\left(x\right)$, equation (1.1) transformed into the following form:

$$\theta_t\left(x,t\right)-\frac{1}{2}\int_{T^2} L\left(y\right)\left[\theta\left(x+y,t\right)-2\theta\left(x,t\right)+\theta\left(x-y,t\right)\right]dy=f\left(x,t\right). \tag{1.4}$$

Now, we analyze this singular integro-differential equation. It is important to note that the kernel $K\left(x\right)$ exhibits a singularity of the form $|x|^{-2-\tau}$. In considered case, where $\tau>0$, this kernel is not integrable over T^2 and the corresponding integral operator is not bounded in $L_2(T^2)$ [1]. In what follows, we assume that

$$0 < \tau < 2. \tag{1.5}$$

For integer $\alpha > 0$, we define the Sobolev space $L_2^\alpha(T^2)$ as the space of vector functions $f \in L_2(T^2)$ for which the following norm is finite [9]

$$\|f\|^2_{L_2^\alpha} = 4\pi^2 \sum_{k\in Z^2} |f_k|^2(1+|k|^2)^\alpha. \tag{1.6}$$

Here

$$f_k = \frac{1}{(2\pi)^2}\int_{T^2} f(x)\, e^{-ikx} dx$$

are the Fourier coefficients of function f.

Set

$$A\theta(x) = \frac{1}{2}\int_{T^2} L(y)\left[\theta(x+y,t) - 2\theta(x,t) + \theta(x-y,t)\right] dy. \tag{1.7}$$

We consider the following Cauchy problem

$$\theta_t(x,t) - A\theta(x,t) = f(x,t), \quad x\in T^2, t>0, \tag{1.8}$$

$$\theta(x,0) = \varphi(x), \quad x\in T^2. \tag{1.9}$$

Definition 1.1. We define the solution of the Cauchy problem (1.8) - (1.9) as a function $\theta(x,t)$ that satisfies the following conditions:
$\theta(x,t)$ belongs to the space $L_2^\tau(T^2)$ for every $t\geq 0$;
it is continuous with respect to t in the norm of this space on the closed half-line $t\geq 0$;
it is continuously differentiable on the open halfline $t>0$ in the norm of $L_2(T^2)$;
it satisfies conditions (1.8) and (1.9).

Note that by imposing condition (1.5) for $L_2^\tau(T^2)$ we allow for the existence of solutions that may be discontinuous in spatial variables.

We prove the following statement:

Theorem 1.2. *Let $\beta \geq 0$. Assume that the initial function $\varphi(x)$ belongs to Sobolev space $L_2^{\beta+\tau}(T^2)$ and that $f(x,t)$ depends continuously on $t \geq 0$ in the norm of $L_2^\beta(T^2)$. Then, the solution of the Cauchy problem (1.8) – (1.9) exists and belongs to $L_2^\beta(T^2)$.*

In Section 2, we transform the hyper-singular operator (1.7) into a regular integro-differential operator. While the proof follows a reasoning similar to that in [5], we outline the key steps and provide additional details for the parts that differ from the referenced work. Section 3 examines the Fourier transform of the kernel under consideration and establishes the proof of estimate (1.8). In Section 4, we derive the solution to the Cauchy problem for the Fourier-transformed version of the original Cauchy problem. Finally, Section 5 presents the proof of Theorem 1.

2. Conversion of the hyper–singular operator

We define the differential operator $\nabla\otimes\nabla$ as follows:

$$\nabla\otimes\nabla = \begin{pmatrix} \dfrac{\partial^2}{\partial x_1^2} & \dfrac{\partial^2}{\partial x_1\partial x_2} \\ \dfrac{\partial^2}{\partial x_2\partial x_1} & \dfrac{\partial^2}{\partial x_2^2} \end{pmatrix}. \tag{2.1}$$

For any function $\theta\in C^\infty(T^2)$, the Fourier coefficients of the function $u = (\nabla\otimes\nabla)\theta$ satisfy the following equation:

$$u_k = (ik \otimes ik)\,\theta_k. \tag{2.2}$$

Let

$$I\Delta \;=\; \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}, \tag{2.3}$$

where Δ is the Laplace operator and I is the identity matrix.
We define the following differential operator:

$$B(\nabla) = \frac{1}{\tau(\tau+2)}\left[(\nabla\otimes\nabla) + \frac{1}{\tau} I\Delta\right]. \tag{2.4}$$

Proposition 2.1. *For any $\tau > 0$ the following holds:*

$$B(\nabla)\frac{1}{|x-y|^{\tau}} = \frac{(x-y)\otimes(x-y)}{|x-y|^{\tau+4}}. \tag{2.5}$$

The validity of this statement is established by direct calculation.

Corollary 2.2. *Without loss of generality, we can assume that the kernel (1.3) takes the following form:*

$$L(x) \;=\; B_\tau(\nabla)\,\frac{\chi(|x|)}{|x|^{\tau}} \;+\; V(x), \quad \tau > 0, \quad x \in \mathbb{T}^2, \tag{2.6}$$

where $V(x)$ is a matrix-function with entries $v^{(ij)} \in C_0^\infty(\mathbb{T}^2)$.

Note that we obtain the matrix

$$B(ik) = \frac{-1}{\tau(\tau+2)}\left[(k\otimes k) + \frac{k\cdot k}{\tau} I\right], \quad k \in Z^2. \tag{2.7}$$

According (2.2), we get

$$v_k = [B(\nabla)\,u]_k = B(ik)\,u_k. \tag{2.8}$$

Using proposition 2.1, we transform the hyper-singular operator A into a regular form.

Proposition 2.3. *Let $g \in C_0^\infty(T^2)$, $u \in C^\infty(T^2)$ and $0 < \varepsilon < 1$. Then following*

$$\int_{T^2}\left(\frac{\partial^2}{\partial y_k \partial y_j}\frac{g(y)}{|y|^{2-\varepsilon}}\right)[u(x+y) - 2u(x) + u(x-y)]\,dy = \int_{T^2}\frac{g(y)}{|y|^{2-\varepsilon}}\left[\frac{\partial^2 u(x+y)}{\partial y_k \partial y_j} + \frac{\partial^2 u(x-y)}{\partial y_k \partial y_j}\right]dy \tag{2.9}$$

is valid.

Proposition 2.4. *For any $u \in C^\infty(T^2)$ the following equation*

$$Au(x) = \int_{T^2}\frac{\chi(|y|)}{|y|^{\tau}}B(\nabla_y)\,u(x-y)\,dy + \frac{1}{2}\int_{T^2}[V(y) + V(-y)]\,u(x+y)\,dy - \left(\int_{T^2} V(y)\,dy\right)u(x) \tag{2.10}$$

is valid.

3. Fourier expansion of a singular kernel

The main part of the hyper-singular operator B , defined by (1.7), is an integro-differential operator on appearing on the right-hand side of equation (2.10).

Consider for $0 < \tau < 2$ kernel

$$H(x) = \frac{\chi(|x|)}{|x|^{\tau}}, \quad x \in T^2. \tag{3.1}$$

Set

$$\Upsilon(k) = (2\pi)^2\left[H_k B(k) - V_k - V_{-k} + V_0\right], \quad k \in Z^2. \tag{3.2}$$

Here H_k are the Fourier coefficients of H

$$H_k = (2\pi)^{-2} \int_{T^2} H(x)\, e^{-ikx} dx.$$

Denote by V_k the matrix which elements $v_k^{(ij)}$ are Fourier coefficients of the elements $v^{(ij)}$ of the matrix $V(x)$.

From (2.10) follows, that for any $u \in C^\infty(T^2)$ the Fourier coefficients of the Au have the form

$$(Au)_k = -\Upsilon(k)\, u_k, \quad k \in Z^2. \tag{3.3}$$

Let estimate the Fourier coefficients of the (3.1).

Proposition 3.1. *For any natural N, the Fourier coefficients H_k of H satisfy the estimate*

$$H_k = \frac{C_\tau}{|k|^{2-\tau}} + O\left(|k|^{-N}\right), \quad k \neq 0, \tag{3.4}$$

where C_τ depends only on τ.

For any 2×2 matrix $M = \left(m^{(ij)}\right)$, we define

$$|M| = \left(\sum_{i,j=1}^{2} |m^{(ij)}|^2 \right)^{1/2}.$$

Proposition 3.2. *For any natural N, the following holds:*

$$\Upsilon(k) = (2\pi)^2 C_\tau B(k)\, |k|^{\tau-2} + V_0 + O(|k|^{-N}), \quad k \in Z^2, \tag{3.5}$$

here $\Upsilon(k)$ is defined by (3.2).

The proof follows from proposition 3.1 and the fact that the matrix function V belongs to $C_0^\infty(T^2)$.

Corollary 3.3. *The following estimate holds:*

$$|\Upsilon(k)| \leq C(1 + |k|^2)^{\tau/2}, \quad k \in Z^2. \tag{3.6}$$

Proposition 3.4. *Let $0 < \tau < 2$. Then for any $u \in C^\infty(T^2)$ the following estimate is valid*

$$\|Au\|_{L_2(T^2)} \leq C \|u\|_{L_2^\tau(T^2)}. \tag{3.7}$$

Proof. According to proposition 2.4, the hyper- singular operator (1.7) can be expressed in a regular form for any function $u \in C^\infty(T^2)$. From (3.3) and (3.6), we obtain:

$$|(Au)_k|^2 \leq C |u_k|^2 (1 + |k|^2)^\tau.$$

Therefore, by Parseval's theorem, we can express it as:

$$\|Au\|^2_{L_2(T^2)} = (2\pi)^2 \sum_{k \in Z^2} |(Au)_k|^2 \leq C \sum_{k \in Z^2} |v_k|^2 (1 + |k|^2)^\tau \leq C \|u\|^2_{L_2^\tau(T^2)}.$$

$\square$

Corollary 3.5. *The hyper-singular operator $A : C^\infty(T^2) \to L_2(T^2)$ can be extended as continuous operator $A : L_2^\tau(T^2) \to L_2(T^2)$.*

4. The Solvability Of Differential Equation In Hilbert Space

Now, we analyze the differential equation (1.8) with the initial condition (1.9).

By transitioning to Fourier coefficients, we obtain the following Cauchy problem for the differential equation:

$$\frac{d\theta_k(t)}{dt} + \Upsilon(k)\,\theta_k(t) = f_k(t), \quad k\in Z^2, t>0, \tag{4.1}$$

with initial condition

$$\theta_k(0) = \varphi_k. \tag{4.2}$$

Matrix $\Upsilon(k)$ defined by (3.2).

To determine the solution of the problem (4.1)-(4.2), we introduce a matrix function that depends on the parameter t:

$$E(t,\Upsilon) = \sum_{m=0}^{\infty}(-1)^m\frac{t^m}{m!}\Upsilon^m. \tag{4.3}$$

The following estimates are obvious.

Proposition 4.1. *Let $\Upsilon = \Upsilon(k) > 0$. Then for any $t \geq 0$ the following estimate holds:*

$$\|E(t,\Upsilon)\|\leq 1. \tag{4.4}$$

Proposition 4.2. *For any $T > 0$ there exists a constant $C_T > 0$ such that the following estimate holds:*

$$\|E(t,\Upsilon(k))\|\leq C_T, \quad k\in Z^2, \tag{4.5}$$

for $0\leq t\leq T$.

Now, the solution of the problem (4.1)-(4.2) can be expressed as follows:

$$\theta_k(t) = E(t,\Upsilon(k))\,\varphi_k + \int_0^t E(t-s,\Upsilon(k))\,f_k(s)\,ds. \tag{4.6}$$

Proposition 4.3. *Let $\varphi(x)$ belongs to $L_2(T^2)$, and let $f(x,t)$ depends continuously on $t \geq 0$ in the norm of $L_2(T^2)$. Then functions (4.6) form a sequence of Fourier coefficients of some function $\theta(x,t)$, which depends continuously on $t \geq 0$ in the norm of $L_2(T^2)$.*

Proof. Using proposition 4.2, we get estimate for functions (4.6)

$$|\theta_k(t)| \leq C_T\,|\varphi_k| + C_T\int_0^t |f_k(s)|\,ds, \quad 0\leq t\leq T. \tag{4.7}$$

By Parseval's equation, the right-hand side belongs to l_2. Therefore, we can conclude that the sequence $\theta_k(t)$ is an element of l_2.

In this case, by the Riesz-Fischer theorem, there exists a function $\theta(x,t)$ whose Fourier coefficients coincide with the sequence $\theta_k(t)$.

Moreover, since the right-hand side of (4.6) depends continuously on $t\geq 0$ the functions $\theta_k(t)$ are also continuous on the half-line $t\geq 0$. By Parseval's equation, it follows that the function $\theta(x,t)$ is continuous on the half-line $t\geq 0$ in the norm of $L_2(T^2)$. □

5. Proof Of Theorem

We prove that, under condition of the Theorem 1.2 , the function

$$\theta(x,t) = \sum_{k\in Z^2}\theta_k(t)\,e^{ikx}, \tag{5.1}$$

where the coefficients $\theta_k(t)$ are given by (4.6), serves as a solution to the Cauchy problem (1.8) - (1.9).

Proposition 5.1. *Let $\beta\geq 0$, $\varphi(x)\in L_2^{\beta+\tau}(T^2)$, and let $f(x,t)$ depend continuously on $t\geq 0$ in the norm of $L_2^{\beta}(T^2)$. Then function (5.1) is continuous function of $t\geq 0$ in the norm of $L_2^{\beta}(T^2)$ and belongs to $L_2^{\beta}(T^2)$ with respect to the variable x.*

Proof. It follows from propositions 4.1 and 4.2 that:

$$|\theta_k(t)|\leq|\varphi_k|+C\int_0^t|f_k(s)|\,ds.$$

Further,

$$|\theta_k(t)|\,(1+k_1^2+k_2^2)^{\beta/2}\leq|\varphi_k|\,(1+k_1^2+k_2^2)^{\beta/2}+(1+k_1^2+k_2^2)^{\beta/2}\int_0^t|f_k(s)|\,ds.$$

According to the given condition, the right-hand side is an element of $L_2(T^2)$. Consequently, the left-hand side also belongs to the same space. This implies that $\theta\in L_2^{\beta}(T^2)$.

The continuity with respect to t is established using the same method as demonstrated in the proof of proposition 4.3. □

Proposition 5.2. *Let proposition 5.1 is fulfilled. Then $\theta(x,t)$, defined by (5.1), is continuously differentiable with respect to t on the half-line $t\geq 0$ in the norm of $L_2^{\beta}(T^2)$.*

Proof. Set

$$w_k(t)=\frac{d\theta_k(t)}{dt}.$$

Then

$$w_k(t)=-\Upsilon(k)\,\theta_k(t)+f_k(t),\quad k\in Z^2, t>0. \tag{5.2}$$

Further,

$$w_k(t)\,(1+k_1^2+k_2^2)^{\beta/2}=-\Upsilon(k)\,\theta_k(t)\,(1+k_1^2+k_2^2)^{\beta/2}+f_k(t)\,(1+k_1^2+k_2^2)^{\beta/2}.$$

From (4.7) follows

$$|w_k(t)|\,(1+k_1^2+k_2^2)^{\frac{\beta}{2}}\leq C\,|\theta_k(t)|\cdot(1+k_1^2+k_2^2)^{\frac{\beta+\tau}{2}}+|f_k(t)|\cdot(1+|k|^2)^{\beta/2}.$$

By proposition 5.1 and the Parseval equality, the right-hand side is an element of l_2. The same holds for the left-hand side. This implies that the function

$$w(x,t)=\sum_{k\in Z^2}\theta_k'(t)\,e^{ikx}$$

belongs to $L_2^{\beta}(T^2)$.

It is clear that $w(x,t)=\theta_t(x,t)$. From (5.2) and proposition 5.1 $w(x,t)$ continuously depends on $t\geq 0$. □

5.1. **Proof of the Theorem 1.2.**

Proof. 1.*Existence*. We show that the function $\theta(x,t)$, defined by (5.1), is a solution to the Cauchy problem (1.8)-(1.9). By proposition 4.3, it follows that $\theta(x,t)$ satisfies equation (1.8).

Similarly, the initial conditions (1.9) can be derived from relations (4.2). The remaining properties of the solution follow from propositions 5.1 and 5.2.

2. *Uniqueness*. Suppose there are two solutions, $\theta_1(x,t)$ and $\theta_2(x,t)$, to the problem (1.8)- (1.9). Then the difference $v(x,t)=\theta_1(x,t)-\theta_2(x,t)$ is the solution of homogeneous problem

$$\frac{\partial v(x,t)}{\partial t}-Av(x,t)=0, x\in T^2, t>0, \tag{5.3}$$

with initial condition

$$v(x,0)=0, x\in T^2. \tag{5.4}$$

From equation (5.3) and the conditions of Theorem 1.2, it follows that the Fourier coefficients of $v(x,t)$ satisfy the homogeneous equation

$$\frac{dv_k(t)}{dt} + \Upsilon(k)\, v_k(t) = 0, k{\in}Z^2, t > 0 \tag{5.5}$$

with initial conditions

$$v_k(0) = 0. \tag{5.6}$$

Integrating (5.5) with respect to t and taking into account (5.6) we get

$$v_k(t) = -\int_0^t \Upsilon(k)\, v_k(s)\, ds.$$

In the obtain Volterra-type integral equation the right-hand side operator is quasinilpotent, which implies that the equation admits only the trivial solution $v_k(t) \equiv 0$. Consequently, $v(x,t) \equiv 0$, and thus $\theta_1(x,t) = \theta_2(x,t)$.

□

References

[1] S. A. Alimov, Y. Cao and O. A. Ilhan.; On the problems of peridynamics with special convolution kernels. J. of Integral Equations and Applications,–2014.–26(3).–P.301–321.

[2] S. A. Alimov and S. Sheraliev.;On the solvability of the singular equation of peridynamics. Complex Variables and Elliptic Equations,–2019.–64.– No.5.–P.873-887.

[3] S. A.Alimov, S. N. Sheraliev.; On the solvability of hypersingular equation of peridynamics, Bulletin of National University of Uzbekistan, Mathematics and Natural Sciences,–2020.–vol.3.–No 4.–P.278-298.

[4] Sh. A. Alimov and A. V. Yuldasheva; On the Solvability of the Peridynamic Equation with a Singular Kernel, Differential Equations, –2021. –57.–No. 3.–P.353–365.

[5] S. Alimov, A. Yuldasheva; Solvability of Singular Equations of Peridynamics on Two-Dimensional Periodic Structures , Journal of Peridynamics and Nonlocal Modeling.–2023.–5.–P.241–259.

[6] Q. Du, J. R. Kamm, R. B. Lehoucq, and Michael L. Parks.; A new approach for a nonlocal, nonlinear conservation law.SIAM J. Appl. Math.,–2012.–72(1).–P.464–487.

[7] M. Gunzburger and R. Lehoucq.; A nonlocal vector calculus with application to nonlocal boundary value problems. Multiscale Model. Simul.,–2010.–8(5). –P.1581–1598.

[8] L. V. Kantorovich, G. P. Akilov.; Functional Analysis (Russian). "Nauka", Moscow.–1977.

[9] S. M. Nikol'skiy.; Approximation of functions of several variables and imbedding theorems, Grundl. Math. Wissensch., 205, Springer-Verlag, New York. –1975.

[10] P. Seleson, Michael L. Parks, M. Gunzburger, and R. B. Lehoucq.; Peridynamics as an upscaling of molecular dynamics. Multiscale Model. Simul.,–2009.–8(1).–P.204–227.

[11] S. A. Silling.; Reformulation of elasticity theory for discontinuities and long-range forces. J. Mech. Phys. Solids,–2000.–48(1). –P.175–209,

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Computer imposition: *A.F. Aliyev*

The journal was registered by the Press and Information Agency of the Republic of Uzbekistan on December 22, 2006. Register. No 0044.

Handed over to the set on 11/03/2022. Signed for printing on 12/04/2023
Format 60×84 1/16. Literary typeface. Offset printing.

V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences 9 University st. 100174

Printed in a printing house "MERIT-PRINT"
Tashkent city, Yakkasaray district, Sh. Rustaveli street, 91